Introduction to Hyperbolic Geometry

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Lecture overview

Disclaimer

These are notes from a course given by Subhojoy Gupta. They have been written and TeX'ed during the lecture and have not been proofread, so there's bound to be a number of typos and mistakes that should be attributed to me rather than the lecturers. Also, I've made these notes primarily to be able to look back on what happened with ease. That being said, feel very free to send any comments and/or corrections to fuglede@qgm.au.dk.

The most recent version of these notes is available at http://home.imf.au.dk/pred.

1st Lecture, August 31st 2012

1 Introduction and motivation

1.1 Uniformization

This course will be an introduction to hyperbolic geometry in dimension 2 and 3. References are "Fuchsian groups" by Katok, "Automorphisms of surfaces of surfaces" by Casson and Bleiler, Thurston's notes on 3-manifold geometry, and Hubbard's book on Teichmüller theory.

We will start with the uniformization theorem, which connects the world of complex analysis with hyperbolic geometry.

Theorem 1.1.1 (Uniformization Theorem, Poincaré, Koebe). *Any simply connected Riemann surface is conformally equivalent to one* $\mathbb{C}P^1$, $\mathbb{C}$, *or* $\mathbb{D}$.

Recall here that a *Riemann surface* is a surface S with a maximal atlas of charts to $\mathbb{C}$ such that the transition maps are conformal (picture). This data is also called a conformal structure on S. Of course, such a surface is not necessarily simply connected, but one can the look at its universal cover. Examples of such surfaces are abundant.

An example of a construction of Riemann surfaces is the following of Euclidean tori/elliptic curves. Consider $\mathbb{C}$ with euclidean metric ($ds^2 = dx^2 + dy^2$). Take a parallelogram P in $\mathbb{C}$, with corners $0, 1, \tau$. One can identify opposite sides by translations. It is a fact that the space of conformal automorphisms is

$$\mathrm{Aut}(\mathbb{C}) = \{az + b \mid a \neq 0\}.$$

In particular, translations are conformal (take $a = 1$). The resulting torus obtained by identifying opposite sides is a Riemann surface. Its universal cover is $\mathbb{C} \to T^2$, with an action of $\pi_1 T^2 = \mathbb{Z} \oplus \mathbb{Z}$.

It is a fact that a closed Riemann surface of genus $g \geq 2$ has universal cover $\mathbb{D}$. The reason hyperbolic geometry enters the picture is that the hyperbolic metric is the unique metric (up to scaling) that is invariant under $\mathrm{Aut}(\mathbb{D})$.

Another way to state the uniformization theorem is the following.

Theorem 1.1.2 (Uniformization Theorem II)**.** *Any closed Riemann surface X of genus $g \geq 2$ is conformally equivalent to $\mathbb{H}^2/\Gamma = \mathbb{D}/\Gamma$, Γ a discrete subgroup of* $\mathrm{Aut}(\mathbb{D}) = \mathrm{PSL}_2(\mathbb{R})$.

1.2 Course plan

The first part of the course covers the topics:

- Geometry of $\mathbb{H}^2$.
- Isometries of $\mathbb{H}^2$ (the group $\mathrm{PSL}_2(\mathbb{R})$).
- Fuchsian groups (the groups Γ above), examples.
- Geometry of hyperbolic surfaces; the collar lemma and the Margulis lemma.

The second part will be:

- Teichmüller space T_g. Fenchel–Nielsen coordinates.
- The mapping class group and its action on T_g, as well as the Nielsen–Thurston classification.

In the third part:

- Measured laminations.
- Earthquakes.
- The solution to the Nielsen Realization Problem (by Kerckhoff).

Finally:

- Hyperbolic 3-manifolds.
- Isometries of $\mathbb{H}^3$ ($\mathrm{PSL}_2\mathbb{C}$).
- Quasifuchsian manifolds.
- Complex earthquakes (a la McMullen).

For the 3-manifolds $M = \mathbb{H}^3/\Gamma$, we have the following:

Theorem 1.2.1 (Mostow rigidity)**.** *IF M, N are closed hyperbolic 3-manifolds and $f : \pi_1 M \to \pi_1 N$ is an isomorphism (e.g. when M, N are homeomorphic), then there is an isometry $F : M \to N$, with $F_* = f$.*

The same theorem holds with "closed" replaced by "finite volume" (a result due to Rasad). It turns out that finite volume hyperbolic manifolds have torus boundary (this folllows from the Margulis lemma).

Theorem 1.2.2 (Ending Lamination Theorem, Minsky)**.** *For a hyperbolic 3-maniofld with boundary of genus ≥ 2, the metric is unique given $\pi_1 M$ and "ending data" on ∂M.*

1.3 Sketch of proof of uniformization

The proof we sketch is loosely based on Hubbard's exposition.

Warm-up case: Consider a simply connected domain with smooth boundary Ω in $\mathbb{C}$. For Ω, the uniformization theorem is also called the Riemann mapping theorem. In this case, there is a conformal map $f : \Omega \to \mathbb{D}$. The idea here comes from physics: Put a point charge p in Ω and ground the boundary $\partial\Omega$, viewing Ω as a metallic plate. Consider the equipotential curves L_c, $0 < c < \infty$ (so the potential is 0 on $\partial\Omega$ and ∞ at p) and the gradient lines l_θ, $0 < \theta \leq 2\pi$. The map f takes the L_c to concentric circles and the l_θ to the lines perpendicular to the circles. That is, we map $L_c \cap l_\theta \mapsto (e^{-c}, \theta)$, which by construction gives a conformal map. Mathematically, this can be described as follows.

Definition 1.3.1. A *Green's function* on Ω with singularity at p is a function $g : \Omega \to \mathbb{R}^+$ such that

1. g is harmonic on $\Omega \setminus \{p\}$, (that is, $\Delta g = 0$),
2. g vanishes on the boundary $\partial\Omega$, and
3. $g + \ln|z - p|$ (which is also harmonic) is bounded around p.

Assume that Ω admits a Green's function g. What we want is that $g = \mathrm{Re}(G)$, where G is holomorphic, and take $f = e^{-G}$. Locally, at $U \subseteq \Omega \setminus p$, one can find a harmonic conjugate of g (i.e. another harmonic function h such that $G = g + ih$ is holomorphic) and then put $f_U = e^{-G}$. Around p, $f_0 = ze^{-G_0}$, where G_0 corresponds to the harmonic function $g + \ln|z - p|$. Since Ω is simply connected, we can define the function e^{-G} by analytic continuation.

Now we sketch in more detail how to build such a Green's function. To do so, one needs to solve the Dirichlet Problem: Given Ω a simply connected domain, we want to find a function $u : \Omega \to \mathbb{R}$ such that

1. $\Delta u = 0$ in Ω,
2. $u|_{\partial\Omega} = f$.

Solving this, one gets the Green's function. Take $f = -\ln|z-p|$ on $\partial\Omega$, $g = -u-\ln|z-p|$. To solve the problem, we need Perron's Method: Given f, consider $\mathcal{F}$, the family of functions $h : \overline{\Omega} \to \mathbb{R}$ such that

1. h is subharmonic in Ω (i.e. $\Delta h \geq 0$),
2. and $h|_{\partial\Omega} \leq f$.

Then $u := \sup \mathcal{F}$ solves the Dirichlet problem. Here,

$$u(z) = \sup_{h \in \mathcal{F}} h(z).$$

The idea/toy example is the following: Consider instead $f : [0, 1] \to \mathbb{R}$ and replace "subharmonic" with "convex" and "harmonic" with "linear"; here it is easy to believe that u does the job. This is referred to as "local harmonic majorization". It works because of the following facts.

Fact 1.3.2. *If f, g are subharmonic, then so is* $\max\{f, g\}$.

Fact 1.3.3. *If f is harmonic in $D \subseteq \Omega$ and subharmonic in $\Omega \setminus D$, then f is subharmonic in Ω.*

We turn now to the general case: Let X be a simply connected Riemann surface. Assume that X is non-compact. We shall show that $X \cong \mathbb{D}$ or $X \cong \mathbb{C}$.

Step 1: Exhaust X by compact subsurfaces $\{X_n\}$ with smooth boundary, i.e. $X_i \subseteq \int(X_{i+1})$, $\bigcup_{n=1}^{\infty} X_n = X$, and X_n is simply connected.

Step 2: Use Preron's method to construct a Green's function on each X_n with a singularity at $x_0 \in X_0$.

Step 3: Get conformal homeomorphisms $\varphi_n : X_n \to \overline{\mathbb{D}}$. Rescale the φ_n to $\tilde{\varphi}_n : X_n \to B_{r_n}$ so that the derivative at x_0 is 1. One shows that the φ_n form a normal family, and the limit $\varphi : X \to \mathbb{C}$ is a conformal homeomorphism.

If X is compact and simply connected, then $X \setminus \{x_0\}$ is simply connected. Now $\mathbb{D} \cup \{x_0\}$ is not a Riemann surface (by Riemann's removable singularity theorem), and $X \setminus \{x_0\}$ has to be $\mathbb{C}$.

Going backwards, recall that solving the Dirichlet problem implying that a Green's functions exists had to do with the complex plane, but it also works in general. Going back to step 1, one way of obtaining the X_n is by considering a proper C^∞ Morse function $g : X \to \mathbb{R}^+$ and let X_n be the union of $g^{-1}([0,n])$ with the pireces of $X \setminus g^{-1}([0,n])$ with compact closure. One way of constructing such a Morse function is to take a locally finite cover of X by compact sets and take a partition of unity $\{\psi_n\}$ and take $g = \sum_{n=1}^{\infty} n\psi_n$. One also needs to show that X is second countable for a locally finite cover to exist; this is always the case for Riemann surfaces.

Let Z be a component of $X \setminus X_n$. The first claim is that ∂Z is a single component; the proof is a homological argument. The second claim is that there exists a continuous retract $r : Z \to \partial Z$. It follows that the X_n are simply connected.

2 The hyperbolic plane

2nd Lecture, September 4th 2012

Today, we will start describing the hyperbolic plane $\mathbb{H}^2$. We will discuss three models; the upper half plane model, $\mathcal{H}$, the Poincaré disk model $\mathbb{D}$, and the hyperboloid model.

2.1 The upper half plane

Let

$$\mathcal{H} = \{(x, y) \mid y \geq 0\}.$$

The hyperbolic metric is a Riemannian metric on this space. The upper half plane is conformally equivalent to $\mathbb{D}$, which we will talk about seperately. The conformal map is $\mathcal{H} \to \mathbb{D} :\mapsto \frac{z-i}{z+i}$. The *upper half plane* is $\mathbb{H} = (\mathcal{H}, ds^2_{\text{hyp}})$, where

$$ds^2_{\text{hyp}} = \frac{ds^2_{\text{euc}}}{y^2} = \frac{dx^2 + dy^2}{y^2}.$$

This means that for a curve γ in $\mathcal{H}$, the length $l(\gamma)$ of the curve is

$$\int_0^1 \frac{\|\gamma'(t)\|}{y(t)}\, dt = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt.$$

The *hyperbolic distance* between two points $z, w \in \mathcal{H}$ is defined to be the length of the shortest path between z and w, i.e.

$$d_h(z, w) = \inf l(\gamma),$$

where the inf is over paths between z and w. We will see that the topology given by this metric coincides with the Euclidean topology.

Lemma 2.1.1. *Vertical lines are geodesics.*

Proof. Consider two points (x_0, a), (x_0, b). Let $\gamma(t) = (x(t), y(t))$, $0 \leq t \leq 1$, be a path connecting (x_0, a), (x_0, b) and assume $b > a$. Then

$$l(\gamma) = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt \geq \int_0^1 \frac{y'(t)}{y(t)}\, dt = \ln y(t)|_0^1 = \ln \tfrac{b}{a}.$$

Consider the path $p(t) = (x_0, t)$, $a \leq t \leq b$. Then $l(p) = \ln \frac{b}{a}$. □

Corollary 2.1.2. *The hyperbolic distance between points on the imaginary axis is*

$$d_h(ia, ib) = |\ln \tfrac{b}{a}|.$$

Definition 2.1.3. A metric space is *complete* if and only if for any sequence of points $P_1, P_2, \ldots$ in X, we have that

$$\sum_{i=1}^{\infty} d(P_i, P_{i+1}) < \infty$$

implies that $P_i \to P_\infty$ for some P_∞ in X.

Lemma 2.1.4. *The hyperbolic metric in $\mathcal{H}$ is complete.*

One should think of this as saying that we can not reach the boundary of $\mathcal{H}$ by finite length paths.

Proof. Let $\sum_{i=1}^{\infty} d(P_i, P_{i+1}) = L < \infty$ (*). If $P_i = (x_i, y_i)$. Then by the triangle inequality, we have $d(P_1, P_i) < L$. Then by 2.1.2,

$$|\ln \frac{y_1}{y_i}| \leq L.$$

This implies that $y_1 e^{-L} \leq y_i \leq y_1 e^L$. In particular, the points P_i are all above some horizontal line. Let $c = y_1 e^{-L}$. In this region $\{y \geq c\}$, the hyperbolic and euclidean metrics are compatible, i.e.

$$\frac{1}{K} ds^2_{\text{euc}} \leq ds^2_{\text{hyp}} \leq K ds^2_{\text{euc}}.$$

Now use the completeness of ds^2_{euc}. □

For later, we will use the following notation: $\partial H^2 = \mathbb{R} = \{(x, 0) \mid x \in \mathbb{R}\} \cup \{\infty\}$ will be called the *boundary at infinity*.

Let us now see what the isometries of this metric are.

Example 2.1.5. If we use z as the coordinate in the upper half plane, we have

$$ds^2_{\text{hyp}} = \frac{|dz|^2}{(\text{im} z)^2}$$

The following are isometries:

- Reflection $z \mapsto -\overline{z}$ about the imaginary axis.
- Translations $z \mapsto z + a$, $a \in \mathbb{R}$.
- Homothetics $z \mapsto \lambda z$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$.
- Inversion $z \mapsto -\frac{1}{z}$.

Inversion can be understood geometrically (picture).

Fact 2.1.6. *Inversion maps the semi-circle $|z - a| = a$ to a vertical line.*

Proof. For points on the semi-circle, we have $(z - a)(\overline{z} - a) = a^2$, so $z\overline{z} - a(z + \overline{z}) = 0$, or

$$\frac{z + \overline{z}}{z\overline{z}} = \frac{1}{a}.$$

This implies that $2\text{Re}(1/z) = 1/a$, so under inversion, these points are mapped to points with the same real part, $-\frac{1}{2a}$. □

We turn now to *linear fractional transformation*, also called *Möbius transformations* or *Möbius maps*: For $A \in \mathrm{SL}_2\mathbb{R}$, we define $T_A : \mathcal{H} \to \mathcal{H}$ by $z \mapsto \frac{az+b}{cz+b}$. This is invariant under $A \mapsto -A$, so we may as well use $\mathrm{PSL}_2\mathbb{R} = \mathrm{SL}_2\mathbb{R}/\{\pm I\}$.

Fact 2.1.7. *We have $T_A \circ T_B = T_{AB}$.*

Fact 2.1.8. *Any Möbius transformation is a composition of translations, homotheties, and inversions.*

Lemma 2.1.9. *We have* $\mathrm{PSL}_2\mathbb{R} \subseteq \mathrm{Isom}(\mathbb{H}^2)$.

Proof. Let T be a Möbius map. Then

$$w = T(z) = \frac{az+b}{cz+d} = \frac{ac|z|^2 + adz + bc\overline{z} + bd}{|cz+d|^2},$$

and

$$\mathrm{Im}(w) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

This implies that Möbius maps preserve the upper half plane. One finds that

$$\frac{dT}{dz} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{1}{(cz+d)^2}. \tag{1}$$

Now if $\gamma(t) = (x(t), y(t))$, $0 \leq t \leq 1$, we want to show that $l(\gamma) = l(T(\gamma))$. Write $T(\gamma(t)) = (u(t), v(t))$. We have

$$l(T(\gamma)) = \int_0^1 \frac{|dT/dt|}{v(t)}\, dt = \int_0^1 \frac{|dT/dz||dz/dt|}{v(t)}\, dt.$$

Equation (1) implies that $|dT/dz| = v/y$, so

$$l(T(\gamma)) = \int_0^1 \frac{|dz/dt|}{y(t)}\, dt = l(\gamma).$$

□

Lemma 2.1.10. *The geodesics in $\mathcal{H}$ are the vertical lines together with all semicircles perpendicular to the real axis.*

Proof. Any semi-circle can be translated to one beginning at 0 which can be inverted to a vertical line, and we saw that those were geodesics before. Any two points of the upper half plane can be transformed by Möbius maps to lie on a vertical lines, and so by uniqueness of geodesics, we have found all possible ones. □

Example 2.1.11. We ask: What are the points on $\mathcal{H}$ that are at distance less that some given $r > 0$ from the vertical line γ through i. This turns out to be a cone from 0 of angle 2θ in $\mathcal{H}$, where

$$r = \frac{1}{2} \ln \frac{1 + \sin\theta}{1 + \sin\theta},$$

or $\sin\theta = \tanh r$.

We could also ask, given some point, say $i \in \mathcal{H}$, what are the points of distance at most $r > 0$ from i? In other words, what is the hyperbolic circles of radius r around i. It turns out that these are euclidean circles. For instance, if $|z - i| \leq r$, then by distance formulas to be mentioned below,

$$\frac{|z-i|^2}{2\mathrm{Im} z} \leq r,$$

which can be written as the formula for a Euclidean disk. This proves the following:

Lemma 2.1.12. *The topology induced by the hyperbolic distance function is the usual Euclidean topology.*

We need some distance formulae: For $z, w \in \mathcal{H}$, the hyperbolic distance is

$$\cosh d(z,w) = 1 + \frac{|z-w|^2}{2\mathrm{Im}z\mathrm{Im}w},$$
$$d(z,w) = \ln\left(\frac{|z-\overline{w}| + |z-w|}{\overline{z-\overline{w}} - \overline{z-w}}\right).$$

To see the first formula, note that the right hand side is invariant under translations, homotheties, and inversions. So it is invariant under all Möbius maps. On the other hand, one can map any two points to a vertical line, and the formula holds for points on a vertical line, say $z = ia$, $w = ib$, which can be seen by just plugging in the numbers. The second formula can be seen by a similar proof.

Remark 2.1.13. The Möbius maps can be viewed as $\mathrm{PSL}_2\mathbb{R} \subseteq \mathrm{PSL}_2\mathbb{C}$, Möbius maps acting on $\hat{\mathbb{C}}$. So we consider maps $z \mapsto \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$. These maps are once again generaeted by translations, homotheties, and inversions. It takes circles and lines to circles and lines, and the action of $\mathrm{PSL}_2\mathbb{C}$ on $\hat{\mathbb{C}}$, and it is triply transitive – i.e. it takes any three points to any three points.

In our case, $\mathrm{PSL}_2\mathbb{R}$ preserves $\overline{\mathcal{H}}$.

Lemma 2.1.14. *The action of* $\mathrm{PSL}_2\mathbb{R}$ *is triply transitive on* $\partial\mathcal{H}$*.*

Proof. Given (a_1, a_2, a_3), these can be mapped to $(0, \infty, 1)$ by the map

$$z \mapsto \frac{z-a_1}{z-a_2}\frac{a_3-a_2}{a_3-a_1}.$$

□

The *cross-ratio* of 4 points (z_1, z_2, z_3, z_4) is the number

$$\rho(z_1, z_2, z_3, z_4) = \frac{z_1-z_2}{z_1-z_3}\frac{z_4-z_3}{z_4-z_2}.$$

The cross-ratio is Möbius invariant.

The following follows from what we discussed before.

Lemma 2.1.15. *The hyperbolic plane is a* homogeneous *and* isotropic *space. That is, given* $z \in \mathcal{H}$ *and a unit tangent vector* $v \in T_z\mathcal{H}$*, there is a unique Möbius map* T*, such that* $Tz = i$*, and* $DTv = (0, 1)$*, the unit tangent vector at* i *along the imaginary axis.*

Corollary 2.1.16. *The unit tangent bundle to* $\mathbb{H}^2$ *is* $T^1\mathbb{H}^2 \cong \mathrm{PSL}_2\mathbb{R}$.

Remark 2.1.17. The hyperbolic plane is a symmetric space. In general, a *symmetric space* is a connected and homogeneous Riemannian manifold with global isometries of order two, $X = G/K$, where G is a Lie group and K a maximal compact subgroup. In our case, $G = \mathrm{PSL}_2(\mathbb{R})$ and $K = \mathrm{SO}(2)$.

2.2 The disk model

Write $\mathbb{H}^2 = (\mathbb{D}, ds^2_{\text{hyp}})$. This time,

$$ds^2_{\text{hyp}} = \frac{4ds^2_{\text{euc}}}{(1-|z|^2)^2}.$$

There is a map $\varphi : \mathcal{H} \to \mathbb{D} : z \mapsto \frac{z-i}{z+i}$ called the *Cayley map* or *Cayley transformation*, which can be shown to be an isometry. Let $z \in \mathcal{H}$, $v \in T_z\mathcal{H}$. We want to show that

$$\|D_z\varphi(v)\|_{\mathbb{D}} = \|v\|_{\mathcal{H}} = \frac{|v|_{\mathrm{euc}}}{\mathrm{Im} z}.$$

We have

$$\begin{aligned}\|D_z\varphi(v)\|_{\mathbb{D}} &= \frac{2|D_z\varphi(v)|}{1-|\varphi(z)|^2} = \frac{2}{1-|\frac{z-i}{z+i}|^2}\left|\frac{-2i}{(z+i)^2}v\right| \\ &= \frac{|v|}{\mathrm{Im} z} = \|v\|_{\mathcal{H}}.\end{aligned}$$

The geodesics in $\mathbb{D}$ are the following: Any circular arc perpendicular to the boundary $\partial\mathbb{D}$ (drawing), including diameters. For $r \in \mathbb{R} \cap \mathbb{D}$, we have

$$\rho = d(r, 0) = \int_0^r \frac{2}{1-t^2}\, dt = \ln \frac{1+r}{1-r},$$

or $r = \tanh \frac{\rho}{2}$.

3rd Lecture, September 7th 2012

The plan for today is to finish the discussion of the disk model, discuss area and curvature, and features of negative curvature.

So, consider again the unit disk $\mathbb{D}$ with metric $ds_h = \frac{2ds_{\mathrm{euc}}}{(1-r^2)}$, in polar coordinates. One thing to note is that two geodesics only intersect if their limit points nest (drawing). We saw that

$$\rho = d(r, 0) = \ln \frac{1+r}{1-r},$$

or $r = \tanh(\rho/2)$. Let us find the circumference of a hyperbolic circle of radius ρ. From the expression for the metric, this is just

$$2\pi \frac{2r}{1-r^2} = 4\pi \tanh(\rho/2)\cosh^2(\rho/2) = 2\pi \sinh(\rho).$$

Note that this grows exponentially in ρ. Note also that for $\rho \ll 1$, the circumference is $C_\rho \approx 2\pi\rho$, the Euclidean circumference. We saw last time that the Cayley map $\varphi : \mathcal{H} \to \mathbb{D} : z \mapsto \frac{z-i}{z+i}$ is an isometry. For any Möbius map T, $\varphi \circ T \circ \varphi^{-1} : \mathbb{D} \to \mathbb{D}$ is an isometry. It is an exercise to show that these are of the form $z \mapsto \frac{\alpha z+\beta}{\overline{\beta} z+\overline{\alpha}}$, where $|\alpha|^2 > |\beta|^2$. This is related to the fact that $\mathrm{SU}(1,1) \cong \mathrm{PSL}_2(\mathbb{R})$. In particular,

1. an isometry taking $w \in \mathbb{D}$ to 0 is given by

$$z \mapsto \frac{z-w}{1-\overline{w}z}.$$

2. Rotations $z \mapsto e^{i\alpha}z$ are isometries.

Lemma 2.2.1. *The orientation preserving isometries of* $\mathbb{H}^2$ *are given by* $\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}_2\mathbb{R}$.

Proof. Last time, we saw $\subseteq$. The proof uses the Schwarz Lemma which says that any conformal map $f : \mathbb{D} \to \mathbb{D}$ such that $f(0)$ satisfies $|f'(0)| \leq 1$ and equality holds only when $f(z) = e^{i\alpha}z$.

An orientation preserving isometry is a conformal automorphism, so it is enough to show that $\mathrm{Aut}(\mathbb{D}) = \mathrm{PSL}_2\mathbb{R}$. Take an $f \in \mathrm{Aut}(\mathbb{D})$. Consider $T_{f(0)} \circ f$ mapping the point 0 to itself. Its derivative at 0 has norm 1. So $T_{f(0)} \circ f = R_\alpha$ is rotation by α, and f is a Möbius map. $\square$

Lemma 2.2.2. *The hyperbolic metric is the unique metric (up to scaling), invariant under* $\mathrm{Aut}(\mathbb{D})$.

Proof. Let $\rho(z)|dz|$ be an invariant metric. Consider again the automorphism $T_z : \mathbb{D} \to \mathbb{D}$ mapping $z \mapsto 0$. This maps $v \in T_z\mathbb{D}$ to $\frac{v}{(1-|z|^2)} \in T_0\mathbb{D}$. So,

$$\rho(z)|dz|(v) = \rho(0)\frac{|v|}{1-|z|^2},$$

and so $\rho(z) = \frac{\rho(0)}{1-|z|^2}$. In our case, we had $\rho(0) = 2$, but any other number would work as well (we discuss the reason to use the factor 2 later). □

2.3 Area and curvature

The area form on $\mathbb{H}$ is given by $\frac{dx\,dy}{y^2}$ in $\mathcal{H}$, and $\frac{4r\,dr\,d\theta}{(1-r^2)^2}$ in $\mathbb{D}$.

Lemma 2.3.1. *Let $\Omega \subseteq \mathcal{H}$. Then for any Möbius map T, we haev*

$$\mathrm{Area}(T(\Omega)) = \mathrm{Area}(\Omega).$$

Proof. Exercise: Check this directly. I.e., if $T(x+iy) = u+iv$, check that

$$\iint_{T(\Omega)} \frac{du\,du}{v^2} = \iint_{\Omega} \frac{\partial(u,v)}{\partial(x,y)}\frac{dx\,dy}{y^2}.$$

□

Definition 2.3.2. An *ideal triangle* in $\mathbb{H}^2$ (unique up to isometry) is one with vertices on the boundary at ∞ (drawing).

Lemma 2.3.3. *The area of an ideal triangle is π.*

Proof. We have

$$\int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\infty} \frac{dy\,dx}{y^2} = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \int_{-\pi/2}^{\pi/2} \frac{1}{\cos\theta}\cos\theta\,d\theta = \pi.$$

□

Lemma 2.3.4. *The area of a hyperbolic triangle with interior angles $\alpha_1, \alpha_2, \alpha_3$ has area $\pi - \alpha_1 - \alpha_2 - \alpha_3$.*

Proof. Case 1. Consider what could be called a “2/3-ideal triangle”, i.e. assume that $\alpha_2 = \alpha_3 = 0$ (drawing) and let $\alpha_1 = \pi - \theta$. Such triangles are again unique up to isometry. We claim that the area $A(\theta)$ of such a triangle is additive, i.e. $A(\theta_1) + A(\theta_2) = A(\theta_1 + \theta_2)$. The proof is a picture in the disk model. So A is $\mathbb{Q}$-linear, and it is continuous, so it is linear. In particular, since $A(\pi) = \pi$, we have $A(\theta) = \theta$, which is what we claimed.

Case 2. Consider now the case where $\alpha_3 = 0$. Here, once again, one obtains the formula by drawing triangles in $\mathbb{D}$.

Case 3. The general case can also be handled by drawing. □

By decomposing a general geodesic polygon into triangles, one obtains the following.

Corollary 2.3.5. *The area of a geodesic polygon with interior angles $\alpha_1, \ldots, \alpha_n$, $n > 2$, is*

$$(n-2)\pi - \alpha_1 - \alpha_2 - \cdots - \alpha_n.$$

Exercise 2.3.6. Such a polygon exists if $\alpha_1 + \cdots + \alpha_n < (n-2)\pi$.

We now turn to curvature.

Fact 2.3.7. *For any conformal metric $\rho(z)|dz|$, or $\rho^2(x,y)(dx^2+dy^2)$, the (Gaussian) curvature is given by*

$$K(z) = -\frac{\Delta \ln \rho}{\rho^2}(z),$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Lemma 2.3.8. *The hyperbolic metric has curvature -1 everywhere ($\rho = 1/y$ in $\mathcal{H}$).*

Recall that the Gauss-Bonnet Theorem says that if M is a Riemannian surface (possibly with boundary), then

$$\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\chi(M).$$

This gives an easy proof of the area formula for polygons from before as the Theorem tells us that

$$\int_M (-1)\, dA + \sum \text{exterior angles} = 2\pi,$$

and so

$$-\text{Area}(M) + \left(\sum_{i=1}^{n} \pi - \alpha_i\right) = 2\pi.$$

Another consequence of Gauss–Bonnet is that there are no bigons in $\mathbb{H}^2$: If a bigon existed, it would have positive area and angles $\alpha_1 + \alpha_2 < 2\pi$.

Some features of negative curvature we want to discuss are area growth, convexity of the distance function, that projections contract, and that triangles are thin.

Fact 2.3.9. *For a disk D_r of radius r at a point x on a surface,*

$$\text{Area}(D_r) = \pi(r^2 - \frac{1}{12}K(x)r^4) + o(r^4).$$

In $\mathbb{H}^2$,

$$\text{Area}(D_r) = \int_0^r 2\pi \sinh \tau\, d\tau = 2\pi(\cosh r - 1) = 4\pi \sinh^2(r/2) \sim e^r.$$

In particular,

$$\frac{\text{Area}(D_r)}{C_r} \to 1.$$

Fact 2.3.10 (Hilbert)**.** *There is no C^2-immersion of $\mathbb{H}^2$ (or any complete surface with constant negative curvature) in $\mathbb{R}^3$.*

Geodesics diversge: Let γ_1, γ_2 be geodesic segments starting and x_0, and let x_t, y_t be at distance t from x_0 on γ_1, γ_2 respectively. Then $d_h(x_t, y_t)$ is strictly convex (drawing) as a function of t. We proof this fact next week. It is enough to show that if m_1, m_2 are the midpoints on the geodesic segments γ_1, γ_2 (going from x_0 to x_1, x_2 respectively), then

$$d_h(m_1, m_2) < \frac{1}{2} d_h(x_1, x_2). \tag{2}$$

Fact 2.3.11. *If γ_1, γ_2 are geodesic rays in $\mathbb{H}^2$ that remain a bounded distance apart, i.e. if $\gamma_i : [0,\infty) \to \mathbb{H}^2$ with*

$$\limsup_{t\to\infty} d(\gamma_1(t), \gamma_2(t)) < \infty,$$

then

$$\lim_{t\to\infty} d(\gamma_1(t), \gamma_2(t+t_0)) = 0$$

for some offset $t_0 \in \mathbb{R}$.

Remark 2.3.12. A geodesic metric space X satisfying (2) is called *negatively curved in the sense of Busemann.*

Lemma 2.3.13. *Let γ be a bi-infinite geodesic in $\mathbb{H}^2$. Let B be any hyperbolic disk disjoint from γ. Then the orthogonal projection of B onto γ has a uniformly bounded diameter.*

The proof is a drawing of $\mathbb{D}$ again.

Definition 2.3.14 (Gromov)**.** A geodesic metric space X is *δ-hyperbolic* if for any geodesic triangle if for any geodesic triangle pqr, the side $\overline{pr}$ is contained in the δ-neighbourhood $N_\delta(\overline{pq} \cup \overline{qr})$ for some fixed δ.

Lemma 2.3.15. *The hyperbolic plane $\mathbb{H}^2$ is δ-hyperbolic with $\delta = \ln(1+\sqrt{2})$.*

This δ comes about as follows: The worst case is when the triangles under consideration are ideal, say it has vertices at $0, 1$, and ∞: Then the distance from from the "top" of the semi-circle from 0 to 1 to the straight lines (drawing) is

$$\rho = \frac{1}{2}\ln\frac{1+\sin(\pi/4)}{1-\sin(\pi/4)} = \ln(1+\sqrt{2}).$$

4th Lecture, September 11th 2012

Today, we will do some hyperbolic trigonometry, discuss the hyperboloid model, and talk about the classification of isometries.

Theorem 2.3.16. *Consider a geodesic triangle with side lengths a, b, c and angles α, β, γ (drawing). The following hold for geodesic triangles in $\mathbb{H}^2$:*

1.
$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma,$$

2.
$$\cos\gamma = \sin\alpha\sin\beta\cosh c - \cos\alpha\cos\beta,$$

3.
$$\frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\gamma}.$$

Remark 2.3.17. These have analogues for spherical and Euclidean triangles. On S^2, we have

$$\cos c = \cos a\cos b + \sin a \sin b \cos\gamma,$$

and

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.$$

On $\mathbb{R}^2$, we have

$$c^2 = a^2 + b^2 - 2ab\cos\gamma,$$

and

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

The second of the three formulas of the Theorem has no analogues. It implies that similar triangles are congruent, which is not true in the other cases. A special case of the first formula, when $\gamma = \pi/2$ is

$$\cosh c = \cosh a \cosh b.$$

This could be proved directly: Any right-angles triangle like that can be put in a standard position (drawing) where the right angle is placed in i, and the other vertices are ie^b and $e^{i\theta}$. Then the distance formula tells us that

$$\begin{aligned} c = 1 + \frac{|ie^b - e^{i\theta}|^2}{2\mathrm{Im}(ie^b)\mathrm{Im}(e^{i\theta})} &= \frac{1}{\sin\theta}\left(\frac{e^b + e^{-b}}{2}\right) \\ = \frac{1}{\sin\theta}\cosh b &= \cosh a \cosh b. \end{aligned}$$

One difference from Euclidean right angle triangles is that it is much shorter to cross the diagonal, that is, for the Euclidean ones, we have $a + b \gg c$ as $a, b \to \infty$, whereas in hyperbolic geometry,

$$\begin{aligned} \cosh(a+b) &= \cosh a \cosh b + \sinh a \sinh b \le 2\cosh a \cosh b = 2\cosh c \\ &\le \cos(c + \mathrm{arccosh} 2), \end{aligned}$$

which implies that $a + b < c + \mathrm{arccosh} 2$. This is related to the fact that triangles are thin, as we discussed last time.

The proof of the theorem we will do in the hyperboloid model.

2.4 Hyperboloid model

Consider $\mathbb{R}^3$ with the inner product

$$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3.$$

Define the *hyperboloid* (drawing)

$$H = \{\vec{x} \in \mathbb{R}^3 \mid \langle \vec{x}, \vec{x} \rangle = 1, x_3 > 0\}.$$

The group

$$\mathrm{SO}(2,1) = \left\{ A \in \mathrm{SL}(3,\mathbb{R}) \;\middle|\; A^t \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \right\}$$

preserves the inner product as well as the set H. We need the following facts:

1) The group acts transitively on H: Let

$$L_\sigma = \begin{pmatrix} \cos\sigma & -\sin\sigma & 0 \\ \sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; M_\rho = \begin{pmatrix} \cosh\rho & 0 & \sinh\rho \\ 0 & 1 & 0 \\ \sinh\rho & 0 & \cosh\rho \end{pmatrix}.$$

These are elements of $\mathrm{SO}(2,1)$. The L_σ are "rotations" that fix $(0,0,1)$. The M_ρ are called "boosts" and map $(0,0,1) \mapsto (\sinh\rho, 0, \cosh\rho)$. This shows that $\mathrm{SO}(2,1)$ acts transitively.

2) The inner product $\langle\,,\,\rangle$ restricted to tangent vectors of H is positive definite. This follows from the following fact:

Exercise 2.4.1. If $\vec{v} \in T_{\vec{x}}H$, where $\vec{x} \in H$, then $\langle \vec{v}, \vec{x}\rangle = 0$.

The inner product is positive definite when restricted to tangent vectors at $(0,0,1)$ and hence by transitivity everywhere.

3) H with this inner product on the tangent vectors is isometric to $\mathbb{H}^2$.

Proof. Project the unit disk from the point $(0,0,-1)$ (drawing). This map

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} \frac{x_1}{1+x_3} \\ \frac{x_2}{1+x_3} \end{pmatrix} \in \mathbb{D}$$

is an isometry, which can be shown by calculation of the derivatives (exercise). □

4) The geodesics of H are intersections with planes through the origin. This can be seen to be true for geodesics passing through $(0,0,1)$: The intersection with the plane $x_2 = 0$ can be parametrized as $\gamma(t) = (\sinh t, 0 \cosh t)$. The derivative of this is $\gamma'(t) = (\cosh t, 0 \sinh t)$, so $\|\gamma'(t)\| = 1$.

This is also easy to believe as we know that the geodesics through the origin of $\mathbb{D}$ are the diameters.

Lemma 2.4.2. *In this model,*

$$\cosh d_h(\vec{v}, \vec{w}) = -\langle \vec{v}, \vec{w}\rangle.$$

Proof. Take, without loss of generality, $\vec{v} = (0,0,1)$, $\vec{w} = (\sinh t, 0, \cosh t)$. Then $d_h(\vec{v}, \vec{w}) = t$, and $\langle \vec{v}, \vec{w}\rangle = -\cosh t$. □

Proof of Theorem 2.3.16. Let one vertex of the triangle (connecting the sides with lengths a,c) be $p_0 = (0,0,1)$, so that c is along $x_2 = 0$ to the left (drawing). Consider the isometry $L_{\pi-\alpha}M_c$. This translates the geodesic $\{x_2 = 0\}$ (containing the side c) and rotates the triangle it so that now the side b is along this geodesic. Similarly, we find that

$$L_{\pi-\beta}M_c L_{\pi-\gamma} M_b L_{\pi-\alpha} M_c = \mathrm{Id}\,.$$

Rearranging these,

$$M_a L_{\pi-\gamma} M_b = L_{\beta-\pi} M_{-c} L_{\alpha-\pi}$$

Writing out these matrices, multiplying them, and comparing coefficients, we obtain the formulas of the Theorem. Namely,

$$\begin{pmatrix} \cosh a & 0 \sinh a & \\ 0 & 1 & 0 \\ \sinh a & 0 & \cosh a \end{pmatrix} \begin{pmatrix} -\cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & -\cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh b & 0 & \sinh b \\ 0 & 1 & 0 \\ \sinh b & 0 & \cosh b \end{pmatrix}$$
$$= \begin{pmatrix} -\cos\beta & -\sin\beta & 0 \\ \sin\beta & -\cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh c & 0 & -\sinh c \\ 0 & 1 & 0 \\ -\sinh c & 0 & \cosh c \end{pmatrix} \begin{pmatrix} -\cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & -\cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

Remark 2.4.3. All of this works in dimension n. That is, we have hyperbolic n-space, where the upper half space model is

$$\mathcal{H}^n = \{x \in \mathbb{R}^n \mid x_n > 0\}$$

with metric

$$\frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

We have the ball model

$$B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\},$$

with metric

$$\frac{4(dx_1^2 + \cdots + dx_n^2)}{(1 - (x_1^2 + \cdots + x_n^2))^2},$$

and the hyperboloid model

$$H^n = \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle = -1\rangle.$$

Later we will see that $\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2\mathbb{C}$ (related to the upper half space model as with $n = 2$), but the isometries in general are best described as $\mathrm{Isom}^+(\mathbb{H}^n) = \mathrm{SO}(n, 1)$.

Another important historical fact is that the parallel postulate fails for $\mathbb{H}^2$ (drawing). This is one reason it arose historically, whereas we still study it for its relation to Riemann surfaces through the uniformization theorem.

2.5 Classification of isometries

Recall that the classification of isometries goes as follows: Any isometry $A \in \mathrm{Isom}^+(\mathbb{H}^2)$, $A \neq \mathrm{Id}$, is of exactly one of the following types:

1. Elliptic: It has exactly 1 fixed point in the interior of $\mathbb{H}^2$, e.g. in $\mathbb{D}$ the rotation $z \mapsto e^{i\theta z}$.
2. Parabolic: 1 fixed point in the boundary of $\mathbb{H}^2$, e.g. in translations $z \mapsto z + a$ in $\mathbb{H}$ fixed ∞ (and the horizontal lines are invariant and are called *horocycles* (drawing of this in $\mathbb{D}$)).
3. Hyperbolic: 2 fixed points on the boundary of $\mathbb{H}^2$ (as well as the geodesic between them, which is the *axis* of the isometry). For example, in $\mathcal{H}$, $z \mapsto \lambda z$ fixes 0 and ∞.

Remark 2.5.1. Note first that any isometry must have fixed point in $\overline{\mathbb{D}}$ by Brouwer's fixed point theorem. If there are more than 3 fixed points in $\partial\mathbb{D}$, the map must be the identity.

We saw that we could think of isometries as elements of $\mathrm{PSL}_2\mathbb{R}$. Solving $z \mapsto Az = \frac{az+b}{cz+d}$, the above characterization turns into:

1. If $|\mathrm{tr}\, A| < 2$, then A is elliptic.
2. If $|\mathrm{tr}\, A| = 2$, then A is parabolic.
3. If $|\mathrm{tr}\, A| > 2$, then A is hyperbolic.

This explains where the terminology comes from: Up to conjugation, the previous examples are the only ones, and they fix either ellipses, parabolas or hyperbolas in $\mathcal{H}$ respectively.

Lemma 2.5.2. *The translation distance d of a hyperbolic isometry satisfies*

$$\mathrm{tr}^2(A) = 4\cosh^2(d/2).$$

Proof. Conjugate the isometry so that the axis is the imaginary axis in $\mathcal{H}$, so the isometry is

$$\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 1/\sqrt{\lambda} \end{pmatrix}$$

for some $\lambda \neq 1$. Then $\mathrm{tr}(A) = \sqrt{\lambda} + 1/\sqrt{\lambda}$. It maps i to λi, and thus the translation distance d is $\ln \lambda$. $\square$

Lemma 2.5.3. *Isometries commute if and only if they have the same fixed point set.*

Proof. If two isometries S, T satisfy $ST = TS$, and say p is fixed by T. Then $S(p)$ is also fixed by T.

For the converse, there are three cases: In the elliptic case, we saw that any isometry fixing $0 \in \mathbb{D}$ is a rotation, and those commute. In the parabolic case, any isometry fixing ∞ is conjugate to

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix},$$

and those always commute. In the hyperbolic case, any isometry fixing $0, \infty$ is of the form $z \mapsto \mu z$, and again, these commute. $\square$

Corollary 2.5.4. *Hyperbolic isometries commute if and only if they have the same axis.*

3 Fuchsian groups

Definition 3.0.5. A *Fuchsian group* is a discrete subgroup of $\mathrm{PSL}_2\mathbb{R}$, i.e. for a sequence $T_n \to \mathrm{Id}$ in the group, we must have $T_n = \mathrm{Id}$ for all large n.

Next time, we will see that $\mathbb{Z} \oplus \mathbb{Z}$ can not be a subgroup of a Fuchsian group so all abelian subgroups of Fuchsian groups are cyclic. We will also see that a subgroup of $\mathrm{PSL}_2\mathbb{R}$ is Fuchsian if and only if all its orbits are discrete.

Example 3.0.6. 1) The group generated by a hyperbolic isometry, $\Gamma = \langle z \mapsto \lambda z \rangle$. This sends a circle with center 0 passing through i to one with center 0 passing through λi (drawing). The half-annulus between these two is a fundamental domain for the action of Γ. The space $\mathbb{H}^2/\Gamma$ is a hyperbolic cylinder (drawing) in such a way that lengths of equidistant curves on the cylinder grow exponentially.

2) The group generated by a parabolic element $z \mapsto z + 1$ has fundamental domain a rectangle (drawing). The quotient will in this case be what will be the model of cusps in hyperbolic surfaces.

We finish with a hængeparti and prove (2), that $d(m, m') < \frac{1}{2} d(x_t, y_t)$. Let $R_{m'}$ be the elliptic rotation of angle π around m' and similarly R_m t he rotation around m. Then $H = R_m \circ R_{m'}$ is an hyperbolic isometry which preserves the axis between m ad m'. The translation distance is $2d(m, m')$. On the other hand, $H(y_t) = x_t$. The translation distance is achieved along the axis of the hyperbolic isometry, i.e.

$$d(x_t, y_t) > 2d(m, m').$$

5th Lecture, September 14th 2012

We will discuss Fuchsian groups further. Recall that those are discrete subgroups of $\mathrm{PSL}_2\mathbb{R}$.

Example 3.0.7. One example we saw was $\Gamma = \langle z \mapsto \lambda z\rangle$. The quotient $\mathbb{H}^2/\Gamma$ was a hyperbolic cylinder, homeomorphic to $S^1 \times \mathbb{R}$. Another one was $\Gamma = \langle z \mapsto z+1\rangle$, where $\mathbb{H}^2/\Gamma$ was homeomorphic to D^*, a punctured disk.

A third example is $\langle z \mapsto e^{i2\pi/n}z\rangle$, which has quotient an orbifold, homeomorphic to D.

Fourthly, take $\Gamma = \mathrm{PSL}_2\mathbb{Z}$. This quotient looks like a sphere with two cone points and a cusp.

Lemma 3.0.8. *A Fuchsian group has no $\mathbb{Z} \oplus \mathbb{Z}$-subgroup.*

Proof. All elements of an abelian subgroup commute so they have the same fixed point set. That is, either all of them are elliptic, parabolic, or hyperbolic (drawings).

In the hyperbolic case, all elements fix 0 and ∞. Then Γ is a discrete subgroup of $\{z \mapsto \lambda z\}$, which as a topological group is $\mathbb{R}$. Now, the only discrete subgroups of $\mathbb{R}$ are infinite cyclic.

In the parabolic case, Γ is a discrete subgroup of $\langle z \mapsto z+a\rangle \cong \mathbb{R}$, and we argue as before.

In the elliptic case, Γ is a discrete subgroup of $\mathrm{SO}(2)$, which implies that Γ is finite. $\square$

Remark 3.0.9. An *elementary* Fuchsian group is one that is virtually abelian (i.e. it is abelian or has a finite index subgroup that is) – it is either cyclic, or generated by a hyperbolic element, together with an element that interchanges end points.

Fact 3.0.10. *A non-elementary Fuchsian group has infinitely many hyperbolic elements with disjoint axes.*

We prove a weaker statement from which it is not hard to deduce the above fact.

Lemma 3.0.11. *A non-elementary Fuchsian group has at least one hyperbolic element.*

Proof. If all elements are elliptic, then they cannot have a fixed point.

If $g, h \in \Gamma$ are elliptic, then the fixed point sets are different, $\mathrm{Fix}(h) \neq \mathrm{Fix}(g)$, and $ghg^{-1}h^{-1}$ is hyperbolic. If g fixed a point x and rotates by and angle 2θ clockwise (drawing), then $hg^{-1}h^{-1}$ fixes $h(x)$ and rotates 2θ anti-clockwise. Let N be the line through $x, h(x)$. Let L, M be lines making angle θ with N, going through x and $h(x)$ respectively. Then $g = \sigma_L\sigma_N$, where σ denotes reflection in a line, and $hg^{-1}h^{-1} = \sigma_N\sigma_M$. Thus

$$ghg^{-1}h^{-1} = \sigma_L\sigma_M,$$

which is hyperbolic as L and M are disjoint. (Actually, all we used was that $h(x) \neq x$, and not that h was elliptic).

If $f \in \Gamma$ is a parabolic element, say $f(z) = z+1$. Let $g(z) = \frac{az+b}{cz+d}$ be any other element of Γ, such that $c \neq 0$. Then

$$f^n \circ g(z) = \frac{(a+nc)z + (b+nd)}{cz+d},$$

and then $\mathrm{tr}^2(f^n g) = (a+nc+d)^2$ which is greater than 4 for large n. $\square$

Definition 3.0.12. A group Γ acts *properly discontinuously* on a locally compact metric space X if for any compact set $K \subseteq X$,

$$\#\{T \in \Gamma \mid T(K) \cap K \neq \emptyset\} < \infty.$$

Equivalently, for any $x \in X$, $\Gamma_x \cap K$ is finite.

Lemma 3.0.13. *A subgroup of* $\mathrm{PSL}_2\mathbb{R}$ *is Fuchsian if and only if it acts properly discontinuously on* $\mathbb{H}^2$*.*

As an example of what might happen in general, let $X = \mathbb{R}^2 \setminus \{0\}$, and consider the group generated by maps $(x, y) \mapsto (2x, y/2)$, acting on X. In this case Γ is discrete, but the action is not properly discontinuous.

Remark 3.0.14. If Γ acts freely and properly discontinuously, then $\mathbb{H}^2/\Gamma$ is a (hyperbolic) surface.

Proof of lemma. If $T_k \to T$, then $T_k z \to z$, which contradicts proper discontinuity.

The converse uses the following key claim: Let $z_0 \in \mathbb{H}^2$, and let $K \subseteq \mathbb{H}^2$ be compact. Then $\{T \in \mathrm{PSL}_2\mathbb{R} \mid T(z_0) \in K\}$ is compact. To see this, consider $\psi : \mathrm{PSL}_2\mathbb{R} \to \mathcal{H} : A \mapsto Az_0$. We want to show that $\psi^{-1}(K)$ is closed and bounded. It is clearly closed. Since K is compact, $|\frac{az_0+b}{cz_0+d}| < M$, for some constant M_1, and $\mathrm{Im}\left(\frac{az_0+b}{cz_0+d}\right) > M_2 > 0$ for some M_2. These inequalities imply that a, b, c, d are bounded. □

Definition 3.0.15. For any group Γ acting properly discontinuously by homeomorphisms on a metric space X, a *fundamental domain* is a closed $C \subseteq X$, such that

1. the interior $\int(C) \neq \emptyset$,
2. $T \neq \mathrm{id}$ implies that $T(\int C) \cap \int C = \emptyset$,
3. Γ-translates of C *tessellate* X, i.e. $\bigcup_{T\in\Gamma} T(C) = X$.

Example 3.0.16. The punctured torus group: Take a quadrilateral in $\mathbb{D}$. Identifying opposite sides gives a punctured torus (drawing). Concretely, consider $A : z \mapsto \frac{z+1}{z+2}$, $B : z \mapsto \frac{z-1}{-z+2}$. Here the quadrilateral consists of vertical lines from -1 to ∞ and 1 to ∞ together with half-circles from -1 to 0 and 0 to -1 (drawing), and A, B send sides to opposite sides. In this case, $\Gamma = \langle A, B\rangle$ is discrete (since it lies in $\mathrm{PSL}_2\mathbb{Z}$, which is discrete). One can draw a tessellation in this case. We have that $BAB^{-1}A^{-1}$ is parabolic.

In general, one can ask the question: When is $\langle A, B\rangle \subseteq \mathrm{PSL}_2\mathbb{R}$ discrete? There is a complete answer to this. In the case of $\mathrm{PSL}_2\mathbb{C}$, Gilman has developed algorithms to decide whether that holds.

Lemma 3.0.17 (Shimuzu)**.** *If Γ is Fuchsian and contains $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then for any $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, either $c = 0$ or $|c| \geq 1$.*

Proof. Let $B_0 = B$, $B_{n+1} = B_n A B_n^{-1}$. Then

$$B_{n+1} = \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - a_n c_n & a_n^2 \\ -c_n^2 & 1 + a_n c_n \end{pmatrix}.$$

Now $c_{n+1} = -c_n^2$ implies that $c_n = -c^{2n}$ so $c_n \to 0$ if $|c| < 1$.

By induction, $|a_n| \leq n + |a_0|$ if $|c| < 1$. This implies that $a_n c_n \to 0$. This tells us that $B_n \to A$, which contradicts discreteness. □

Corollary 3.0.18. *If A, B are as in the Lemma, then*

$$|\mathrm{tr}(ABA^{-1}B^{-1}) - 2| - 2 \geq 1.$$

Fact 3.0.19 (Jørgensen's inequality)**.** *If A, B are elements of a discrete subgroup of $\mathrm{PSL}_2\mathbb{R}$ (or $\mathrm{PSL}_2\mathbb{C}$), then*

$$|\mathrm{tr}^2(A) - 4| + |\mathrm{tr}([A, B]) - 2| \geq 1.$$

Example 3.0.20. Surface groups: For any $g > 1$, take a regular $4g$-gon with interior angles $\frac{\pi}{2g}$ (drawing of why this is possible). Now, pair opposite sides by isometries. The result is a genus g surface. The group $\Gamma = \langle s_i\rangle$ generated by these side pairings is Fuchsian and isomorphic to the fundamental group of the surface. The area is $(4g - 2)\pi - 2\pi$. By Gauss-Bonnet,

$$\int K dA = 2\pi\chi(S).$$

For side pairings of a general polygon, we have the Poincaré Polygon Theorem (drawing), which tells us that for a general polygon, possibly with ideal vertices, and pairings between sides, such that the pairings satisfy vertex conditions, that the sum at (non-ideal) vertex cycles is $\frac{2\pi}{n}$ for some integer n, and that the isometry associated with an ideal vertex cycle is parabolic (as for the punctured torus group), we have that the group Γ generated by side pairings is Fuchsian, and it has a presentation $\Gamma = \langle s_i \mid \text{vertex relations}\rangle$. Furthermore, the polygon P is a fundamental domain, and the quotient $\mathbb{H}^2/\Gamma$ is complete.

Example 3.0.21. Triangle groups: If P is a triangle with interior angles $\alpha = \pi/p, \beta = \pi/q, \gamma = \pi/r$, where p, q, r are integers (possibly infinite if a vertex is ideal). Then the group Γ generated by rotations

$$\Gamma = \langle R_{2\alpha}, R_{2\beta}, R_{2\gamma} \mid (R_{2\alpha})^p = (R_{2\beta})^q = (R_{2\gamma})^q = \mathrm{Id}\rangle$$

is discrete. $\mathrm{PSL}_2\mathbb{Z}$ arises in this way with $(p, q, r) = (2, 3, \infty)$.

Example 3.0.22. There are examples of an incomplete surface: Take $\Gamma = \langle z \mapsto az + 1\rangle$. If $a \neq 1$, say $a > 1$, then the images of vertical lines under negative iterates of this map will accumulate and not tile the hyperbolic plane. The quotient is an incomplete surface (drawing): One can come up with a path going to infinity with finite length.

Another example is the following: Consider the setup from the punctured torus with $A = \frac{z+1}{z+a}$, $B = \frac{z-1}{-z+b}$. Then

$$[A, B] = \left(\frac{b-1}{a-1}\right)^2 z + \frac{(ab-1)(a+b-2)}{(a-1)^2},$$

which is parabolic only if $a = b$

Example 3.0.23. *Schottky groups*: Choose $C_1, C_2, \ldots, C_{2k}$ geodesics, forming disjoint regions $D_1, D_2, \ldots, D_{2k}$ (drawing). Choose isometries $\{g_1, \ldots, g_k\}$ pairing them, $g_i(C_{2i-1}) = C_{2i}$ and $g_i(D_{2i-i}) = D_{2i}^c$. The group $\Gamma = \langle g_i\rangle$ is Fuchsian and *free*. This follows from the Poincaré Polygon Theorem. Next time we will discuss how this comes from the Ping Pong Lemma. One consequence of this construction is that every non-elementary Fuchsian group contains a free subgroup (drawing): If g, h are hyperbolic elements of Γ with distinct axes, then $\langle g^n, h^m\rangle$ is free for large n, m.

6th Lecture, September 18th 2012

Today, we will finish the discussion of Fuchsian groups and move on to hyperbolic surfaces.

Let Γ be a non-elementary Fuchsian group. Take $g, h \in \Gamma$ hyperbolic elements with distinct axes (drawing) (consider regions D_1, D_2, D_3, D_4 in $\mathbb{D}$ as last time with $g(D_1) = D_2^c$, $h(D_3) = D_4^c$). For sufficiently large n, m, $\langle g^n, h^m\rangle$ is a free subgroup. This follows from the

Lemma 3.0.24 (Ping Pong Lemma)**.** *Let G be a group generated by a, b of infinite order. Let G act on X such that there are non-empty subsets $A, B \subseteq X$, $B \not\subseteq A$, such that $a^n B \subseteq A$, $b^n A \subseteq B$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then $G = \langle a, b\rangle$ is freely generated.*

Proof. Let $F(a, b)$ be the free group in a, b. We have the natural homomorphism $\varphi : F(a, b) \to G$, which we want to show is injective. Let w be a reduced word in a, b with $\varphi(w) = e$.

Case 1. Assume $w = a^{n_0} b^{m_1} a^{n_1} b^{m_2} \cdots b^{m_k} a^{n_k}$. Note that

$$B = e \cdot B = \varphi(w) \cdot B = a^{n_0} \cdots a^{n_k} B \subseteq a^{n_0} \cdots b^{m_k} A \subseteq a^{n_0} B \subseteq A,$$

which is a contradiction (here we see the reason for the name of the lemma).

Case 2. Assume $w = a^{n_1} \cdots b^{n_k}$. Consider then awa^{-1} or $a^{-1}wa$, which puts us in the first case.

Case 3. Assume $w = b^{n_1} \cdots a^{n_k}$. This works as in case 2.

Case 4. Assume $w = b^{n_1} \cdots b^{n_k}$. Consider again awa^{-1}. □

In our setup, $A = D_1 \cup D_2$, $B = D_3 \cup D_4$.

Remark 3.0.25 (Tits alternative). A finitely generated linear group either has a finite index solvable subgroup or has a free subgroup.

We turn now to fundamental domains of Fuchsian groups.

Definition 3.0.26. Let Γ be a Fuchsian group. The *Dirichlet domain* of Γ centered at $z_0 \in \mathbb{H}^2$ is

$$\begin{aligned} D_\Gamma(z_0) &= \{z \in \mathbb{H}^2 \mid d(x, z_0) \le d(x, Tz_0) \text{ for all } T \in \Gamma\} \\ &= \bigcap_{T \in \Gamma} \text{half-planes determined by the bisector } \overline{z_0, Tz_0}. \end{aligned}$$

Lemma 3.0.27. *The Dirichlet domain $D_\Gamma(z_0)$ is a convex fundamental domain for the action of Γ on $\mathbb{H}^2$.*

Proof. By definition, $D_\Gamma(z_0)$ is an intersection of closed half-planes, so it is closed and convex. It is non-empty as $z_0 \in \operatorname{int}(D_\Gamma(z_0))$ since Γ is discrete. Suppose $z, T(z)\operatorname{int}D_\Gamma(z_0)$ where $T \neq \mathrm{id}$. By definition, $d(z, z_0) < d(z, T^{-1}z_0) = D(Tz, z_0)$. Similarly, $d(Tz, z_0) < d(Tz, Tz_0) = d(z, z_0)$, which is a contradiction. Lastly, let $z' \in \mathbb{H}^2$. Since Γ is discrete, there exists $T \in \Gamma$ such that $d(z', Tz_0) = \inf_{S \in \Gamma} d(z, Sz_0)$. One can check that $T^{-1}z' \in D_\Gamma(z_0)$. Namely,

$$d(T^{-1}z', z_0) = d(z', Tz_0) \le d(z', TSz_0) = d(T^{-1}z', Sz_0)$$

for all $S \in \Gamma$. □

Example 3.0.28. We claim that the Dirichlet domain of the $\mathrm{SL}_2\mathbb{Z}$-action on $\mathbb{H}^2$ centered at $2i$ is (drawing)

$$D_{\mathrm{SL}_2\mathbb{Z}}(2i) = \{z \in \mathbb{H}^2 \mid |\mathrm{Re}z| \le 1/2, |z| \ge 1\} =: \Delta.$$

We have two elements $T : z \mapsto z + 1$, $S : z \mapsto -1/z$ in $\mathrm{SL}_2\mathbb{Z}$. The inclusion "$\subseteq$" follows from the definition, as it is the intersection of two half-lines. To show the other one, it is enough to show that translates of $\operatorname{int}(\Delta)$ are disjoint. Suppose this is not the case. Then there are points $z, \frac{az+b}{cz,d} \in \Delta$ with $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$ (perhaps $c \neq 0$ or $d \neq 1$ in which cases the claim is obviuos). We have

$$|cz + d|^2 = c^2|z|^2 + 2cd\mathrm{Re}(z) + d^2 > c^2 + d^2 - |cd| = (|c| - |d|)^2 + |cd| > 1,$$

which implies that

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right)\frac{\mathrm{Im}z}{|cz+d|^2} < \mathrm{Im}z.$$

The same argument with h^{-1} gives the other inequality, and hence we get a contradiction.

Recall that the Poincaré Polygon Theorem tells us when gluings of sides in a polygon with finitely many sides generate a Fuchsian group, what a fundamental domain for the action is, and what the relations in the group are: For every vertex in the polygon, one can define vertex cycles (the equivalence classes of vertices that end up in the same vertex after gluing). The vertex conditions are as follows: For interior (non-ideal) vertices, the sum of interior angles of vertices in a vertex cycle is

$$\sum_{i=1}^{n-1} \alpha_i(v_i) = 2\pi/m$$

for some integer $m \ge 1$. This is equivalent to the composition of side gluings involving the interior vertex has order m – this is the relation going into the group generated by the side pairings. For ideal vertices, the requirement is that this composition is a parabolic element.

The conditions of the Poincaré Polygon Theorem holds for our example of $\mathrm{SL}_2\mathbb{Z}$, where we break the circular curve in two with vertices at $v_2 = i$, $v_0 = (1+i\sqrt{3})/2$, $v_1 = (-1+i\sqrt{3})/2$, and with $v_0 \overset{S^{-1}}{\to} v_1 \overset{T}{\to} v_0$, with the relation $(TS^{-1})^3 = \mathrm{id}$ and $v_2 \overset{S}{\to} v_2$ with $S^2 = \mathrm{id}$. We can conclude that

$$\mathrm{SL}_2\mathbb{Z} = \langle S, T \mid (TS^{-1})^3 = S^2 = \mathrm{id}\rangle.$$

Fact 3.0.29. *Let $\Lambda \subseteq \Gamma$ be a finite index subgroup with a coset decomposition $\Gamma = \Lambda T_1 \cup \Lambda T_2 \cup \cdots \cup \Lambda T_n$. If F is a fundamental domain for Γ, we have the fundamental domain for Λ*

$$F_1 = T_1(f) \cup T_2(F) \cup \cdots \cup T_n(F).$$

For instance, for

$$\Gamma_0 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2\mathbb{Z} \;\middle|\; c \equiv 0 \bmod 2 \right\},$$

one gets the *Farey Tessellation* (drawing) with vertices at $\mathbb{Q} \cup \infty$, and there is an edge between two points p/q, r/s if $ps - qr = \pm 1$.

Definition 3.0.30. A Fuchsian group is called *cocompact* if the quotient $\mathbb{H}^2/\Gamma$ is compact, or, equivalently, it has a compact fundamental domain.

Lemma 3.0.31. *A cocompact Fuchsian group has no parabolic elements.*

Proof. Let F be the Dirichlet domain for Γ. Then F is compact, so the function

$$\eta(z) = \inf\{d(z, Tz) \mid T \in \Gamma \setminus \mathrm{id} \text{ not elliptic}\}$$

is continuous, positive, and achieves its minimum $c > 0$ in F. We claim that in fact $\inf_{z\in\mathbb{H}^2} \eta(z) = c > 0$: For any $z \in \mathbb{H}^2$, there exists an $S \in \Gamma$ so that $w := Sz \in F$, and $d(z, Tz) = d(Sz, STz) = d(w, STS^{-1}w) \geq c$. On the other hand, if there were a parabolic in Γ, say by conjugation $z \mapsto z+1$, then $d(z, z+1) \to 0$ as $z \to \infty$, which contradicts the claim. □

Definition 3.0.32. A Fuchsian group is *geometrically finite* if it has a convex fundamental domain with finitely many sides.

Lemma 3.0.33. *If* $\mathrm{Area}(\mathbb{H}^2/\Gamma) < \infty$*, then* Γ *is geometrically finite.*

Proof. Suppose for instance that there are infinitely many ideal vertices. This is impossible (by drawing). □

Definition 3.0.34. A cocompact Fuchsian group with a fundamental domain with k elliptic cycles, r of which are "non-accidental" with orders $m_1, \ldots, m_r$ and quotient a genus g surface has *signature* $(g; m_1, m_2, \ldots, m_r)$.

Lemma 3.0.35. *Let* Γ *have signature* $(g; m_1, \ldots, m_r)$*. Then*

$$\mathrm{Area}(\mathbb{H}^2/\Gamma) = 2\pi((2g-2) + + \sum_{i=1}^{r}(1 - 1/m_i)).$$

This follows from the area formula for polygons. By considering various values of g and m_i, one gets the following.

Lemma 3.0.36. *Let* Γ *be a cocompact Fuchsian group. Then the area of* $\mathbb{H}^2/\Gamma$ *is greater than* $\pi/21$.

Remark 3.0.37. A triangle group $\Delta(p, q, r)$ has signature $(0; p, q, r)$. The minimum area $\pi/21$ is achieved by $\Delta(2, 3, 7)$.

From the Lemma, one can derive the classical fact that for a hyperbolic surface X, we have $|\mathrm{Aut}(X)| \leq 84(g-1)$.

4 Hyperbolic surfaces

7th Lecture, September 22nd 2012

Note: This lecture was TeXed by Jens Kristian Egsgaard.
So far:

1. models for and geometry of $\mathbb{H}^2$
2. Γ Fuchsian groups
3. and today: $\mathbb{H}/\Gamma$ qoutients, hyperbolic surfaces

Definition 4.0.38. A hyperbolic surface S or hyperbolic structure on a topological surface S, is

1. covering by open sets U_i with charts $\varphi_i : U_i \to \mathbb{H}$
2. transition maps $\varphi_j \circ \varphi_i^{-1} \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are restrictions of isometries

Example 4.0.39. $\mathbb{H}^2/\Gamma$ where the Fuchsian group Γ acts freely.

Theorem 4.0.40. *Let X be a complete hyperbolic surface. Then $X \cong \mathbb{H}/\Gamma$ for some torsion-free Fuchsian group Γ.*

Proof. This will only be the raw strategy.

1. construct the developing map, a local isometry $\mathrm{Dev} : \tilde{X} \to \mathbb{H}^2$
2. is a lemma:

   Lemma 4.0.41. *If X is a complete hyperbolic surface, then* $\mathrm{Dev} : \tilde{X} \to \mathbb{H}^2$ *is surjective covering map*

   Proof. Show that Dev satisfies the path lifting property: for all $z_0 \in \mathrm{Dev}(\tilde{X})$ and $\tilde{z}_0 \in \mathrm{Dev}^{-1}(z_0)$, paths $\gamma : [0,1] \to \mathbb{H}^2$ with $\gamma(0) = z_0$there exists paths $\tilde{\gamma} : [0,1] \to \tilde{X}$ such that $\tilde{\gamma}(0) = \tilde{z}_0$, $\mathrm{Dev} \circ \tilde{\gamma} = \gamma$. Consider

   $$t_0 = \sup\{t \in [0,1] \mid \exists \tilde{\gamma} : [0,t]\tilde{X}, \tilde{\gamma}(0) = \tilde{z}_0, \mathrm{Dev} \circ \tilde{\gamma} = \gamma|_[0,t]\}$$

   and show that $t = 1$ by showing open/closedness. Open becaus Dev is local isometry, closed because $\tilde{X}$ is complete. ☐

   Corollary 4.0.42. *Since $\mathbb{H}^2$ is simply connected, this implies that* Dev *is a global isometry.*

   This gives us that $\tilde{X}$ is isometric to $\mathbb{H}^2$

3. Construct the holonomy map, $\mathrm{hol} : \pi_1(X,p) \to \mathrm{PSL}_2\mathbb{R}$, defined as $[\alpha] \mapsto T_1 \circ \cdots \circ T_n$. It can be shown that this is injective. because $\mathbb{H}^2$ is contractible and Dev is a covering so homotopies can be lifted.

for 1, fix a basepoint $p \in U_0$, where U_0 is one of the charts. Then the universal cover can be identified with the set of homotopy classes of paths starting in p. Let $[\alpha] \in \tilde{X}$. Cover the image of $\tilde{\alpha}$ by the pullbacks of U_i's such that $U_i \cap U_{i+1}$ is connected. We define $\mathrm{Dev}|_{U_0 \cap \tilde{\alpha}} = \varphi_0|_{U_0 \cap \tilde{\alpha}}$. Note that $\varphi_0 \circ \varphi_1^{-1}|_{U_0 \cap U_1} = T_1 \in \mathrm{PSL}_2\mathbb{R}$. Set $\mathrm{Dev}|_{U_1 \cap \tilde{\alpha}} = T_1 \circ \varphi_1|_{U_1 \cap \tilde{\alpha}}$. Repeating this process, w eoptain maps $T_1, T_2, \ldots, T_n \in \mathrm{PSL}_2\mathbb{R}$ and we have that $\mathrm{Dev}([\alpha]) = \mathrm{Dev}(\tilde{\alpha}(1))$. This construction is independent of the homotopy class of α ☐

Remark 4.0.43. more generally, a (G, X)-structure on a surface S: X homogeneous space, G lie group with action on X. Everything said today is true as well, and examples are $\mathbb{C}P^1$-structures and $G = \mathrm{PSL}_2\mathbb{C}$.

We want to figure out how these surfaces look like.

Lemma 4.0.44. *X is a hyperbolic surface and γ is a nontrivial (in π_1) primitive simple closed curve (so its class in π_1 is not a power). Then either:*

1. *there exists a unique geodesic on X homotopic to γ or*

2. *γ is homotopic to a simple closed curve bounding a puncture*

(the word homotopic in the second item is now really neccesary, but we will later find a standard horocycle that it is homotopic to.)

Proof. cosnider the cover associated with the subgroup genereated by γ. Then $\pi_1 X\gamma = \mathbb{Z}$ so either a annulus or a punctured finite-radius disc or $\mathbb{C}^*$ but the last option is not hyperbolic, so is ruled out.

□

Lemma 4.0.45. 1. *If $\gamma \subset X$ geodesic homotopic to a* simple *closed curve γ', then γ is simple*

2. *Two geodesic curves interset minimally among curves in their homotopy classes.*

3. *A simple closed geodesic has a neightborhood containg no point of a non-intersecting simple closed geodesic.*

Proof. 1. if two curves interset, the lifts to $\mathbb{D}$ intersects, and if they dont intersects the lifts remain disjoint. We want to show that two lifts of γ, $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, dont intersect. If they intersect, their end points are crossed. A homotopy from γ to γ' can be lifted to the universal cover, and the lift $\tilde{\gamma}'$ is bounded distance from $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$

2. there are no geodesic bigons in $\mathbb{H}^2$. Removable intersections are bigons, and their lift to $\mathbb{D}$ find their endpoints and the geodesic connecting these are the geodesic representative and these wont have those bigons.

3. draw some annulus neightbourshood of γ. Lift geodesic to $\mathbb{D}$. If some geodesic comes very close to γs lift (the diameter), then the translation along $\tilde{\gamma}$ will make this close geodesic intersect itself (its endpoints will be crossed)! but then it cant be simple. There is a quantative version which is called the collar lemma. We will get to that later. □

Definition 4.0.46. A pair of pants is a 3-holed sphere (punctures are counted as holes).

Fact 4.0.47. *If X is a compact connected hyp surface with geodesic boundary such that all s. c. c. of $X$are the boundary components of X , then X is a pair of pants.*

Corollary 4.0.48. *Any compact hyperbolic surface has a pants decomposition.*

Proof. Choose a maximal multicurve. this will be a pants decomposition because of the fact.

for a closed surface of genus g, we have that the number of curves are $3g - 3$ and number of pants are $2g - 2$, just by euler char. A pair fo pants decomposition can be encoded as an trivalent graph – vertices are the pants, and edges the legs. □

Two pair of paints are MCG-realted if their graphs are isomorphic.

Lemma 4.0.49. *Given $a, b, c > 0$, there exists a unique hyperbolic pair of pants such that the boundary lengths are a, b, c*

This is proved with the help of the following lemma:

Lemma 4.0.50. *In $\mathbb{H}^2$, a right angled hexagon is uniquely determined by the lentghs of 3 alternating sides, and these lengths can be any 3 positive numbers.*

because we can glue two of thse hexagons with side lenghts $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$. Conversely, we cna cut a pair of paints along geodesics connecting the boundary and hitting the boundary in right angles. Alot of drawing, see the primer.

8th Lecture, September 25th 2012

4.1 The Collar Lemma

Last time, we discussed hyperbolic surfaces and pants decompositions. Today we will finish our discussion of hyperbolic surfaces, talking about the collar lemma. We will go on to a discussion of Teichmüller space.

The setup for the collar lemma is the following: Consider a complete hyperbolic surface X and a simple closed non-peripheral, non-trivial curve γ. One can find a unique geodesic representative of γ. In $\mathbb{H}^2$, this curve lifts to geodesic lines (drawing). On $\mathbb{H}$, $\Gamma = \pi_1 X$ acts by deck transformation.

Definition 4.1.1. The *δ-collar* of a simple closed geodesic γ is

$$A_\delta(\gamma) = \{x \in X \mid d(x, \gamma) < \delta\}$$

Definition 4.1.2. Define

$$\eta(l) = \frac{1}{2} \ln \frac{\cosh(l/2) + 1}{\cosh(l/2) - 1}.$$

This function can be interpreted as measuring particular distances in $\mathbb{H}^2$ (drawing).

Lemma 4.1.3 (The Collar Lemma)**.** *For any collection of disjoint simple closed geodesics $\gamma_1, \ldots, \gamma_n$, of lengths $l_1, \ldots, l_n$, the collars $A_{\eta(l_i)}(\gamma_i)$ are disjoint.*

Proof. Consider disjoint γ_1, γ_2. Extend $\{\gamma_1, \gamma_2\}$ to a pants decomposition (drawing of a pair of pants where two of the legs are γ_1, γ_2 and the last one is C). Cut along curves δ_1, δ_2 (to obtain $\delta_1, \delta_2, \delta_1', \delta_2'$), geodesic arcs minimizing the distance between γ_1 and C, and γ_2 and C respectively, and cut along δ_1, δ_2 to obtain an octogon. In the universal cover (drawing of $\mathcal{H}$), extending the geodesics $\delta_1, \delta_2, \delta_1', \delta_2'$, one obtains disjoint curves whose collars are disjoint as well (drawing). $\square$

Corollary 4.1.4. *If γ_1, γ_2 are simple closed geodesics of lengths l_1, l_2 respectively, and if $l_2 < 2\eta(l_1)$, then either $\gamma_1 = \gamma_2$ or $\gamma_1 \cap \gamma_2 = \emptyset$.*

Example 4.1.5. The Collar Lemma is sharp: The $\eta(l)$ is realized for a once punctured torus. Let X be a punctured torus, and let A, B be hyperbolic isometries with translation lengths $l, 2\eta(l)$ (drawing). Solving, $l = 2\eta(l)$, one finds $l = \ln(3 + 2\sqrt{2})$.

Corollary 4.1.6. *If γ_1, γ_2 have lengths less that $\ln(3 + 2\sqrt{2})$, then either $\gamma_1 = \gamma_2$ or $\gamma_1 \cap \gamma_2 = \emptyset$.*

Definition 4.1.7. The *injectivity radius* of a point $x \in X$ is

$$\mathrm{inj}(x) = \sup_r \{r \mid D(x, r) \text{ isometrically embedded disk}\}.$$

Equivalently, this is

$$\mathrm{inj}(x) = \frac{1}{2} \inf\{\text{length of essential closed curves that contain } x\}.$$

Theorem 4.1.8 (Thick-thin decomposition)**.** *Let X be a hyperbolic surface without boundaries. Then there exists $\varepsilon > 0$ such that any component the "ε-thin" part*

$$X^{<\varepsilon} = \{x \in X \mid \mathrm{inj}(x) < \varepsilon\}$$

is either

1. *an annulus neighbourhood of a simple closed geodesic of length $< 2\varepsilon$, or*
2. *the neighbourhood of a puncture.*

Fact 4.1.9. *There exists a constant $B > 0$, depending on the genus of the surface and called the* Bers constant*, such that any hyperbolic surface has a pants decomposition with all pants curves of length $\leq B$.*

The idea is that, starting at any point x, one can find an essential closed loop through the point of length B, as the area of the embedded disk is

$$2\pi(\cosh(\mathrm{inj}(x)) - 1) \leq 4\pi(g-1).$$

5 Teichmüller space

5.1 Definition

We give three definitions of Teichmüller space. Let $S_{g,n}$ be a (topological) surface of genus g with n punctures.

Definition 5.1.1 (Teichmüller space, 1)**.** Let $\mathcal{T}(S_{g,n})$ be the set of all marked conformal structures on $S_{g,n}$ up to equivalence, i.e. the set of (f, Σ), where $f : S_{g,n} \to \Sigma$ is a homeomorphism, and Σ is a Riemann surface, up to equivalence, where $(f, \Sigma_1) \sim (g, \Sigma_2)$ if and only if there is a conformal mapping $h : \Sigma_1 \to \Sigma_2$ such that $f \circ h \sim g$.

Fact 5.1.2. *(Drawing of markings on the torus T^2). It is a fact that* $\mathrm{MCG}(T^2) = \mathrm{SL}_2\mathbb{Z}$ *and* $\mathrm{Teich}(S_{1,0}) = \mathcal{H}$*. Later, we will see that this is isometric to $\mathbb{H}^2$ in the Teichmüller metric. For every point τ in the upper half plane, let $\mathbb{T}_\tau$ be the torus with the corresponding conformal structure.*

Lemma 5.1.3. *The tori $\mathbb{T}_\tau$ and $\mathbb{T}_\sigma$ are conformally equivalent if and only if*

$$\tau = \frac{a\sigma + b}{c\sigma + d}.$$

Proof. Suppose that $g : \mathbb{T}_\tau \to \mathbb{T}_\sigma$ is a conformal map. Lift this to the universal cover, $\tilde{g} : \mathbb{C} \to \mathbb{C}$, so $\tilde{g}(z) = \alpha z + \beta$, preserving the lattices in $\mathbb{C}$, so

$$\begin{aligned} \alpha\tau &= \tilde{g}(\tau) - \tilde{g}(0) = a\sigma + b, \\ \alpha &= \tilde{g}(1) - \tilde{g}(0) = c\sigma + d, \end{aligned}$$

for integers a, b, c, d, and so $\tau = \frac{a\sigma+b}{c\sigma+d}$. For the converse, one does the same with g^{-1}. $\square$

Definition 5.1.4 (Teichmüller space, 2)**.** If $3g - 3 + n > 0$, then $\mathcal{T}(S_{g,n})$ is the set of marked hyperbolic structures on $S_{g,n}$ up to equivalence. That is, one replaces "Riemann surface" with "hyperbolic surface" and "conformal map" with "isometry" in the earlier definition. The two definitions are equivalent by the uniformization theorem.

Definition 5.1.5 (Teichmüller space, 3)**.** Assume again $3g - 3 + n > 0$. Let $\mathcal{T}(S_{g,n})$ be the set of $\rho : \pi_1 S_{g,n} \to \mathrm{PSL}_2\mathbb{R}$ up to equivalence, where ρ is a faithful representation whose image is discrete, and $\rho_1 \sim \rho_2$ if and only if there exists $A \in \mathrm{PSL}_2\mathbb{R}$ such that $\rho_1 = A\rho_2 A^{-1}$.

Example 5.1.6. It turns out (drawing) that $\mathcal{T}(S_{1,1}) = \mathcal{T}(S_{1,0}) = \mathcal{H}$ (note that in general, $\mathcal{T}(S_{g,1})$ is not $\mathcal{T}(S_{g,0})$). To get an idea about this, we need two parameters giving different punctured tori. One concrete example we have seen before is $\rho : \pi_1 S_{1,1} = \langle A, B \rangle \to \mathrm{PSL}_2\mathbb{R}$ mapping

$$A = \begin{pmatrix} \sqrt{2} - 1 & 0 \\ 0 & \sqrt{2} + 1 \end{pmatrix}, \; B = \begin{pmatrix} \sqrt{2} & 1 + \sqrt{2} \\ \sqrt{2} - 1 & \sqrt{2} \end{pmatrix}.$$

One way to change the torus is by considering a pants decomposition where one glues the two free legs together by a twist. Then B changes to

$$B = \begin{pmatrix} \sqrt{2}e^{t/2} & (1 + \sqrt{2})e^{t/2} \\ (\sqrt{2} - 1)e^{-t/2} & \sqrt{2}e^{-t/2} \end{pmatrix}$$

Remark that the length of the curve B on the new X_t (which is now no longer a geodesic as it has become "broken") is an increasing strictly convex function. Namely, it is the trace of B_t which is

$$\operatorname{tr} B_t = B_0 \cosh(t/2),$$

for some constant B_0. On the other hand, by a prior calculation, recall that for a hyperbolic element a, $\operatorname{tr}(a)$ is $2\cosh(\text{translation length of } a/2)$.

5.2 The topology on Teichmüller space

Based on the third definition of Teichmüller space, the topology on $\mathcal{T}(S_g)$ is the one induced from $\operatorname{Hom}(\pi_1 S_g, \mathrm{PSL}_2\mathbb{R})$ which has a topology induced from $(\mathrm{PSL}_2\mathbb{R})^{2g}$. The discrete faithful representations form an open subset of this Hom-space, and we take the quotient topology on $\mathcal{T}(S_g)$. Equivalently, fix a base point $x_0 \in \mathbb{H}^2$, and let $\gamma_1, \ldots, \gamma_{2g}$ be the standard generators of $\pi_1 S_g$. Two representations $[\rho]$,$[\sigma]$ are *close* in $\mathcal{T}(S_g)$ if for some isometry $A \in \mathrm{PSL}_2\mathbb{R}$, the points $\rho(\gamma_i)x_0$ and $A \circ \sigma(\gamma_i)x_0$ are close for all i.

Similarly, one defines a quotient for non-closed surfaces $S_{g,n}$

Lemma 5.2.1. *Length functions are continuous: I.e. for α a non-peripheral, non-trivial simple closed curve on S, we have a function $l_\alpha : \mathcal{T}(S) \to \mathbb{R}^+$, defined by taking the trace of the appropriate image in* $\mathrm{PSL}_2\mathbb{R}$.

5.3 Fenchel–Nielsen coordinates on Teichmüller space

Fix a pants decomposition $\mathcal{P}$ on S_g. We can define a map $\varphi_{\mathrm{FN}} : \mathbb{R}^{3g-3} \times \mathbb{R}_+^{3g-3} \to \mathcal{T}(S_g)$. The second factor gives the length parameters of the legs in the pants. The first $3g-3$ parameters determine how much the pants are twisted when glued (drawing).

Theorem 5.3.1. *The map φ_{FN} is a homeomorphism.*

Proof. Claim 0. The map is surjective. This follows by construction. The twist parameters can be read off in the universal cover.

Claim 1. The map is injective. We prove this in the next lecture.

9th Lecture, September 28th 2012

Definition 5.3.2. Let X_0 have parameters $(l_1, \ldots, l_{3g-3}, \theta_1, \ldots, \theta_{3g-3})$ and let $\gamma_1 \in \mathcal{P}$. Let $\mathrm{tw}_{t\gamma_1} X_0$ denote the surface determined by $(l_1, \ldots, l_{3g-3}, \theta_1 + t, \theta_2, \ldots, \theta_{3g-3})$.

The proof of injectivity is based on the following Lemma:

Lemma 5.3.3. *If α is a simple closed geodesic on X, and β is a simple closed geodesic such that $i(\alpha, \beta) > 0$ (i.e. $\alpha \cap \beta \neq \emptyset$). Then, the length $l_\beta(t) = l(\beta, \mathrm{tw}_{t\alpha} X)$ of β on $\mathrm{tw}_{t\alpha}(X)$ is a strictly convex function of t (drawing).*

To see that the lemma implies injectivity, one just needs to check that different sets of twists give the marked surface: Let $\mathcal{P} = \{\gamma_1, \ldots, \gamma_{3g-3}\}$. Consider curves α_i intersecting γ_i, disjoint from $\mathcal{P} \setminus \gamma_i$. Consider the curve $\beta_i = \mathrm{tw}_{\gamma_i} \alpha_i$ (i.e. the result of applying a Dehn twist along γ_i to α_i). Define

$$A_i(t) = l(\alpha_i, \mathrm{tw}_{t\gamma_i} X), \;\; B_i(t) = l(\beta_i, \mathrm{tw}_{t\gamma_i} X).$$

The functions A_i, B_i are strictly convex by the lemma.

We claim that after twisting, at least one of the lengths of $\{\alpha_1, \beta_1, \ldots, \alpha_{3g-3}, \beta_{3g-3}\}$ will change. This will prove injectivity. Notice that by definition, $A_i(t + 2\pi) = B_i(t)$. By convexity, there exists at most one non-zero s such that $A_1(s) = A_1(0)$. Now, we claim that $B_1(s) \neq B_1(0)$, i.e. that $A_1(s + 2\pi) \neq A_1(2\pi)$. This follows from strict convexity of A_i by considering three cases, $s < 2\pi$, $s > 2\pi$, and $s = 2\pi$. A corollary of this argument is that lengths of a particular set of $9g-9$ curves determine the marked hyperbolic structure. □

Proof of Lemma 5.3.3. We first claim that

$$\frac{d}{dt}\Big|_{t=0} l_\beta(t) = \sum_{p\in\alpha\cap\beta} \cos\theta_p, \tag{3}$$

where θ_p is the interior angle between α and β (drawing of the (non-trivial) proof using the universal cover in this case).

Secondly, the angles of intersection $\theta_p(t)$ are strictly decreasing. This follows since the endpoints of the lift of the new geodesic representative of β are strictly on one side of the previous endpoint (drawing of universal cover). This implies the lemma. ☐

Remark 5.3.4. We have tw_α vector fields in moduli space satisfying the reciprocity $\mathrm{tw}_\alpha(l_\beta) = -\mathrm{tw}_\beta(l_\alpha)$. This is related to Wolpert's formula for the Kähler form on moduli space,

$$\omega = \sum_{\alpha\in\mathcal{P}} dl_\alpha \wedge d\theta_\alpha.$$

Really, we argued that φ_{FN} is bijective, but it is continuous because lengths vary continuously with Fenchel–Nielsen paramters (as they are defined in terms of traces of matrices, or, i.e. by (3)). Similarly, the inverse $\varphi_{\mathrm{FN}}^{-1}$ is continuous.

Corollary 5.3.5. *Teichmüller space $\mathcal{T}(S_g)$ is homeomorphic to $\mathbb{R}^{6g-6}$.*

5.4 Mapping class group action and moduli space

The *mapping class group* is $\mathrm{MCG}(S_g) = \mathrm{Homeo}^+(S_g)/\mathrm{Homeo}_0(S_g)$, where Homeo^+ denotes orientation preserving homeomorphisms, and Homeo_0 consists of those homeomorphisms isotopic to the identity. this acts on $\mathcal{T}(S_g)$ as follows: For a marking $f : S \to \Sigma$ and $\varphi \in \mathrm{MCG}$, $\varphi\cdot(f,\Sigma) = (f\circ\varphi^{-1},\Sigma)$.

Definition 5.4.1. A set of simple closed curves $\alpha_1,\dots,\alpha_n$ *fill* S_g if $S_g \setminus (\alpha_1\cup\cdots\cup\alpha_n)$ is a union of disks.

In fact, one can always find two curves that fill.

Fact 5.4.2. *A mapping class is determined up to a finite choice ambiguity by the images of a filling set of curves.*

This follows from

Lemma 5.4.3 (Alexander lemma)**.** *We have* $\mathrm{MCG}(\mathbb{D}) = \{\mathrm{Id}\}$.

Lemma 5.4.4. *The mapping class group acts properly discontinuously on $\mathcal{T}(S_g)$.*

Proof. Let $K \subseteq \mathcal{T}(S_g)$ be compact. We want to rule out that there exists a sequence $\psi_n \in \mathrm{MCG}$, all distinct, such that $\psi_n(K)\cap K \neq \emptyset$. Fix a filling set of curves, say $\{\alpha,\beta\}$. Since K is compact, $l(\alpha,X), l(\beta,X) \le L$ for all $X \in K$. Let $X_n \in \psi_n(K)\cap K$. We claim that $l(\alpha,\psi_n X_n) = l(\psi_n^{-1}(\alpha), X_n) \to \infty$. This claim will be the contradiction. To see the claim, note that $\{\psi_n^{-1}(\alpha), \psi_n^{-1}(\beta)\}$ is infinite. This follows from Fact 5.4.2, since the ψ_n are distinct. Suppose $\alpha_n = \psi_n^{-1}(\alpha)$ are all distinct. Now we claim that there is a pants decomposition $\mathcal{P}$ of S_g such that $i(\alpha_n,\mathcal{P}) \to \infty$. To see this, suppose that $i(\alpha_n,\mathcal{P})$ is bounded. Then the α_n must differ by twists along $\mathcal{P}$. In particular, there is a pants curve $\gamma\in\mathcal{P}$, such that the twisting number of α_n across γ is an unbounded sequence. Replace γ by another γ' to get a new pants decomposition such that the α_n intersect the new pants decomposition more and more. Now by the Collar Lemma, every pants curve in $\mathcal{P}$ has a definite collar around it for any surface in $K\subseteq\mathcal{T}(S)$. This implies the first lemma. ☐

The quotient of the action is *moduli space* $M_g = \mathcal{T}_g/\mathrm{MCG}$.

For example, $\mathcal{T}(S_{1,0}) = \mathcal{H}$, $\mathrm{MCG}(S_{1,0}) = \mathrm{SL}_2\mathbb{Z}$, and the quotient is the fundamental domain we have considered before.

Lemma 5.4.5. *Moduli space M_g has one end (drawing).*

Here, a topological space X has *one end* if there is an increasing sequence (K_n) of compact sets exhausting X such that $X \setminus K_n$ is connected. The lemma will follow from the following two lemmas.

Lemma 5.4.6 (Mumford Compactness Criterion)**.** *The thick part $M_g^{\varepsilon>0}$ is compact. Here,*

$$M_g^{\varepsilon>0} = \{X \in M_g \mid \textit{injectivity radius of } X \textit{ is} \geq \varepsilon\}.$$

Proof. Let $X_n \in M_g^{\varepsilon>0}$. Fix a Bers pants decomposition, i.e. one where all pants curves have length bounded from above by some constant B depending only on g. Now they also have lengths $\geq \varepsilon$. The twist parameters can be assumed to be in $[0, 2\pi]$ by applying Dehn twists to the pants curves. Since the length and twist parameters lie in a compact set, the surfaces converge. □

Lemma 5.4.7. *The thin part $M_g^{<\varepsilon}$ is connected.*

The proof of this involves the *curve complex* $C(S_g)$, where vertices are isotopy classes of non-trivial simple closed curves on S_g. A set of such vertices $\{\gamma_1, \dots, \gamma_n\}$ span a simplex, if the γ_i are pairwise disjoint. The mapping class group acts on $C(S_g)$.

For example, $C(S_1)$ is the Farey tessellation (where one modifies the definition of C and assumes minimal intersection instead of disjointness).

Fact 5.4.8. *The curve complex $C(S_g)$ is connected for $g \geq 2$.*

Proof of Lemma 5.4.7. Suppose now we want to connect two points X, Y in the thin part of moduli space. On X there exists a curve γ_0 that is short, and on Y there is a curve γ_1 that is short. For ε small enough, these curves must be disjoint by the Collar Lemma. Now, by varying pants decompositions, there is a path connecting γ_0 and γ_1 in $C(S_g)$ with the resulting surfaces staying in the thin part. □

10th Lecture, October 2nd 2012

Last time, we introduced the complex of curves $C(S_{g,n})$ with $3g - 3 + n > 0$. The vertices are isotopy classes of non-peripheral simply closed curves. A set $\{\gamma_n\}$ span a simplex (drawing) if they are pairwise disjoint (except in a few cases: For $S_{0,4}$ the curves intersect twice, and for $S_{1,1}$ they intersect once).

For example, for a torus or $S_{1,1}$, simply closed curves are given by a pair of integers (p, q) (drawing). Two such curves (p, q), (r, s) intersect once if and only if $|\det \begin{pmatrix} p & q \\ r & s \end{pmatrix}| = 1$. In this case, it turns out that the curve complex is the Farey tessellation; simplices are $\{\alpha, \beta, D_\alpha\beta\}$.

Fact 5.4.9. *Pants decompositions give maximal simplices, so $\dim C(S_g) = 3g - 3$. The complex is locally infinite and has infinite diameter (in the metric where each edge has length 1). The mapping class group $\mathrm{MCG}(S_{g,n})$ acts on $C(S_{g,n})$. The curve complex is δ-hyperbolic (a theorem due to Maser–Minsky). For $3g - 3 + n > 1$, $C(S_{g,n})$ is connected.*

We indicate a proof of connectedness of $C(S_g)$ for $g \geq 2$.

Proof. Let $\alpha, \beta \in C(S_g)$. The proof goes by induction on $i(\alpha, \beta)$, where $i(\alpha, \beta)$ is the *intersection number* of α and β, i.e. the minimal number of intersections between representatives of α and β. If $i(\alpha, \beta) = 0$, we are done. Similarly, if $i(\alpha, \beta) = 1$ (drawing), one can takes a thickening $N(\alpha \cup \beta)$

of $\alpha \cup \beta$. Then the boundary of $N(\alpha \cup \beta)$ is a non-trivial curve disjoint from α, β (here we use that $g \geq 2$), and so we get a path between α, β. If $i(\alpha, \beta) = n + 1$, we need to create a new curve γ with $i(\alpha, \gamma) \leq n$, $i(\alpha, \beta) \leq n$. The idea is to look at two successive intersections and do a simple surgery (drawing). □

This fact we used in the proof of

Lemma 5.4.10. *The moduli space M_g has one end.*

Proof. For $g = 1$, $M_g = H/\mathrm{SL}_2\mathbb{Z}$ (drawing). For $g \geq 2$ (drawing), we saw that the thin part $M_g^{<\varepsilon}$ is connected: Suppose X, Y are surfaces with injectivity radius less that ε. Then there exist curves α, β such that $l(\alpha, X) < \varepsilon$, $l(\beta, Y) < \varepsilon$. Consider a path $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ in $C(S_g)$. On X, we extend α_0, α_1 to a pants decomposition P_0. Shrink the length parameter of α_1 to less than ε. This gives a path in M_g from X to some point X_1. On X_1, extend α_1, α_2 to a pants decomposition and shrink α_2. Continuing like that gives a path from X to Y in the thin part $M_g^{<\varepsilon}$. □

Lemma 5.4.11. *On a hyperbolic surface X, there are only finitely many simple closed curves of length $\leq l$.*

Proof. Fix a pants decomposition P on X and a curve α with length less than L. The Collar Lemma implies that $\sum_{\gamma \in P} i(\alpha, \gamma)$ is bounded. Similarly, by the strict convexity of the length function along twists, $\sum_{\gamma \in P} |\mathrm{tw}(\alpha, \gamma)|$ is bounded. There are only finitely many curves α satisfying this. □

5.5 Thurston compactification of Teichmüller space

There are several ways to diverge in $\mathcal{T}_g$.

- One way is to shrink curves.
- One can hit by infinitely many distinct mapping classes, e.g. by taking powers of a mapping class.
- Combinations of these.

The general picture is the following: For a diverging sequence X_n in $\mathcal{T}_g$, there exists a sequence of simple closed curves γ_n such that

$$l(\gamma_n, X_n) \leq B,$$

where B is the Bers constant. There exist scalars λ_n, $\lambda_n \to 0$ such that the weighted curves "$\lambda_n \gamma_n \to \lambda$", where λ is a *measured lamination*. We will discuss a notion of *projective measured laminatures* of measured laminations up to scaling, that will compactify $\mathcal{T}_g$.

Example 5.5.1. For $S_{1,1}$, we saw that Teichmüller space is the upper half plane. It is a fact that the shortest curves on X are the vertices of the Farey triangle X is in.

We sketch now the Thurston compactification: Let $\mathcal{S}$ be the set of all isotopy classes of simple closed curves on S_g. Consider $\mathbb{R}_+^{\mathcal{S}}$, the infinite-dimensional space of functionals on $\mathcal{S}$ with values in $\mathbb{R}_+$ with the weak topology, and $P\mathbb{R}_+^{\mathcal{S}} = \mathbb{R}_+^{\mathcal{S}} \setminus \{0\}/\mathbb{R}_+$. Intersection number induces a map $i_* : \mathcal{S} \to \mathbb{R}_+^{\mathcal{S}}$, mapping $\alpha \mapsto i(\alpha)(\beta) := i(\alpha \cap \beta)$. This extends to a map $i_* : \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}_+^{\mathcal{S}}$ by $i_*(\lambda\alpha) = \lambda i_*(\alpha)$. Let $\pi : \mathbb{R}_+^{\mathcal{S}} \to P\mathbb{R}_+^{\mathcal{S}}$ be projection. Then we claim that $\pi \circ i_*$ is injective. This follows from the fact that if α, β are distinct curves, then there exists a curve γ such that $i(\alpha, \gamma) \neq 0$, $i(\beta, \gamma) = 0$. We claim also that $\pi \circ i_*(\mathbb{R}_+ \times \mathcal{S})$ is precompact in $P\mathbb{R}_+^{\mathcal{S}}$. It will turn out that $\mathbb{R}_+ \times \mathcal{S} \hookrightarrow M\mathcal{L}$, the space of measured laminations, such that i_* extends to $I_* : M\mathcal{L} \to \mathbb{R}_+^{\mathcal{S}}$ such that $\pi \circ I_*(M\mathcal{L}) = \overline{\pi \circ i_*(\mathbb{R}_0 \times \mathcal{S})}$. Moreover, the length functional $l_* : \mathcal{T}_g \to \mathbb{R}_+^{\mathcal{S}}$, mapping $X \mapsto l_X(\alpha) := l(\alpha, X)$, is proper and homeomorphic to its image. We claim that $\pi \circ l_* : \mathcal{T} \to P\mathbb{R}_+^{\mathcal{S}}$ is an embedding. In fact, there can not exist two hyperbolic surfaces X, Y such that $l(\alpha, X) \geq l(\alpha, Y)$ for all $\alpha \in \mathcal{S}$. We will see a prove of that later, when we talk about the Earthquake Theorem.

Now, $\pi \circ l_*(\mathcal{T}_g)$ and $\pi \circ I_*(M\mathcal{L})$ are disjoint in $P\mathbb{R}^+_{\mathcal{S}}$: E.g. for $\alpha \in \mathcal{S}$, $i_*(\alpha)$ is distinct from $l_*(\mathcal{T}_g)$ in $\mathbb{R}^{\mathcal{S}}_+$, as $i_*(\alpha)(\alpha) = 0$, but $l_X(\alpha) > 0$. The embeddings of $\mathcal{T}_g$ and $M\mathcal{L}$ in $P\mathbb{R}^{\mathcal{S}}_+$ fit together and give a closed ball $\overline{\mathbb{R}^{6g-6}}$.

Remark 5.5.2. The mapping class group acts on $\mathbb{R}^{\mathcal{S}}_+$: Given $f \in \mathbb{R}^{\mathcal{S}}_+$, $\varphi \in \mathrm{MCG}(S_g)$, let $\varphi \circ f(\alpha) = f(\varphi^{-1}\alpha)$.

Toy Lemma 5.5.3. *Consider $X_n \in \mathcal{T}(S_g)$ obtained by pinching a curve γ, i.e. $l(\gamma, X_n) \to 0$. Then for any $\alpha, \beta \in \mathcal{S}$,*

$$\frac{l(\alpha, X_n)}{l(\beta, X_n)} \to \frac{i(\alpha,\gamma)}{i(\beta,\gamma)}.$$

Proof. We claim that there exist $\lambda_n \in \mathbb{R}$, $\lambda \to 0$ such that for any $\beta \in \mathcal{S}$, $\lambda_n l(\beta, X_n) \to i(\beta,\gamma)$. This follows from the Collar Lemma, which tells us that $C_n i(\beta,\gamma) \le l(\beta, X_n) \le C_n i(\beta,\gamma) + C$, for some C independent of n and $C_n \to \infty$. Take $\lambda_n = 1/C_n$. □

We will see that in general, one has this lemma with γ replaced by a lamination λ.

5.6 Laminations

We first discuss geodesic laminations.

Definition 5.6.1. Let X be a hyperbolic surface. A *geodesic lamination* λ on X is a closed set foliated by geodesics. I.e. for any point in λ, there is a neighbourhood U and a homeomorphism of pairs

$$(U, \lambda \cap U) \to ((0,1) \times (0,1), (0,1) \times K),$$

where K is a compact subset of $(0,1)$.

Example 5.6.2. A union of simple clsoed disjoint geodesics is a geodesic lamination.
A second example is a geodesic together with a spiralling leaf (drawing).
A third example, which we will construct next time, is a "minimal" lamination with each leaf dense in the λ, and K is the Cantor set.

Fact 5.6.3. *We have* $\mathrm{Area}(\lambda) = 0$.

Sketch of proof. The metric completion $\widehat{X \setminus \lambda}$ is a hyperbolic surface with geodesic boundary (drawing). We claim that

$$\mathrm{Area}(\widehat{X \setminus \lambda}) = \mathrm{Area}(X \setminus \lambda) = \mathrm{Area}(X)$$

There is a notion of *reduced Euler characteristic*,

$$\chi'(S) = \chi(S) - \frac{k}{2},$$

where k is the number of cusps (drawing). It is a fact that $\mathrm{Area}(S) = -2\pi\chi'(S)$. On the other hand, on a surface with cusps, consider a foliation $\mathcal{F}$ parallel to the boundary, "standard" at cusps (drawing), one has

$$\chi'(S) = \sum_{p \in \text{singularities of } \mathcal{F}} \mathrm{ind}_{\mathcal{F}}(p).$$

Combining these,

$$\begin{aligned}\mathrm{Area}(X \setminus \lambda) &= -2\pi\chi'(X) = -2\pi \sum_{p\in\mathrm{sing}} \mathrm{ind}_{\mathcal{F}}(p) = -2\pi \sum_{p\in\mathrm{sing}} \mathrm{ind}_{\tilde{\mathcal{F}}}(p)\\ &= -2\pi\chi(X) = \mathrm{Area}(X),\end{aligned}$$

where $\tilde{\mathcal{F}}$ is a continuation of $\mathcal{F}$ to X, with the same singularity set of $\mathcal{F}$. □

11th Lecture, October 5th 2012

Today, we will talk more on laminations and

- Construct minimal laminations using train tracks and
- talk about measured laminations $M\mathcal{L}$, and how
- $\overline{\mathcal{T}_g} = \mathcal{T}_g \cup PM\mathcal{L}$.

Last time, we saw what a geodesic lamination was a closed set foliated by geodesics. The space of all geodesic laminations will be denoted $\mathcal{GL}(S)$, and we saw that $\text{Area}(\lambda) = 0$, $\lambda \in \mathcal{GL}(S)$.

Fact 5.6.4. *The space $\mathcal{GL}(S)$ is compact in the Hausdorff metric on the space $C(S)$ for closed subset of S.*

Recall here that for $A, B \in C(S)$, $d_{\text{Haus}}(A, B) \leq \varepsilon$ if and only if the ε-neighbourhoods satisfy $A \subseteq N_\varepsilon(B)$, $B \subseteq N_\varepsilon(A)$. The proof of the fact uses the fact that λ is "uniformly sparse", $\text{Area}(N_\varepsilon(\lambda)) = C(\varepsilon)$, and $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, where $C(\varepsilon)$ is independent of λ. One consequence of the fact is that an infinite sequence of simple closed geodesics converge (after passing to a subsequence) to a geodesic lamination (drawing of lift to the universal cover). A geodesic lamination can be thought of as a closed, Γ-invariant of $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2 \setminus \Delta$, where Δ is the diagonal, where each pair of elements satisfy the "non-crossing" condition.

We will now construct a minimal lamination.

Lemma 5.6.5. *Let $p(t)$ be a priecewise geodesic path in $\mathbb{H}^2$. Suppose each segment has length $\geq l$. Then there exists $\theta(l)$ such that if the exterior angles of p are less than $\theta(l)$, then $p(t)$ convergers as $t \to \pm\infty$ to $\xi_\pm \in S_\infty$, with $\xi_+ \neq \xi_-$. Furthermore, the Hausdorff distance $d_{\text{Haus}}(p, [\xi_+, \xi_-]) < C_\theta$, where $[\xi_+, \xi_-]$ is a geodesic line connecting ξ_+, ξ_- (drawing).*

Proof. (Drawings). □

Let X be a hyperbolic surface. Take a trivalent graph $\tau \subseteq X$ such that the edges are geodesics of length $> l > 0$ and the interior angles at vertices are less than $\theta(l)$ (drawing).

Definition 5.6.6. Such a graph is called a *train track*, the vertices are *switches*, edges are *branches*, and a *train route* is a paht along the branches with no sharp turns.

Fact 5.6.7. *The lift of any train route satisfies the conditions of Lemma 5.6.5.*

In particular, we obtain a particular geodesic curve in $\mathbb{H}^2$ from the second part of the lemma. To get a lamination from this, thicken the train track (drawing) such that at a switch, the thickening goes from having width 1 to having widths $r, 1-r$ (drawing), and corresponding to every $0 < \alpha < 1$, we have a train route σ_α (drawing). Each σ_α lifts to a path with distinct endpoints, satisfying the non-crossing condition and form a geodeosic laminiation

$$\tilde{\lambda} = \bigcup_{\alpha \in [0,1]} \overline{\sigma_\alpha},$$

where $\overline{\sigma_\alpha}$ is the gedeosic line with the same endpoints as σ_α. We claim that for $r \notin \mathbb{Q}$, we get a *minimal lamination* (drawing of a square torus where the lines). A proof of the following can be found in Casson–Bleiler.

Theorem 5.6.8 (Structure theorem)**.** *Any $\lambda \in \mathcal{GL}(S)$ is a finite union*

$$\lambda = \lambda_1 \cup \cdots \cup \lambda_m \cup \gamma_1 \cup \cdots \cup \gamma_n \cup \delta_1 \cup \ldots \delta_{m'},$$

where the λ_i are minimal, the γ_i are simple closed geodesics, and the δ_i are spiralling leaves that accumulate on λ_i or γ_j (drawing), and n, m, m' are bounded by g.

5.7 Measured laminations

Definition 5.7.1. A *transverse measure* on a geodesic lamination λ is a nonnegative function μ on transversal arcs (drawing) which is

1. additive: $\mu(t_1 \cup t_2) = \mu(t_1) + \mu(t_2)$, if t_1, t_2 are disjoint,
2. holonomy invariant: $\mu(t) = \mu(t')$, if t and t' are isotopic through transverse arcs, and
3. μ is supported on λ: $\mu(t) = 0$ if $t \cap \lambda = \emptyset$.

Example 5.7.2. If λ is a simple closed geodesic, the "counting measure" works, i.e. $\mu(t) = \# t \cap \lambda$.

If λ is a minimal lamination, we can define an "average" of the counting measure (drawing); fix a transversal arc τ_0. Define

$$\mu_t(\tau) = \frac{\# \tau \cap l_t}{\# \tau_0 \cap l_t},$$

where l_t is a leaf of length t. Then it is a fact that the μ_t converge (after passing to a subsequence) to μ.

For a general lamination λ, one can choose a "train track"-neighbourhood $N_\varepsilon(\lambda)$ and assign weights to the branches satisfy an additivity condition at the switches (drawing).

Let $M\mathcal{L}(S)$ be the set of geodesic laminations with transverse measure with full support (i.e. if $t \cap \lambda \neq \emptyset$, then $\mu(t) \neq 0$). A non-example is a lamination containing a spiralling leaf (drawing of train track argument). The set $M\mathcal{L}(S)$ has a natural topology: For any set of transverse arcs $a_1, a_2, \ldots, a_n$ on S, λ is close to λ' if and only if $\mu_\lambda(a_i)$ is close to $\mu_{\lambda'}(a_i)$.

Example 5.7.3. In $M\mathcal{L}$, for α_1 and α_2 disjoint, $\alpha_1 + \frac{1}{n}\alpha_2 \to \alpha_1$. If $\alpha \cap \beta \neq \emptyset$, $\frac{1}{n}\mathrm{tw}_\alpha^n \beta \to \alpha$.

Theorem 5.7.4 (Thurston). $M\mathcal{L} \cong \mathbb{R}^{6g-6}$.

The rough idea is the following: Train tracks give coordinates: Take a (recurrent, i.e. each branch has a closed train route through it) train track τ which is *maximal* (i.e. complementary regions are triangles). Let

$$P(\tau) = \{\omega : B \to \mathbb{R}_{>0} \mid \text{additivity condition } \omega_1 + \omega_2 = \omega_3\},$$

where B is the set of branches of τ, and the ω are the weights. This is a cone in $\mathbb{R}^B$. There is a map $P(\tau) \to M\mathcal{L}$ given by thickening train tracks. One claim is that this map is homeomorphic to its image, giving coordinate charts. Another one is that $\dim P(\tau) = 6g - 6$, which follows from the fact that the number of branches minus the number of vertices is $6g - 6$, which follows from an (reduced) Euler characteristic argument.

12th Lecture, October 9th 2012

The plan for today is to

- finish the discussion of $M\mathcal{L}$,
- talk again about $\overline{\mathcal{T}_g} = \mathcal{T}_g \cup PM\mathcal{L}$,
- talk about the Nielsen–Thurston classification and give an idea of the proof.

Last time, we saw considered the space $M\mathcal{L}$ of geodesic laminations with a transverse measure, with the weak topology. On a minimal component λ' of a lamination , the measure can be described as a weighted train track, so that in $N_\varepsilon(\lambda)$, the measure of a transverse arc τ is the weight of the branch. One could ask what happens to the measure of a shorter arc τ' that doesn't go across all of $N_\varepsilon(\lambda')$. To work this out one comes up with a splitting the branch into thinner branches (drawing).

An example of convergence in $M\mathcal{L}$ we talked about last time is the following: Take two geodesics α, β, $\alpha \cap \beta \neq \emptyset$, and consider $\mathrm{tw}_\alpha^n \beta$. This converges to λ in $\mathcal{GL}$. In $M\mathcal{L}$, $\frac{1}{n}\mathrm{tw}_\alpha^n \beta \to \alpha$ (drawing).

Let τ be a maximal recurrent train track (there are finitely many of those), and consider like last time

$$P(\tau) = \{\omega : B \to \mathbb{R}_{>0} \mid \text{additivity condition af switches}\}.$$

There exists a map $\varphi : P(\tau) \to M\mathcal{L}$ "tightening" the foliated neighbourhood of a realisation of τ on the surface, and this map is a homeomorphism onto an open set in $M\mathcal{L}$. Other facts are that $M\mathcal{L} \cong \mathbb{R}^{6g-6}$, and from this $M\mathcal{L}$ gets a PL-structure and a (Lesbegue-)measure. One more useful fact is that weighted simple closed curves are dense in $M\mathcal{L}$ (the rational points in the polyhedral cone $P(\tau)$, drawing).

5.8 Thurston compactification revisited

Recall that for (weighted) simple closed curves, we have an *intersection number* $i : (\mathbb{R}_+ \times \mathcal{S}) \times (\mathbb{R}_+ \times \mathcal{S}) \to \mathbb{R}_+$, $(\alpha, \beta) \mapsto i(\alpha, \beta)$, defined by

$$i(c_1\alpha, c_2\beta) = c_1c_2 i(\alpha, \beta).$$

This extends to $i : M\mathcal{L} \times M\mathcal{L} \to \mathbb{R}_+$, $(\lambda, \mu) : i(\lambda, \mu)$ through the mreasured lamination: The rough idea is that at any point of transverse intersection between λ and μ, one has a product picture (drawing) with a locally defined measure $da \times db$ obtained through the transverse measure by taking infinitesimal transverse arcs, and we let

$$i(\lambda, \mu) = \iint_S da \times db.$$

Properties 5.8.1. *For any $\alpha, \beta \in M\mathcal{L}$,*

1. $i(c_1\alpha, c_2\beta) = c_1c_2 i(\alpha, \beta)$,
2. $i(\lambda, \lambda) = 0$,
3. $i(\lambda, \mu) = i(\mu, \lambda)$.

Now i induces a map

$$\begin{aligned} I_* : M\mathcal{L} &\to \mathbb{R}_+^{\mathcal{S}} \\ \lambda &\mapsto i(\lambda, \cdot). \end{aligned}$$

One should think of $I_*(M\mathcal{L})$ as a cone in $\mathbb{R}_+^{\mathcal{S}}$. Furthermore, $\mathcal{T}(S_g)$ embeds in $\mathbb{R}_+^{\mathcal{S}}$ through the length functional

$$\begin{aligned} l_* : \mathcal{T}(S_g) &\to \mathbb{R}_+^{\mathcal{S}} \\ X &\mapsto l(\cdot, X). \end{aligned}$$

Claim 5.8.2. *This image of $\mathcal{T}(S_g)$ is asymptotic to the cone given by the embedding of $M\mathcal{L}$.*

Consider the projectivisation $\pi : \mathbb{R}_+^{\mathcal{S}} \setminus \{0\} \to P\mathbb{R}_+^{\mathcal{S}}$.

Theorem 5.8.3 (Thurston compactification)**.** *The set*

$$\pi \circ I_k(PM\mathcal{L}) \sqcup \pi \circ l_*(\mathcal{T}(S_g))$$

is a closed ball.

We want to make the statement of Claim 5.8.2 more precise. Recall that by definition of the topology on $\mathbb{R}_+^{\mathcal{S}}$, $f_n \to f$ in $\mathbb{R}_+^{\mathcal{S}}$ if for all $\alpha \in \mathcal{S}$, $f_n(\alpha) \to f(\alpha)$. In $P\mathbb{R}_+^{\mathcal{S}}$, $[f_n] \to [f]$ if there exist c_n such that $c_n f_n \to f$. Equivalently, for all $\alpha, \beta \in \mathcal{S}$,

$$\frac{f_n(\alpha)}{f_n(\beta)} \to \frac{f(\alpha)}{f(\beta)}$$

We have seen that for the diverging sequence $X_n \in \mathcal{T}$ obtained by pinching a curve γ and leaving the rest of the surface intact, we saw that for all $\alpha, \beta \in \mathcal{S}$,

$$\frac{l(\alpha, X_n)}{l(\beta, X_n)} \to \frac{i(\alpha, \gamma)}{i(\beta, \gamma)}.$$

Claim 5.8.4. *More generally, for any diverging sequence $X_n \in \mathcal{T}_g$ there exists a measured lamination $\lambda \in M\mathcal{L}$ such that for $\alpha, \beta \in \mathcal{S}$,*

$$\frac{l(\alpha, X_n)}{l(\alpha, X_n)} \to \frac{i(\alpha, \lambda)}{i(\beta, \lambda)}. \tag{4}$$

Example 5.8.5. Let $X_n = \mathrm{tw}_\gamma^n X$. We claim that this converges to $X_n \to [\gamma] \in PM\mathcal{L}$. To see (4) in this example, take $\alpha = \gamma$ and β anything that intersects α. Noe $l(\alpha, X_n)$ stays fixed, by $l(\beta, X_n) \to \infty$, so

$$\frac{l(\alpha, X_n)}{l(\beta, X_n)} \to 0 = \frac{i(\alpha, \gamma)}{i(\beta, \gamma)}.$$

In general,

$$l(\alpha, X_n) \approx n i(\alpha, \gamma) l(\gamma) + C.$$

In general, one takes a set of Bers curves for the surfaces X_n, and these sets will converge to the measured lamination of λ.

5.9 Nielsen–Thurston classification

For every $f \in \mathrm{MCG}(S_{g,n})$, one of the following holds:

1. (Elliptic) f has finite order,
2. (Reducible) f has infinite order, and there exists a set C of disjoint nontrivial simple closed curves such that $f(C) = C$ (up to isotopy).
3. f has a pseudo-Anosov representative in its isotopy class.

Definition 5.9.1. A *pseudo-Anosov* mapping preserves two transvere minimal filling (i.e. complementary regions are polygonal) measured laminations λ^+ and λ^- and $f(\lambda^+) = c\lambda^+$, $f(\lambda^-) = \frac{1}{c}\lambda^-$, $c > 1$.

Example 5.9.2. Consider the torus T^2. There are only finitely many finite order elements. Those have orders $1, 2, 3, 4, 6$. One is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Reducible: For T^2, $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2\mathbb{Z}$, the Dehn twist about a curve is reducible.

Anosov: One example is $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$. Such matrices leave two lines of irrational slope in the universal cover $\mathbb{R}^2$ invariant (the eigenvectors).

In general, for $g \geq 2$, one way of constructing pseudo-Anosov maps is to take branched covers of an Anosov map.

Example 5.9.3. Let T be a square torus, $A : T \to T$, given by

$$A = \begin{pmatrix} 4 & 1 \\ 7 & 2 \end{pmatrix}.$$

Let τ be the slit from $(0,0)$ to $(\frac{1}{2}, \frac{1}{2})$. Let Σ be a genus 2 branched double cover of T (drawing), branched over τ. We claim that A lifts to a homeomorphism $\hat{A} : \Sigma \to \Sigma$, the lamination (foliation) on Σ has 2 (order 2) singularities and lifts to the one on T preserved by A (drawing).

Definition 5.9.4. A *Markov partition* for a mapping class $h : S \to S$ is a decomposition of S into a finite union of rectangles $R_1, \ldots, R_n$ such that

1. if $i \neq j$, $\mathrm{Int} R_i \cap \mathrm{Int} R_j = \emptyset$,
2. $h(\bigcup_{j=1}^n \partial_v R_i) \subseteq \bigcup_{j=1}^n \partial_v R_i$,
3. $h(\bigcup_{j=1}^n \partial_h R_i) \subseteq \bigcup_{j=1}^n \partial_h R_i$,

where ∂_h, ∂_v denote the horizontal and vertical boundaries of the rectangles respectively. Then $h(R_i) \cap R_j$ consists of finitely many subrectangles (drawing). Let $l(i,j)$ be this finite number and form the matrix $A = (a_{ij}) = (l(i,j))$. Then (by Perron–Frobenius) A has a positive eigenvector with eigenvalue $\lambda \geq 1$. Widths of the rectangles are given by the entries of the eigenvector.

A pseudo-Anosov map has a Markov partition with $\lambda > 1$. So what happened to the invariant laminations? One can build an invariant weighted train track corresponding to the Markov partition (drawing): For every rectangle, one takes a branch, the widths give the weights, and adjacency determines the switches. To get the other lamination, one switches horizontal with vertical in that description.

We mention some words on the proof of the Nielsen–Thurston classification theorem: There are several proofs of the theorem, and we discuss one that we will call the hyperbolic geometry proof (Casson–Bleiler). If a mapping class f is irreducible and non-periodic, then every isotopy class of simple closed curves c, $\{f^n(c)\}$ is infinite. Since $\mathcal{GL}(S)$ is compact, there is an accumulation point λ of $\{f^n(c)\}$. We can split $\lambda = \lambda' \cup \delta$ into a measurable part and a spiralling part. In fact, $f(\lambda') = \lambda'$ and λ' is filling. This defines one of the invariant laminations, $\lambda^+ = \lambda'$.

The next step is to get the transverse lamination λ^-. To get this, one studies the dynamics of $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^2$. It turns out that this extends continuously to a map $\tilde{f} : \overline{\mathbb{H}^2} \to \overline{\mathbb{H}^2}$. One uses this to construct λ^-. Finally, one builds a Markov partition t o get the transverse measure from eigenvectors of A.

Thurston's original proof of the theorem: The map $f : \overline{\mathcal{T}_g} \to \overline{\mathcal{T}_g}$ has a fixed point by Brouwer's fixed point theorem.

Case 1. If $p \in \mathcal{T}_g$, then f is of finite order.

Case 2. If $p = [\lambda] \in PM\mathcal{L}$, and λ has no closed leaves, then f is reducible.

Case 3. If $[\lambda]$ has no closed leaves, $f(\lambda) = c\lambda$ for some c. If $c = 1$, then f is again reducible.

Case 4. If $c \neq 1$, then f is pseudo-Anosov.

13th Lecture, November 6th 2012

Today, we will discuss

- quasiconformal maps
- the Teichmüller metric
- Bers' proof of the Nielsen–Thurston classification

Recall, that we considered Fenchel–Nielsen coordinates $\mathcal{T}_g \cong \mathbb{R}^{6g-6}$ with a proper continuous action of the mapping class group. Recall here that for $\varphi \in \mathrm{MCG}$, $\varphi(f, \Sigma) = (f \circ \varphi^{-1}, \Sigma)$. The quotient of $\mathcal{T}^g$ by this action is the one-ended moduli space $\mathcal{M}_g$. Recall also the Nielsen–Thurston classification saying that a map is either of finite order, is reducible (i.e. it preserves a system of disjoint curves), or it is pseudo-Anosov.

In fact, the mapping class group acts by isometries of $(\mathcal{T}_g, d_\tau)$, Teichmüller space with the Teichmüller metric. The Nielsen–Thurston classification trichotemy is equivalent to φ being elliptic, parabolic, or hyperbolic, analogously to what happened for $\mathrm{Isom}^+(\mathbb{H}^2)$. For instance, in the hyperbolic case, the action of φ on $\mathcal{T}_g$ will preserve to boundary points $\lambda^+, \lambda^- \in \partial\mathcal{T}_g = PM\mathcal{L}$ and a geodesic axis connecting them, and acts by translation along this axis. In the parabolic case, there is one fixpoint on the boundary of $\mathcal{T}_g$ and some sort of translation around this point. In the elliptic case, φ fixes a point in the interior. This will be a consequence of the Nielsen realization problem.

We turn now to quasiconformal maps. Recall that for $f : \mathbb{C} \to \mathbb{C}$ conformal/holomorphic/analytic, f sends infinitesimal circles to circles (drawing), $f(z + \Delta z) = f(z) + f'(z)\Delta z + O(\Delta z^2)$. In (x, y)-coordinates, write $f(x, y) = (u, v)$ so

$$df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

and f is conformal if and only if df is a similarity (i.e. a translation composed with a rotation), which in terms of u, v means that the Cauchy–Riemann equations hold, $u_x = -v_y$, $u_y = v_x$. In z-coordinates, define

$$f_z = \frac{1}{2}(f_x - if_y), \quad f_{\overline{z}} = \frac{1}{2}(f_x - if_y),$$

so that

$$\begin{aligned} f_{\overline{z}} &= \frac{1}{2}(u_x - u_y) + \frac{i}{2}(v_x + u_y), \\ f_z &= \frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y), \end{aligned}$$

and so f is conformal if and only if $f_{\overline{z}} = 0$.

Definition 5.9.5. Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be domains, and let $f : \Omega_1 \to \Omega_2$ be an orientation-preserving C^1-homeomorphism such that

$$\mu(z) := \sup_{z \in \Omega_1} |\frac{f_{\overline{z}}}{f_z}| \leq k.$$

Then we say that f is K-*quasiconformal* where $K = \frac{1+k}{1-k}$.

Remark 5.9.6. That f is orientation-preserving means that the Jacobian $J = u_x v_y - u_y v_x > 0$, or $|f_z|^2 - |f_{\overline{z}}|^2 > 0$, and so in the above definition, $k \leq 1$ and $K \geq 1$.

A map is 1-quasiconformal if and only if it is conformal.

Infinitesimally, f is a stretch whose major axis stretch is $1 + |\mu|$, and the minor axis stretch is $1 - |\mu|$ (drawing).

Example 5.9.7. We can always write $df(z, \overline{z}) = Az + B|z|$, $A, B \in \mathbb{C}$. If f itself is of this form (i.e. f is affine), locally the map is $A(z + \frac{B}{A}\overline{z})$, where $(z + \frac{B}{A}\overline{z})$ is a stretch map. Here, $\mu(z) = B/A$.

In the actual definition of quasi-conformality, f is only required to be a homeomorphism with locally integrable distributional derivatives. I.e. there exist functions $f_1, f_2 \in L^2_{\mathrm{loc}}(\Omega_1)$ such that

$$\begin{aligned} \iint_{\Omega_1} f_1 h \, dx \, dy &= -\iint_{\Omega_1} f h_x \, dx \, dy, \\ \iint_{\Omega_1} f_2 h \, dx \, dy &= -\iint_{\Omega_1} f h_y \, dx \, dy, \end{aligned}$$

for all $h \in C_c^\infty(\Omega_1)$. In this case, it is harder to see that 1-quasiconformality implies conformality, and the proof uses Weyl's Lemma which says that if $f_{\overline{z}} = 0$ weakly, then f is conformal.

Lemma 5.9.8. *Let R, R' be rectangles with sides (a, b) and (a', b') respectively. Assume WLOG that $\frac{a}{b} \leq \frac{a'}{b'}$. Let $f : R \to R'$ be a K-quasiconformal map (preserving the sides). Then*

$$K \geq \frac{a'/b'}{a/b}.$$

Proof. Firstly,

$$a' \leq \int_0^a |f_x(x+iy)|\, dx \leq \int_0^a |f_z| + |f_{\overline{z}}|\, dx.$$

Integrating again, we have

$$a'b \leq \int_0^b \int_0^a |f_z| + |f_{\overline{z}}|\, dx\, dy.$$

Squaring this and applying Cauchy–Schwarz,

$$(a')^2 b^2 \leq \int_0^a \int_0^b \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|}\, dx\, dy \cdot \int_0^b \int_0^a |f_z|^2 + |f_{\overline{z}}|^2\, dx\, dy \leq Kaba'b'.$$

This implies the Lemma. □

The minimum of the Lemma is attained for the affine map.

Example 5.9.9. If $a = b = a' = 1$, $b' = L$, $K \geq L$, and we have equality for the map $f(x, y) = Lx + iy$.

The *modulus* of a rectangle of length a and height b is a/b. More generally, a *topological rectangle* is a Jordan region $Q \subseteq \mathbb{C}$ together with a pair of disjoint arcs on the boundary, with a unique conformal map to a rectangle with side lengths a, b, mapping the arcs to the sides (by the Riemann mapping theorem), and we define the *modulus* of Q to be $m(Q) = a/b$.

Fact 5.9.10. *A map f is K-quasiconformal on Ω if and only if for every topological rectangle Q in Ω,*

$$\frac{1}{K} m(Q) \leq m(f(Q)) \leq K m(Q)$$

On reference for this is Ahlfors' Lectures on quasiconformal maps.

Similarly as for rectangles, one can define the modulus for an annulus: For a circular annulus A with radii $r < R$, we let $m(A) = \frac{1}{2\pi} \ln \frac{R}{r}$. This is compatible with the previous definition via the exponential map (drawing). Again, there is a definition for topological annuli.

Fact 5.9.11. *A map f is K-quasiconformal if and only if for every annulus A in Ω,*

$$\frac{1}{K} m(A) \leq m(f(A)) \leq K m(A).$$

Basic properties of quasiconformal maps:

1. f is K-quasiconformal if and only if f^{-1} is K-quasiconformal.
2. If f is K_1-quasiconformal and g is K_2-quasiconformal, then $g \circ f$ is $K_1 K_2$-quasiconformal.
3. Removability/extendability: If f is K-quasiconformal on $\Omega \setminus \gamma$, for γ an image of $[0, 1]$ by an analytic map (draaing), then f is K-quasiconformal on Ω.

Example 5.9.12. If $f : \mathbb{D}^* \to \mathbb{C}$ is quasiconformal, then f extends to $\mathbb{D}$.

Proof. Let C_r be the circle of radius r around 0. It is enough to prove that $\mathrm{diam} f(C_r) \to 0$ as $r \to 0$. Suppose that this is not the case. We know that $m(A(r, \frac{1}{2})) \to \infty$ as $r \to \infty$. But

$$m(f(A(r, \tfrac{1}{2}))) \geq \frac{1}{K} m(A(r, \tfrac{1}{2})) \to \infty.$$

If $\mathrm{diam}(f(C_r)) \geq \eta > 0$ (which can be proved to be the case using extremal lengths), then $m(f(A(r, \frac{1}{2}))) \leq M$, which is a contradiction.. $\square$

Another important property is

4. Compactness: K-quasiconformal maps from the unit disk to itself normalized by $f(0) = 0$ form a sequentially compact family with respect to uniform convergence.

Proof. The proof uses Arzelà–Ascoli and some equicontinuity, namely Mori's Theorem: $|f(z_1) - f(z_2)| \leq 16|z_1 - z_2|^{1/K}$ where $z_1, z_2 \in \mathbb{D}$. $\square$

Definition 5.9.13. Let X, X' be Riemann surfaces, a map $f : X \to X'$ is called K-*quasiconformal* if it is a homeomorphism which is locally K-quasiconformal, i.e. that

$$\mu = |\frac{\overline{\partial} f}{\partial f}| \leq k < 1,$$

and again we let $K = (1 + k)/(1 - k)$. The map $\mu : X \to \mathbb{R}$ is the *Beltrami differential* on X. In local coordinates,

$$\mu(z) = \frac{f_{\overline{z}}}{f_z} \frac{d\overline{z}}{dz}.$$

Definition 5.9.14 (Teichmüller metric)**.** For $X, Y \in \mathcal{T}_g$, we define the Teichmüller distance

$$d_\tau(X, Y) = \inf_f \ln K(f),$$

where the infimum is over all quasiconformal maps $f : X \to Y$ preserving the marking, and $K(f)$ is the quasiconformal dilatation. That f preserves the marking means that if we have markings $g_1 : S_g \to X$, $g_2 : S_g \to Y$, then f is homotopic to $g_2 \circ g_1^{-1}$.

To see that this is a metric, first note that if $d_\tau(X, Y) = 0$ if and only if X and Y are conformally equivalent. That it is symmetric and satisfies the triangle inequality follows from properties 1. and 2. above. Note that by definition, the metric is mapping class group invariant, i.e.

$$d_\tau(\varphi \cdot X, \varphi \cdot Y) d_\tau(X, Y).$$

Fact 5.9.15 (Teichmüller (Teichmüller's Theorem))**.** *The distance $d_\tau(X, Y)$ is uniquely realized by a* Teichmüller map $F : X \to Y$.

There exists a decomposition of X and Y into rectangles and a conformal metric that makes them euclidean rectangles with cone singularities at the vertices (drawing), and F restricts to a stretch map on each rectangle.

Lemma 5.9.16 (Wolpert)**.** *Let X, Y be hyperbolic surfaces and $f : X \to Y$ a K-quasiconformal map. Then for any simple closed curve γ, we have that*

$$\frac{1}{K} l_X(\gamma) \leq l_Y(\gamma) \leq K l_X(\gamma).$$

Proof. Consider the annular cover $A_X \to X$, the cover of X corresponding to the subgroup $\langle[\gamma]\rangle \in \pi_1 X$ (drawing). We claim that $l_X(\gamma) = \frac{\pi}{m(A_X)}$, where $\lambda = l_X(\gamma)$. To see this, note that A_X is isometric to $\mathbb{H}^2/\langle z \mapsto e^\lambda z\rangle$ (drawing). A K-quasiconformal map $f : X \to Y$ lifts to a K-quasiconformal map $\tilde{f} : A_X \to A_Y$, and the Lemma follows from the fact about the modulus of annuli. □

Corollary 5.9.17. *The topology induced by d_τ is the same as the topology on $\mathcal{T}_g$.*

Lemma 5.9.18. *The metric d_τ is complete.*

Proof. If $\{X_i\}$ is a Cauchy sequence in $\mathcal{T}_g$. Then by the compactness property of quasiconformal maps, if we have a sequence $f_i : X_0 \to X_i$ of K-quasiconformal maps, then this sequence as a limit, so there exists a surface X_∞ and a K-quasiconformal map $f : X_0 \to X_\infty$ so that $f_i \to f$. □

14th Lecture, November 9th 2012

Last time, we saw the definition of the Teichmüller metric,

$$d_\tau(X, Y) = \inf_f \ln K,$$

for K-quasiconformal maps $f : X \to Y$. Recall that a K-quasiformal map is one that infinitesimally send circles to ellipses with major axis/minor axis $\leq K$ (drawing) or equivalently the same statement for rectangles. We saw that the metric was complete and defines the same topology as the one defined earlier. Moreover, the Teichmüller metric is invariant under the mapping class group action.

Remark 5.9.19.

1. Teichmüller space is a complex manifold of $\dim_{\mathbb{C}} = 3g-3$. Teichmüller space is the so-called Kobayashi metric (Royden).
2. Teichmüller space of the torus with the Teichmüller metric is isometric to $\mathbb{H}^2$. Recall that $\mathcal{T}_{1,1} = \mathcal{T}_{1,0}$ is the set of marked lattices in $\mathbb{R}^2$ (drawing), which we saw was $\mathbb{H}^2$ as a set. One can check that the action of $\mathrm{PSL}_2\mathbb{R}$ on marked lattices is the same as the action on $\mathbb{H}^2$. The point $i \in \mathbb{H}^2$ corresponds to the square torus, and along the imaginary axis, the tori are stretched vertically, and the Teichmüller distance between ia and ib is, by the growth lemma, $\ln b/a$, which is also the hyperbolic distance between ia and ib. So the metrics agree on the imaginary axis and thus since they are preserved by the action, they agree everywhere.

We will now some idea of Bers' proof of the Nielsen–Thurston classification (following Farb–Margalit). Let $\varphi \in \mathrm{MCG}$. Since φ is an isometry of $\mathcal{T}_g$, one can define the *translation distance*,

$$\tau(\varphi) = \inf_{X \in \mathcal{T}_g} d(X, \varphi \cdot X).$$

There are three cases.

1. Elliptic: $\tau(\varphi)$ is achieved and equals zero.
2. Parabolic: $\tau(\varphi)$ is not achieved.
3. Hyperbolic: $\tau(\varphi)$ is achieved and positive.

Claim 5.9.20. *The first possibility implies that φ is periodic, the second one that φ is reducible, and the last that φ is pseudo-Anosov.*

This gives another proof of the Nielsen–Thurston classification. We will prove the claim in the first two cases and discuss the third one.

Case 1: That $\tau(\varphi) = 0$ is achieved means that there is $(f, \Sigma) \in \mathcal{T}_g$ such that $\dot{\varphi(f, X)}(f, X)$. That is, there is an isometry $h : \Sigma \to \Sigma$ such that $h \sim f \circ \varphi^{-1} \circ f^{-1}$. But $h \in \mathrm{Isom}(\Sigma)$ is periodic (recall that $|\mathrm{Isom}(X)| \leq 84(g-1)$), so $f \circ \varphi^{-1} \circ f^{-1}$ is periodic in MCG, and so is φ.

Case 2: Let $\{X_i\} \subseteq \mathcal{T}_g$ such that $d_\tau(X_i, \varphi X_i) \to \tau(\varphi)$.

We claim that the projection of $\{X_i\}$ to $\mathcal{M}_g$ leaves every compact set: Suppose this is not the case; i.e. that there exists a compact set $K \subseteq \mathcal{T}_g$ and mapping classes $h_i \in \mathrm{MCG}(S_g)$ s.t. $h_i \cdot X_i = Y_i \in K$. After passing to a subsequence, $Y_i \to Y \in K$. Now,

$$\lim_{i\to\infty} d(Y_i, h_i\varphi h_i^{-1}Y_i) = \lim_{i\to\infty} d(h_i^{-1}Y_i, \varphi(h_i^{-1}Y_i)) = \lim_{i\to\infty} d(X_i, \varphi X_i) = \tau(\varphi).$$

We (sub)claim that

$$\lim_{i\to\infty} d(Y_i, h_i\varphi h_i^{-1}Y_i) = d(Y, h_k\varphi h_k^{-1}Y) = \tau(\varphi)$$

for some k: By the triangle inequality,

$$d(Y, h_i\varphi h_i^{-1}Y) \leq d(Y, Y_i) + d(Y_i, h_i\varphi h_i^{-1}Y_i) + d(h_i\varphi h_i^{-1}Y_i, h_i\varphi h_i^{-1}Y) \to 0 + \tau(\varphi) + 0,$$

and the subclaim follows from the fact that the MCG action is proper discontinuous. But then

$$d(h_k^{-1}Y, \varphi h_k^{-1}Y) = \tau(\varphi),$$

which means that $h_k^{-1}Y$ achieves the translation distance, which is a contradiction.

Let $l(X)$ denote the length of the shortest non-trivial simple closed curve on X. By Mumford's compactness criterion, that the sequence $\{X_i\}$ leaves every compact set in moduli space means that $l(X_i) \to 0$. By the Collar Lemma, there exists δ such that any two simple closed curves of length $< \delta$ are distinct, and the K comes from Wolpert's Lemma and satisfies $l_X(\gamma) < Kl_Y(\gamma)$ if $d_\tau(X, Y) < \tau(\varphi) + 1$.. Choose M large enough that $d(X_M, \varphi \cdot X_M) < \tau(\varphi) + 1$ and $l(X_M) < (1/K)^{3g-3}\delta$, where here we recall that. Let c_0 be the shortest curve on X_M and let $c_i = \varphi^{-i}c_0$ for $1 \leq i \leq 3g-3$. Then

$$l_{X_M}(c_i) = l_{X_M}(\varphi^{-1}c_0) = l_{\varphi^i X_M}(c_0) \leq K^i l_{X_M}(c_0) < \delta,$$

where the first inequality follows from Wolpert's Lemma. Then by the Collar Lemma, the curves $\{c_0, \dots, c_{3g-3}\}$ are disjoint. This means that $c_i \sim c_j$ for some $i \neq j$. This means that $\varphi^k c_0 = c_0$ for some k, and hence φ is reducible. This concludes the second case.

Case 3. Let $X \in \mathcal{T}_g$ such that $d_\tau(X, \varphi X) = \tau(\varphi)$. We claim that there is a geodesic γ in $\mathcal{T}_g$ connecting X and φX and that the geodesic axis is preserved by φ, and that φ acts by translation on this axis, similarly to what happens for isometries of $\mathbb{H}^2$. In fact, γ is unique. It turns out that one can prove from this picture that φ is pseudo-Anosov from the picture of φ stretching families of rectangles in the surface. To see that the axis is preserved by φ, let Y be the midpoint of the geodesic segment $\overline{X\varphi(x)}$. Then

$$d(Y, \varphi Y) \leq d(Y, \varphi X) + d(\varphi X, \varphi Y) = \frac{1}{2}d(X, \varphi X) + \frac{1}{2}d(X, \varphi X) = d(X, \varphi X),$$

so $d(Y, \varphi Y) = \tau(\varphi)$. This implies that φY lies on the axis: If it does not, one obtains a contradiction using the triangle inequality (drawing).

6 The Nielsen Realization Problem

6.1 The theorem

Theorem 6.1.1 (Nielsen Realization (Kerckhoff, 1983))**.** *Any finite subgroup G of* $\mathrm{MCG}(S_g)$ *fixes a point in* $\mathcal{T}_g$.

Theorem 6.1.2. *Any finite subgroup of* $\mathrm{MCG}(S_g)$ *arises as a group of isometries of some hyperbolic surface.*

These two theorems are equivalent, as we will now see.

Lemma 6.1.3. *Let X be a hyperbolic surface. If $f : X \to X$ is an isometry isotopic to the identity, then $f = \mathrm{Id}$.*

Proof. Lift f to an isometry $\tilde{f} : \overline{\mathbb{H}^2} \to \overline{\mathbb{H}^2}$. That f is isotopic to the identity means that $\tilde{f}|_{\partial\mathbb{H}^2} = \mathrm{id}$. Now, the only such isometry is the identity map. $\square$

Suppose G fixes $(f, X) \in \mathcal{T}_g$. To see that Theorem 6.1.1 implies Theorem 6.1.2, we shall define an injective homomorphism $i : G \to \mathrm{Isom}(X)$. As before, that G fixes (f, X), we have that for $\varphi \in G$, $(f \circ \varphi^{-1}, X) \sim (f, X)$; we have $f^{-1} \circ \varphi^{-1} \circ f \sim h$, and we put $i(\varphi) = h$. The Lemma above implies that i is well-defined, and following the definitions, the map is a homomorphism as well as an injective one (exercise).

So in particular, a finite subgroup of $\mathrm{MCG}(S_g)$ can be "realized" as a subgroup of $\mathrm{Diff}^+(S_g)$.

Remark 6.1.4. We have an exact sequence

$$0 \to \mathrm{Diff}_0(S_g) \to \mathrm{Diff}^+(S_g) \to \mathrm{MCG}(S_g) \to 0,$$

and this does not split if $g \geq 2$ (Morita was the first to give examples of this; other people involved in this problem are Marković and Šarić).

15th Lecture, November 13th 2012

Recall from last time that (drawing)

$$\mathcal{T}_{1,0} = \{\text{marked lattices in } \mathbb{R}^2\}/\sim = \{\sigma\mathbb{Z} \oplus \sigma'\mathbb{Z} \subseteq \mathbb{C}\} = \mathbb{H}^2,$$

and along the imaginary axis the corresponding rectangles get longer.

The action of a Dehn twist tw_γ on $\mathcal{T}_g$ is by parabolic isometry, fixing $[\gamma] \in PM\mathcal{L}$ (drawing), and the translation distance is $\tau(\mathrm{tw}) = 0$; one can see this by considering $X \in \mathcal{T}_g$ with γ very short: If $l(\gamma, X)$ is small, so is $d_\tau(X, \mathrm{tw}_\gamma X)$. The translation distance 0 is not achieved but can be arbitrarily small.

There are reducible elements φ with positive translation distance that is not achieved (drawing): For example, the surface could be split into two parts, A and B, so that $\varphi|_A$ is the identity, and $\varphi|_B$ is pseudo-Anosov.

We return now to

Theorem 6.1.5 (Kerckhoff)**.** *Any finite subgroup of* $\mathrm{MCG}(S_g)$ *fixes a point in* $\mathcal{T}_g$*. That is,* G *realized as a subgroup of* $\mathrm{Isom}(X)$*, for some hyperbolic structure on* S_g*.*

Question (for the listener/reader): Is any finite group a subgroup of $\mathrm{MCG}(S_g)$ for some g?

Lemma 6.1.6 (E. Cartan)**.** *Let X be a geodesic metric space which is negatively curved in the sense of Busemann, G is a finite group acting on X by isometries. Then G fixes a point.*

Recall here that X is Busemann-negatively curved if and only if for geodesics from a point x_0 to points x, x', the midpoints m, m' of the geodesics satisfy

$$d(m, m') \leq \frac{1}{2} d(x, x').$$

This property implies that for any geodesic γ parametrized by arc length and $p \in X$, the map $d : \mathbb{R} \to \mathbb{R}^+$ given by $d(t) = d(\gamma(t), p)$ is convex function.

Proof of Lemma 6.1.6. Take an arbitrary point $p \in X$, and let $D = \{p = p_1, \ldots, p_n\}$ be its G-orbit. Consider the function $d : X \to \mathbb{R}^+$ mapping $x \mapsto \sum_{i=1}^n d(x, p_i)^2$. Since X is Busemann-negatively curved, each $d(x, p_i)^2$ is strictly convex, and thus so is d (here a map $X \to \mathbb{R}^+$ is strictly convex if it is strictly convex along any geodesic path). We claim that the function d has a unique minimum $q \in X$. It is not difficult to see that d is proper, so there is a minimum. Suppose that there are two minima x_0, x_1 with minimum value m, and consider the geodesic path $\gamma : [0,1] \to X$, with $\gamma(0) = x_0, \gamma(1) = x_1$. Then

$$d(\gamma(\tfrac{1}{2})) < \frac{d(\gamma(0)) + d(\gamma(1))}{2} = m$$

by strict convexity of D. This contradiction proves the claim. Now, since G fixes the set D, G fixes the minimum q (q is sometimes called the *barycenter* of D). □

However, $(\mathcal{T}_g, d_\mathcal{T})$ is *not* negatively curved when $g > 1$ (as shown by Maser around 1972). Examples of geodesics that do not diverge are given by *Strebel rays* (drawing).

The solution to this problem by Kerckhoff is the following: Instead of geodesics one considers (1) "earthquake paths" and convexity of (2) "length functions".

Fact 6.1.7 (Theorem by Masur/Minsky)**.** *Given $\alpha \in S$ a simple closed curve, the ε-thin part*

$$\mathrm{Thin}_\varepsilon(\alpha) = \{X \in \mathcal{T}_g \mid l(\alpha, X) < \varepsilon\}$$

can be "electrified" so that any two points in the thin part are at distance 1 *from one another: The electrified $\mathcal{T}_g$ is δ-hyperbolic (in the sense of Gromov).*

Lemma 6.1.8. *Let X, X' be a hyperbolic surface. The lift of any bi-Lipschitz map $f : X \to X'$ extends to the boundary, $\overline{\tilde{f}} : \overline{\mathbb{H}^2} \to \overline{\mathbb{H}^2}$, and the extension is independent of the isotopy class of f.*

Proof. Assume f is a diffeomorphism ($\|df\|_\infty \leq B$). We already have a lift $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^2$. Consider the radial rays $r : [0,\infty) \to \mathbb{H}^2$ s.t. $r(0)$, parametrized by arc length. Under $\tilde{f}$ this maps to We claim that $\tilde{f}(r(t))$ has a well-defined endpoint on $\partial\mathbb{H}^2$. To see this, first note that $\tilde{f}$ is also K-bi-Lipschitz, $d(\tilde{f}(r(t)), 0) > t/K$. In the disk model,

$$|\frac{ds_{\mathrm{euc}}}{ds_{\mathrm{hyp}}}(x)| < Ce^{-d_{\mathrm{hyp}}(x,0)}.$$

Hence, the euclidean length of $\tilde{f}(r(t)))$, $(0 < t < \infty)$ is finite:

$$\begin{aligned}\int_0^\infty \|\tilde{f}'(r(t))r'(t)\|_{\mathrm{euc}}\, dt &\leq \int_0^\infty B\|r'(t)\|_{\mathrm{euc}}\, dt \\ &\leq \int_0^\infty BCe^{-d_{\mathrm{hyp}}(x,0)} \leq \int_0^\infty BCe^{-t/K}\, dt < \infty.\end{aligned}$$

(That it is independent of the isotopy class can be seen almost by drawing). □

Remark 6.1.9. More generally, a quasi-isometry $\mathbb{H}^2 \to \mathbb{H}^2$ (or more generally $\mathbb{H}^n \to \mathbb{H}^n$ (or between negatively curved spaces where one has a notion of Gromov boundary)) extends to a homeomorphism of $\partial\mathbb{H}^n$; here a quasi-isometry is a map h satisfying

$$\frac{1}{K}d(x,y) - C \leq d(h(x), h(y)) \leq Kd(x,y) + C.$$

6.2 Earthquakes

Intuitively, a left *earthquake map* $E : \mathbb{H}^2 \to \mathbb{H}^2$ is a (possibly discontinuous) piecewise Möbius map. The pieces, called *strata*, are domains separated by non-intersecting geodesics (drawing). On each strata, E is a hyperbolic isometry that shifts "left" as seen from any adjacent strata.

Example 6.2.1. Consider a geodesic γ in $\mathbb{H}^2$ separating $\mathbb{H}^2$ into two pieces A and B. An example of an earthquake map is a map fixing A and is a hyperbolic isometry translating by some amount in B (drawing). (Drawn example of a case with several geodesics: Split $\mathbb{H}^2$ into A_i by geodesics γ_i. To each such we associate a positive real number λ_i defining a map E. Then $E|_{A_i}^{-1} \circ E|_{A_{i+1}}$, thought off as a map $\mathbb{H}^2 \to \mathbb{H}^2$ is the hyperbolic translation by λ_i acting γ_i.)

Let $\mathcal{L}$ be a (finite) collection of non-intersecting geodesics γ_i with weights λ_i. Let L be the union of the geodesics of $\mathcal{L}$. We can associate a dual tree having vertices given by components of $\mathbb{D} \setminus L$. To this we can associate a left earthquake (drawing).

Finally, if $\mathcal{L}$ is a set of countably many disjoint geodesics, for instance, the lifts of a curve on a surface, one can again define an earthquake map. Then the twist of the curve lifts to an earthquake map.

16th Lecture, November 16th, 2012

Thanks to Brendan McLellan for supplying the following hand-written notes.

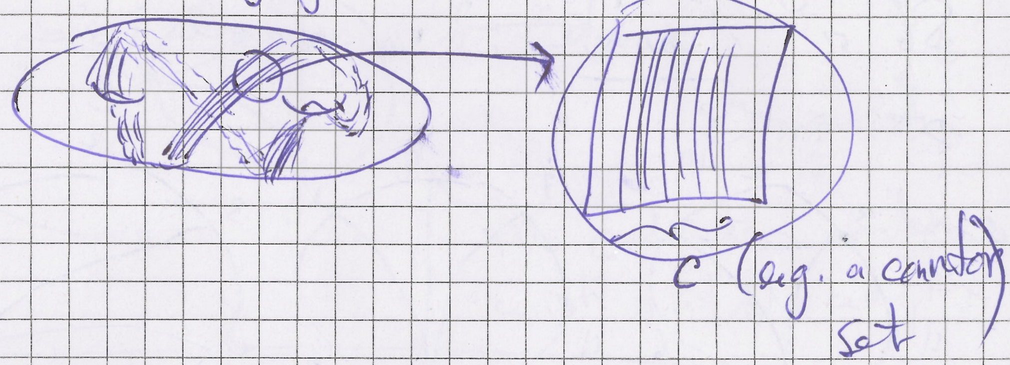
LECTURE #16

Today

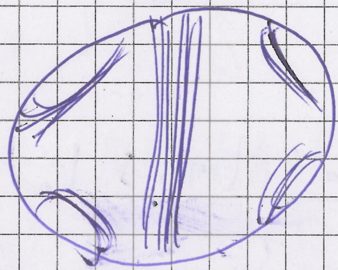
- General geodesics (in $\mathbb{H}^n$)

[redacted]

- More generally, a geodesic lamination on a surface is a closed set foliated by geodesics.



- Lamination λ on $\mathbb{H}^2$.



  - Components of $\mathbb{H}^2 \setminus \lambda$ are called "gaps"

- gaps + boundary leaves: "strata"

DEF// A λ-left earthquake $E: \mathbb{H}^2 \to \mathbb{H}^2$ is a map that is:

1) an isometry on each stratum.
2) for two strata $A \neq B$ of λ, the comparison isometry $\mathrm{comp}(A,B) = E|_A^{-1} \circ E|_B$ – is hyperbolic with axis separating A & B, translating to the left as viewed from A.

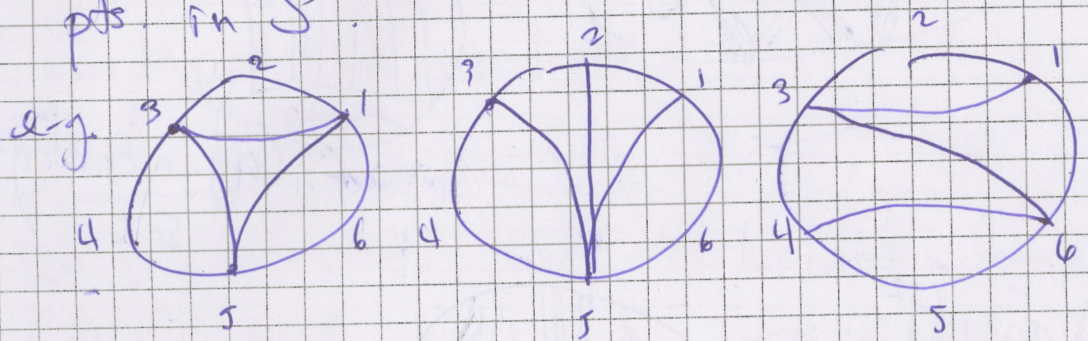
- Amount of shear is determined by transverse measure μ.

FACT// Every earthquake map extends to an orientation-preserving homeomorphism of $\partial \mathbb{H}^2$.

THM// (Earthquake theorem, Thurston) Every orientation preserving homeomorphism of $\partial \mathbb{H}^2$ is an extension of an earthquake map.

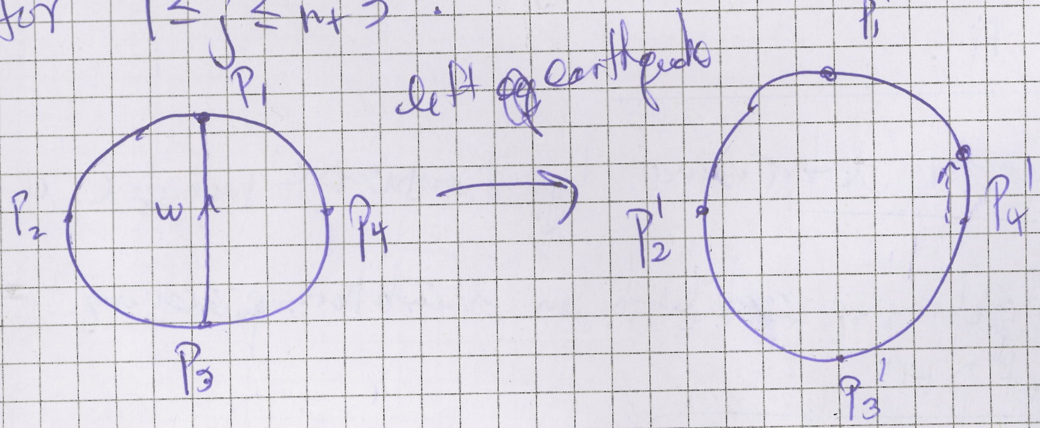
DEF// Given a set S of finitely many pts. on S^1, we say a finite lamination $\mathcal{L}$ with endpts. in S is "allowable" if no geodesic joins adjacent pts. in S. ③

e.g.



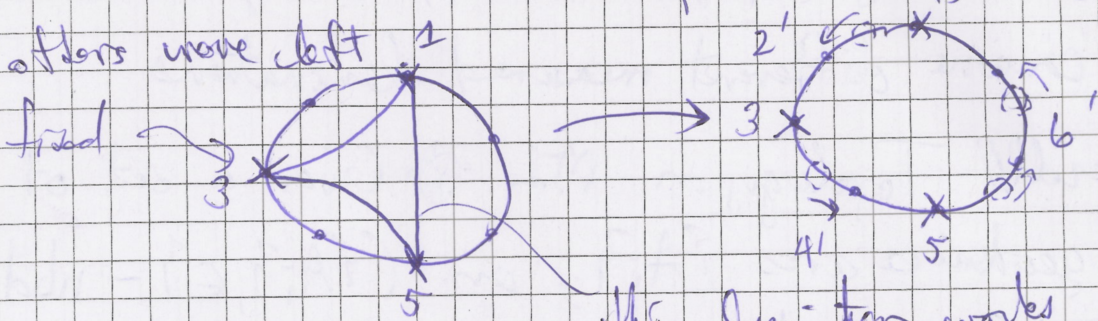
Finite Earthquake thm: Let $S = \{p_1, \ldots, p_{n+3}\}$, $S' = \{p_1', \ldots, p_{n+3}'\}$ be two sets of points on S^1 in counter-clockwise order.

Then $\exists!$ allowable lamination $\mathcal{L}$ for S and a unique set of weights σ on geodesics of $\mathcal{L}$ such that, up to post composition with a Möbius map, the left earthquake map $E_{(\mathcal{L},\sigma)}$ maps p_j to p_j' for $1 \leq j \leq n+3$.



– Algorithm to compute weights – Gardiner, Lakić, Hu.

- Suppose there are at least 3-fixed points in S^1, and all others move left



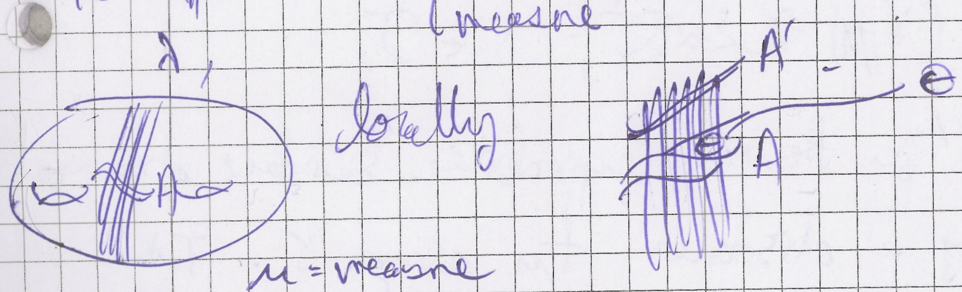
this lamination works with suitable weights.

Key Lemma// Let h be the map from S to S' $\ni$: $h(p_i)=p_i'$, Then $\exists$ an orientation preserving isometry (Möbius map) $A: H^2 \to H^2$, such that $A \circ h$ fixes at least 3-pts, and all others move left.

Proof// for next time. $\boxed{R}$

Earthquakes on surfaces

Recall// $ML = \{$set of geodesic laminations with a transverse measure$\}$



- Given a transverse arc A, can define two real numbers:

 * Intersection number $i(A, \lambda) = \inf_{A' \sim A} \int_{A'} d\mu$.

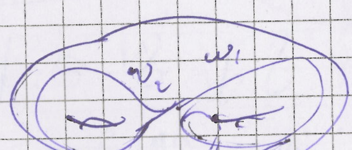
 arc lamination

 * Total angle $\int_A \theta \, d\mu$

- $\tau_A(\lambda) = (i(A,\lambda), \Theta(A,\mu)) \in \mathbb{R}^2$; can be used to compare different measured laminations

Recall// Topology on ML: Given a set of open geodesic arcs $\{A_i\}$, an $(\{A_i\}, \epsilon)$ - nbd. of λ
$= \{ \nu \in ML \mid | v_{A_i}(\nu) - v_{A_i}(\lambda) | < \epsilon, \forall i \}$.

- Topology



If you fix a train track τ, can build a lamination carried on τ.
- and the weights on branches determine the
- transverse measure
  ~ Varying weights gives an open nbd.

Recall// Thm (Thurston): - $ML \cong \mathbb{R}^{6g-6}$. $\mathcal{L} \times \mathbb{R}^+$
- Weighted simple closed curves are dense in ML.

- DEF// For any $(\gamma, \mu) \in \mathcal{L} \times \mathbb{R}^+$, $X \in T_g$
(Time-t twist) Set X' be the new hyperbolic surface obtained by twisting a distance $t\mu$ along γ. Then X' is the time-t twist deformation of X.

(time t - earthquake)
DEF// For $\nu \in ML$, the earthquake of X at time t determined by ν is the limit in T_g of the time-t twist deformation of any sequence $\{(\gamma_i, \mu_i)\} \to \nu$ in ML.
- limit exists
- well defined
} Next time.

17th Lecture, November 20th, 2012

Today we will end our discussion of the proof of the Nielsen realization theorem and talk about hyperbolic 3-manifolds.

Last time, we talked about earthquakes on surfaces. We talked about twist deformations; given X, consider (γ, μ), γ a simple closed curve, $t \in \mathbb{R}^+$, μ a weight. A time t twist twists around γ by an amount $t\mu$ (drawing). In Fenchel–Nielsen coordinates, the twist parameter θ for γ gets mapped to $\theta + t\mu$.

More generally, earthquakes are defined for laminations as limits of twist deformations: Given X and a measured lamination (λ, μ), one can define a time t left earthquake giving a surface $X' = \mathcal{E}_\lambda(t)$. Namely, take a sequence $(\gamma_i, \mu_i) \to \lambda$. We ask: a) Why does this limit exist? and b) Why is this well-defined? The geodesic lamination λ lifts to $\tilde{\lambda}$ on $\mathbb{H}^2$, which can be thought of as a closed subset C of $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2 \setminus \Delta$ (by taking the endpoints). The lifted transverse measure $\tilde{\mu}$ is a Borel measure on C.

Fact 6.2.2. *Any such measure is a limit of finite sums of Dirac measures $\sum_{i=1}^N \delta_{x_i} \to \tilde{\mu}$ in the weak topology. This gives an approximation to (λ, μ): A finite lamination.*

We can now define an earthquake based on this finite lamination.

Definition 6.2.3. An earthquake on the measured lamination $(\tilde{\lambda}, \tilde{\mu})$ is the limit of the earthquake maps for an approximating sequence of $\tilde{mu}$.

We claim that this is convergent (drawing).

Key Lemma 6.2.4 (Kerckhoff)**.** *Suppose we have geodesics (λ_i, μ_i). Replace these by a single geodesic l such that for a transversal arc A making an angle ϑ with l, and the μ_i by $\mu = \sum \mu_i$, and*

$$\vartheta = \frac{1}{i(A, \lambda)} \int_A \theta \, d\mu.$$

Fix ε, T. If $l_{\mathbb{H}^2}(A) < \varepsilon$,

$$d(\mathcal{E}_{\overline{\gamma}}(t)y, \mathcal{E}_l(t)y) < Kt\varepsilon$$

for all $t \leq T$, and K depends on $\sum \mu_i$.

This leads to

Proposition 6.2.5. *Given $X \in \mathcal{T}_g$, $v \in M\mathcal{L}$, $T, \delta > 0$. Then there is a neighbourhood U of v in $M\mathcal{L}$ such that for every $\gamma, \gamma' \in (\mathcal{S} \times \mathbb{R}_+) \cap U$ and for every $t \leq T$, $\mathcal{E}_\gamma(t)$ and $\mathcal{E}_{\gamma'}(t)$ are in the same δ-neighbourhood in $\mathcal{T}_g$.*

Corollary 6.2.6. *Earthquakes are well-defined on laminations. More precisely, left earthquake along any lamination $v \in M\mathcal{L}$ are well-defined for all $t \in \mathbb{R}$.*

Thus, for every $v \in M\mathcal{L}$ we get a path in $\mathcal{T}_g$.

Theorem 6.2.7 (Earthquake Theorem (Thurston))**.** *For every $x, y \in \mathcal{T}_g$, there is a unique earthquake path from X to Y ("geology is transitive").*

Recall from last time we saw

Theorem 6.2.8 (Earthquake Theorem)**.** *Any orientation-preserving homeorphism of $S^1 = \partial\mathbb{H}^2$ is the boundary extension of some left-earthquake map $E_\mathcal{L} : \mathbb{H}^2 \to \mathbb{H}^2$ (up to some post-composition by a Möbius map), and $\mathcal{L}$ is uniquely determined.*

Let us see that this Theorem implies Theorem 6.2.7. Let $X, Y \in \mathcal{T}_g$. Take any bi-Lipschitz map f between them and lift this to $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^2$. We saw that this extends to the boundary to a map equivariant with respect to the actions of the Fuchsian groups defining X, Y, $f(\gamma x) = \gamma' f(x)$. By the Earthquake Theorem, the map on the boundary is realized by an earthquake given by some equivariant lamination $\mathcal{L}$. By equivariance, $\mathcal{L}$ descends to a measured lamination λ on X and defines $\mathcal{E}_\lambda(t)$. We claim that $Y' = \mathcal{E}_\lambda(1)$ is conformal to Y. The lift of the map $\mathcal{E}_\lambda(1) \circ f^{-1} : Y \to Y'$ lifts to a map of $\overline{\mathbb{H}^2} \to \overline{\mathbb{H}^2}$ that is the identity on the boundary.

Recall the geodesic length functions: Given a simple closed curve γ, $l_\gamma : \mathcal{T}_g \to \mathbb{R}^+, X \mapsto l(\gamma, X)$. Similarly, given a collection of simple closed curves $\overline{\gamma} = \{\gamma_i\}_{i=1}^n$, define $l_{\overline{\gamma}}(X) = \sum_{i=1}^n l(\gamma_i, X)$.

Recall from two times ago that if a finite group G acting on a negatively curved metric space X by isometries, then G has a fixed point.

Lemma 6.2.9. *If $\overline{\gamma}$ fills the surface, then the map $l_{\overline{\gamma}} : \mathcal{T}_g \to \mathbb{R}^+$ is proper. In particular, it realizes its minimum in $\mathcal{T}_g$.*

Recall here that $\overline{\gamma}$ *fills* a surface S if $S \setminus \bigcup \gamma_i$ is a union of disks.

Proof. It is enough to show that the set

$$B_{\overline{\gamma}}(K) = \{X \in \mathcal{T}_g \mid l_{\overline{\gamma}}(X) \leq K\}$$

is bounded. Since $\overline{\gamma}$ fills S, any simple closed curve c can be homotoped to a curve in $\bigcup \gamma_i$ (homotoping to the boundary of each of the disks) more than N times, where N depends only on c (and not on X). This implies that $l(c, X) \leq N l_{\overline{\gamma}}(X) \leq NK$. This imples that $B_{\overline{\gamma}}(K)$ is bounded. □

Definition 6.2.10. A function $f : \mathcal{T}_g \to \mathbb{R}^+$ is *convex along earthquake paths* if for any earthquake path $\mathcal{E}_\nu(t)$, $t \in (0, 1)$,

$$f \circ \mathcal{E}_\nu(t) \leq t f \circ +(1-t) f \circ \mathcal{E}_\nu(1).$$

It is called *strictly convex* if the equalities are sharp.

Proposition 6.2.11. *The geodesic length function l_γ of a simple closed curve γ is convex along an earthquake path $\mathcal{E}_\gamma(t)$. It is strictly convex if and only if $i(\gamma, \lambda) \neq 0$.*

Recall that for twist deformations (i.e. λ is a weighted multicurve), we have

$$\frac{dl_\gamma}{dt}|_{t=0} = \sum_{i=1}^n \cos\theta_i,$$

where the θ_i are the angles at intersections of γ and λ. This generalizes to

$$\frac{dl_\gamma}{dt}|_{t=0} \int_\gamma \cos\theta \, d\mu$$

along an earthquake path $\mathcal{E}_\gamma(t)$. Here μ is the transverse measure on λ, and θ is the measure angle from γ to λ. Moreover, $\theta(t)$ is strictly decreasing as a function of t along the earthquake path. These facts together imply Proposition 6.2.11.

Proposition 6.2.12. *If $\overline{\gamma}$ fills S, then $l_{\overline{\gamma}}$ has a* unique *minimum in $\mathcal{T}_g$.*

Proof. Suppose we have two minima X, Y. By the Earthquake Theorem, there exists an earthquake path $\mathcal{E}_\lambda(t)$ from X to Y. Since $\overline{\gamma}$ fills S, $i(\gamma_i, \lambda) \neq 0$ for some $\gamma_i \in \overline{\gamma}$. This implies that $l_{\overline{\gamma}}$ is strictly convex along $\mathcal{E}_\lambda(t)$. This implies that $X = Y$. □

Theorem 6.2.13 (Nielsen Realization)**.** *Every finite subgroup G of* $\mathrm{MCG}(S_g)$ *acting on $\mathcal{T}_g$ has a fixed point.*

Proof. Let $\overline{\gamma}$ be a collection of a simple closed curves that fill S. Then $G\overline{\gamma}$ also fills the surface, and then $l_{G\overline{\gamma}}$ also has a unique minimum by Proposition 6.2.12. But $l_{G\overline{\gamma}}$ is G-invariant, hence this minimum is a fixed point,

$$l_{G\overline{\gamma}}(g_0X) = \sum_{g\in G, \gamma_i\in\overline{\gamma}} l(g\gamma, g_0X) = \sum_{g\in G} l(g_0^{-1}g\gamma_i, X) = l_{G\gamma}(X).$$

□

18th Lecture, November 23rd 2012

7 Hyperbolic 3-manifolds

Recommendable book on the following: The Hyperbolization Theorem and Fibered 3-manifolds.

7.1 Hyperbolic 3-space

The upper half space $\mathbb{H}^3 = \{(z,t) \mid z \in \mathbb{C}, t > 0\}$ has a hyperbolic metric

$$\frac{dz^2 + dt^2}{t^2}$$

and has a boundary at infinity, $\mathbb{C} \cup \infty = \hat{\mathbb{C}}$. We also have the ball model; the unit ball in $B_1 \subseteq \mathbb{R}^3$ has a hyperbolic metric with boundary at infinity $\mathbb{C}P^1$, or $\hat{\mathbb{C}}$. (Drawings of geodesics and hemispheres of geodesics (totally geodesic planes).)

Isometries $\mathrm{Isom}(\mathbb{H}^3)$ of $\mathbb{H}^3$ are Möbius transformations (generated by inversions about hemispheres). For the orientation preserving ones, we can identify $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}_2\mathbb{C}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{C})$ acts by

$$z \xrightarrow{\gamma} \frac{az+b}{cz+d}$$

on the boundary and extends to a unique isometry of $\mathbb{H}^3$ (called a Poincaré extension).

Remark 7.1.1. If you consider quaternions $\mathbb{H}^3 = \{z + tj \mid z = x + iy, t > 0\}$, then γ extends to

$$T(z+tj) = \frac{a\omega + b}{c\omega + d},$$

where $\omega = z + tj$.

We have a classification of isometries into

- The identity
- Elliptic: Finite order, fixed point in the interior,
- Loxodromic/hyperbolic: Fixes an axis and two points at ∞ and is translation along axis by a skew motion (drawing). For instance, the Möbius map $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ of the boundary extends to $T((z,t)) = (\lambda^2 z, |\lambda|^2 t)$.
- Parabolic: Fixes one point at ∞.

7.2 Hyperbolic 3-manifolds

We are interested in discrete subgroups of $\mathrm{Isom}(\mathbb{H}^3)$. A discrete subgroup $\Gamma \subseteq \mathrm{Isom}(\mathbb{H}^3)$ is called a *Kleinian group*. We shall consider subgroups of $\mathrm{Isom}^+(\mathbb{H}^3)$ that are torsion-free. That the groups are discrete means that the identity is isolated, and so the orbit Γx is discrete for any $x \in \mathbb{H}^3$. Moreover, just as for Fuchsian group, Γ will act properly discontinuously, i.e. there is a ball B around a point x such that if $\gamma B \cap B \neq \emptyset$ for $\gamma \in \Gamma$, then $\gamma x = X$. Then the space $M^3_\Gamma = \mathbb{H}^3/\Gamma$ is a manifold with a complete hyperbolic metric.

Suppose M^3 is a hyperbolic manifold. Then the universal cover $\tilde{M}$ is isomorphic to $\mathbb{H}^3$, $\pi_1(M)$ acts on $\tilde{M}$, and Γ acts on $\mathbb{H}^3$. The bottom line is that there is some correspondance between complete hyperbolic metrics on M and the space $\mathcal{DF}(\pi_1(M)) \subseteq \mathrm{Hom}(\pi_1(M), \mathrm{PSL}_2\mathbb{C})$ of representations $\rho : \pi_1(M) \to \mathrm{PSL}_2\mathbb{C}$ that have discrete image and are faithful. We want to see that $\mathcal{DF}(\pi_1(M))$ is closed in $\mathrm{Hom}(\pi_1(M), \mathrm{PSL}_2\mathbb{C})$.

The goal here will be to consider fibered manifolds

$$\begin{tikzcd} \Gamma \arrow[r] & M^3 \arrow[d] \\ & S^1 \end{tikzcd}$$

I.e. $M^3 = \times I/\Sigma \times \{0\} \xrightarrow{\varphi} S \times \{1\}$ and discuss the following Theorem.

Theorem 7.2.1 (Thurston)**.** *If φ is pseudo-Anosov, then M^3 has a hyperbolic metric.*

Example 7.2.2. Examples of Kleinian groups and hyperbolic manifolds are the following.

1. *Elementary groups*: A Kleinian group is called elementary if it contains an abelian group of finite index (it is *virtually abelian*). Examples are the parabolics $\langle z \mapsto z+1 \rangle \cong \mathbb{Z}$ and $\langle z \mapsto z+\mu, z \mapsto z+\lambda \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. These are models for *cusps*. In the first case (drawing of the fundamental domain), the quotient $\mathbb{H}^3/\langle z \mapsto z+1 \rangle \cong S^1 \times \mathbb{R} \times (0,\infty)$ (drawing). The translation preserves horoballs (drawing e.g. everything above a horizontal plane in $\mathbb{H}^3$), and the annuli $S^1 \times \mathbb{R}$ have euclidean metrics, but scales with the height; it is

   $$e^{-2t} dE^2 + dt^2.$$

   In the rank 2 case, write $\Gamma = \langle z \mapsto z+\mu, z \mapsto z+\lambda \rangle$. The quotient $\mathbb{H}^3/\Gamma$ is homeomorphic to $S^1 \times S^1 \times (0,\infty)$. The tori are euclidean, and the metric is again of the form

   $$e^{-2t} dE^2 + dt^2.$$

   Another example of an elementary group is $\Gamma = \langle z \mapsto \lambda z \rangle$ and $\mathbb{H}^3/\Gamma \cong S^1 \times D^2$.

   In general, an elementary group comes from one of the examples together with elliptics.

2. "Fuchsian" manifolds: Recall that $\mathbb{H}^2 \hookrightarrow \mathbb{H}^3$ as the totally geodesic planes (e.g. vertical planes over a line in the boundary in the upper half space model, or say the equator in the ball model). Now $\mathrm{PSL}_2\mathbb{R} \hookrightarrow \mathrm{PSL}_2\mathbb{C}$ preserves these planes. Taking a Fuchsian group $\Gamma \subseteq \mathrm{PSL}_2\mathbb{R}$, we can consider $\mathbb{H}^3/\Gamma$. This is homeomorphic to $\Sigma \times (-\infty, \infty)$, where Σ is a topological surface homeomorphic to $\mathbb{H}^2/\Gamma$. In particular, $\pi_1 M = \pi_1 \Sigma$.

3. Schottky groups: Consider n pairs of circles in $\mathbb{C}$, C_i, C_i', $0 \leq i \leq n$ (drawing). Choose loxodromies γ_i that take the interior of C_i to the exterior of C_i' (drawing). Then it is a fact that $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ generate a Kleinian group that is free, and $\mathbb{H}^3/\Gamma$ is a genus n handlebody.

4. Gluings of ideal tetrahedra: An ideal tetrahedron is a tetrahedron in $\mathbb{H}^3$ with vertices at ∞ and totally geodesic faces in $\mathbb{H}^3$ (drawing). One can for instance obtain the complement of the figure eight knot gluing two tetrahedra (drawing); this is also an example of a fibered manifolds (where the surface is a once-punctured torus). There is an algorithm for figuring out how to obtain hyperbolic metrics on knot complements implemented in the software SnapPea (due to J. Weeks).

5. Poincaré polyhedron theorem: Consider a polyhedron in $\mathbb{H}^3$, faces identified by isometries $\langle \gamma_1, \ldots, \gamma_n \rangle$ satisfying some "cycle conditions". Then $\gamma_1, \ldots, \gamma_n$ generate a Kleinian group. One example is the *Seifert–Weber space* gluing opposite sides of a dodecahedron with a $\frac{3}{10}$ twist: There exists a regular dodecahedron in $\mathbb{H}^3$ with all dihedral angles $2\pi/5$. The quotient is a closed hyperbolic 3-manifold (examples of such are very few).

6. *Hyperbolic Dehn filling*: Let M^3 be a hyperbolic 3-manifold with torus boundary (e.g. the figure eight knot complement). One can glue in a solid torus $S^1 \times D^2$ (drawing) to get a closed manifold. Sending the meridian to p times the meridian and q times the longitude, this is called (p, q)-surgery.

   Theorem 7.2.3 (Thurston)**.** *All except finitely many choices of p and q gives a manifold that admits a hyperbolic metric.*

7. The following theorem is a special case of the Geometrization Theorem.

   Theorem 7.2.4 (Hyperbolization Theorem, Perelman)**.** *If M^3 is closed and irreducible (i.e. any sphere bounds a ball; this fails for example when one considers direct sums of manifolds, and one talks about prime manifolds), $\pi_1 M^3$ is infinite, and is* atoroidal*, i.e. it has no torus subgroup (i.e. $\mathbb{Z} \oplus \mathbb{Z}$), then M^3 admits a hyperbolic metric.*

   Thurston first proved this Theorem in the case where M^3 is compact, irreducible, atoroidal, *Haken* (or *sufficiently large*) with boundary a union of tori. That it is Haken means that exists a surface (closed of genus $g > 1(?)$) subgroup of the fundamental group, $\pi_1 S \hookrightarrow \pi_1 M$. In this case, the complete hyperbolic metric has finite volume. The fibered manifold case is part of this story (although treated differently).

   One corollary of this is that knot complements are hyperbolic if and only if the knot is not a satellite or a torus knot (and most are not).

19th Lecture, November 27th 2012

7.3 The Margulis Lemma

The plan for today is to consider the Margulis Lemma and its consequences, and the limit set of a Kleinian group.

Fact 7.3.1. *Let $g, h \in \mathrm{Isom}(\mathbb{H}^3) \setminus \{\mathrm{Id}\}$ such that $\langle g, h \rangle$ is a Kleinian group. Then*

1. *$\langle g, h \rangle$ is elementary if and only if $\mathrm{Fix}(g) = \mathrm{Fix}(h)$.*

2. *If $\mathrm{Fix}(g) \cap \mathrm{Fix}(h) \neq \emptyset$, then $\mathrm{Fix}(g) = \mathrm{Fix}(h)$.*

3. *If $\langle g, h \rangle$ is non-elementary, then it contains a free group with all loxodromic elements.*

Proof. If g, h commute, then $\mathrm{Fix}(g) = \mathrm{Fix}(h)$: Let $p \in \mathrm{Fix}(h)$. Then $hg(p) = gh(p) = g(p)$, so $g(p) \in \mathrm{Fix}(h)$, so g fixes $\mathrm{Fix}(h) = \{p, p'\}$, so $g^2(p) = p$, and so $p \in \mathrm{Fix}(g)$.

1) Now, if $\langle g, h \rangle$ is elementary, then g^n, h^m commute, so $\mathrm{Fix}(g^n) = \mathrm{Fix}(h^m)$ and $\mathrm{Fix}(g) = \mathrm{Fix}(h)$. On the other hand, suppose that g, h have the same fixed points. There are two cases: If g, h are parabolic, then $\langle g, h \rangle \subseteq \mathrm{Euc}(\mathbb{R}^2)$ and so it is abelian. If g, h are loxodromic fixing the same axis. Then since $\langle g, h \rangle$ is discrete and is a subgroup of $\mathbb{R}$ (by the translation length $\delta : \langle g, h \rangle \to \mathbb{R}$), it must be cyclic.

2) If g, h are both parabolic, there is nothing to prove. If g is loxodromic, and h is parabolic, suppose ∞ is the common fixed point. One can normalize so that $h(z) = \pm\tau z$, $g(z) = \lambda z$. Then $g^{-i} h g^i$ will converge to Id, which contradicts discreteness. If g, h are both loxodromic, one can reduce to the previous case, as $[g, h]$ is parabolic.

3) This follows from the Ping Pong Lemma. ☐

Lemma 7.3.2 (Margulis Lemma)**.** *For $\mathbb{H}^3$, there exists a constant $\varepsilon_3 > 0$ such that if $G \subseteq \mathrm{Isom}(\mathbb{H}^3)$ is a Kleinian group, and $x \in \mathbb{H}^3$, then the group generated by*

$$\Gamma = \{g \in G \mid d(x, gx) < \varepsilon_3\}$$

is elementary.

Remark 7.3.3. This holds true for $\mathbb{H}^n$ (there exists ε_n, ...).

One precursor to the Margulis Lemma is the following:

Lemma 7.3.4 (Zassenhaus–Kazhdan–Margulis)*. More generally, for a connected Lie group G, there exists a neighbourhood U of* Id *such that if $\Gamma < G$ is discrete, then $\Gamma \cap U$ generates a nilpotent group.*

In $\mathbb{H}^3$, one consequence of the Margulis Lemma is Jørgensen's inequality: If a, b are non-elementary,

$$|\mathrm{tr}^2\, a - 4| + |\mathrm{tr}[a, b] - 2| \geq 1.$$

Proposition 7.3.5 (Chuckrow)**.** *Let $G \subseteq \mathrm{Isom}(\mathbb{H}^3)$ be a finitely generated non-elementary Kleinian group. Then the limit of a sequence of discrete faithful representations of G in $\mathrm{PSL}_2\mathbb{C}$ is a discrete faithful representation in $\mathrm{PSL}_2\mathbb{C}$.*

Let $\mathcal{R}(G)$ be the set of representations of G in $\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2\mathbb{C}$. In here, we have the set $DF(G)$ of discrete faithful ones, and the set $DF(G)$ is closed.

Proof of Proposition. Let $\rho_i \to \rho$ (in $\mathcal{R}(G)$). 1) Then ρ is faithful: If not, let $g \neq \mathrm{Id}$, such that $\rho(g) = \mathrm{Id}$. Then for every $h \neq \mathrm{Id}$ in G, $\rho([g, h]) = \mathrm{Id}$. So by the Margulis Lemma, $\rho_i(g)$ and $\rho_i([g, h])$ generate an elementary group for i sufficiently large. Then by the first fact above, $\rho_i(g)$ and $\rho_i([g, h])$ have the same fixed point set, so $\rho_i(g)$ and $\rho_i(h)$ have the same fixed point set (this follows from the general exercise that if $a, b \in \mathrm{Isom}^+(\mathbb{H}^3)$, and $a, [a, b]$ have the same fixed points, then so do a and b). This means that G is elementary, but G was assumed to be non-elementary.

2) ρ is discrete: Let $\{g_1, \ldots, g_k\}$ be a generating set for G. Let $K \subseteq \mathrm{Isom}^+(\mathbb{H}^3)$ such that $\rho_i(g_j) \in K$ for all $j = 1, \ldots, k$ and all i. Let $U \subseteq \mathrm{Isom}^+(\mathbb{H}^3)$ be a neighbourhood of Id such that 1) if $u \in U$, $k \in K$, then $[u, k]$ moves $0 \in \mathbb{H}^3$ a distance less than ε_3, and 2) any $u \in U$ moves 0 a distance less than ε_3. If ρ is not discrete, there exists $h \neq \mathrm{Id}$ in G with $\rho_i(h) \in U$ s.t. $\rho_i(h)$ and $\rho_i(g_j)$ have the same fixed points. In particular, $\rho_i(g_j)$ all have the same fixed points, so G is elementary. ☐

Recall that the ε-thin part of a hyperbolic manifold M is

$$\begin{aligned} M^{<\varepsilon} &= \{x \in M \mid \text{injectivity radius} < \varepsilon\} \\ &= \{x \in M \mid \text{there exists a non-trivial loop through } x \text{ of length} < 2\varepsilon\}. \end{aligned}$$

Recall that if X is a hyperbolic surface, there exists an $\varepsilon_2 > 0$ such that each component of $X^{<\varepsilon_2}$ is either an annulus or a cusp. This follows from the Margulis Lemma: Let $\Gamma \subseteq \mathrm{Isom}(\mathbb{H}^2)$ be a Fuchsian group. Consider

$$\Gamma_{p,\varepsilon} = \langle \{\gamma \in \Gamma \mid d(p, \gamma(p)) < \varepsilon\} \rangle < \Gamma.$$

For $\varepsilon < \varepsilon_2$, $\Gamma_{p,\varepsilon}$ is abelian (by the Margulis Lemma) and so cyclic. There are two cases here: If it is generated by a hyperbolic element, the thin part will be an annulus, and if it is generated by a parabolic, it will be a cusp.

Now, for hyperbolic 3-manifolds, we have the following.

Fact 7.3.6. *A non-trivial abelian torsion-free Kleinian group is one of the following:*

1. *A parabolic group of rank* 1

2. A parabolic group of rank 2

3. An infinite cyclic group generated by a loxodromic element.

Theorem 7.3.7. *Let M be a hyperbolic* 3*-manifold. The ε_3-thin part $M^{<\varepsilon_3}$ is one of 3 types:*

1. The quotient of a horoball by a parabolic group of rank 1*. This is an annulus $\times(0,\infty)$.*

2. The quotient of a horoball by a parabolic group of rank 2*. This is a torus $\times(0,\infty)$.*

3. An r-neighbourhood of a simple closed geodesic of length less than $2\varepsilon_3$. This is a solid torus.

Consider again the closed set $DF(G) \hookrightarrow \mathrm{Rep}(G)$.

Theorem 7.3.8 (Mostow Rigidity Theorem)**.** *If M^3, N^3 are closed hyperbolic* 3*-manifold and $f : \pi_1 M^3 \to \pi_1 N^3$ an isomorphism. Then there exists an isometry $F : M^3 \to N^3$ such that $F_* = f$.*

Prasad showed the same result when boundary are tori.

7.4 Limit sets

Let Γ be a Kleinian group and $x \in \mathbb{H}^3$. Consider the discrete orbit Γx. Let *limit set* is

$$\Lambda(\Gamma) = \overline{\Gamma_x} \cap \partial \mathbb{H}^3.$$

This is independent of x (drawing): If y is another point, the Euclidean distance $d(\gamma x, \gamma y) \to 0$ as $\gamma \to \infty$ in Γ.

There are a couple of cases that we will ignore: If Γ is finite, $\Lambda = \emptyset$. If Γ is elementary, $|\Lambda| \leq 2$.

If Γ is non-elementary, then Λ is uncountable; for example, for the Schottky groups, the limit set has a Cantor set like construction. If Γ is compact (so $\mathbb{H}^3/\Gamma$ is closed), then $\Lambda = S^2$.

20th Lecture, November 30th 2012

We begin with a leftover claim.

Claim 7.4.1. *Let $X, Y \in \mathcal{T}_g$ be hyperbolic surfaces such that*

$$l(\gamma, X) \geq l(\gamma, Y)$$

for all simple closed curves. Then $X = Y$.

Sketch of proof. One can proof this using the Earthquake Theorem. Suppose $X \neq Y$. Then there is an earthquake path $\mathcal{E}_\lambda(t)$ in $\mathcal{T}_g$ with $\mathcal{E}_\lambda(0) = X$, $\mathcal{E}_\lambda(1) = Y$, where λ is some measured foliation. Then there exists a γ such that $l(\gamma, \mathcal{E}_\lambda(t))$ is decreasing (drawings; uses the formula $\frac{dl}{dt}|_{t=0} = \sum \cos\theta$). □

Now, we turn back to Kleinian groups and limit sets. Let Γ be a non-elementary Kleinian group. Recall that the limit set is

$$\Lambda(\Gamma) = \overline{\Gamma_x} \cap \partial \mathbb{H}^3.$$

Facts 7.4.2. *We have the following facts about the limit set:*

1. Λ is the smallest non-empty closed Γ-invariant subset of $\partial\mathbb{H}^3 = \hat{\mathbb{C}} = \mathbb{C}P^1$.

2. Λ is the closure of the set of fixed points of loxodromic elements of Γ.

Now, $\Omega = \hat{\mathbb{C}} \backslash \Lambda$ is also Γ-invariant (and probably empty, e.g. if the Kleinian group is cocompact).

Proposition 7.4.3. *The Kleinian group Γ acts discontinuously on Ω.*

For this reason, Ω is called the *domain of discontinuity*.

Corollary 7.4.4. *The quotient Ω/Γ is a Riemann surface.*

Sketch of proof of Proposition 7.4.3. Consider the convex hull $C(\Lambda)$ of the limit set, i.e. the smallest convex set in $\mathbb{H}^3$ whose closure in $\overline{\mathbb{H}^3}$ contains Λ (here, a set X is called convex if for $p, q \in X$, the geodesic line $\overline{pq}$ is contained in X). Now, $C(\Lambda)$ is the intersection of all closed half-spaces whose extension to $\partial\mathbb{H}^3$ contains Λ. Note also that $\overline{C(\Lambda)} \cap \partial\mathbb{H}^3 = \Lambda$, that $C(\Lambda)$ is Γ-invariant, and that Γ acts properly discontinuously on $C(\Lambda)$

We claim that we can construct a Γ-equivariant retraction $r : \overline{\mathbb{H}^3} \to C(\Lambda)$: For $x \in \mathbb{H}^3$, let $r(x)$ be the closest point of $C(\Lambda)$ (drawing of why this is well-defined). For $x \in \partial\mathbb{H}^3$, one considers expanding horoballs centered at x and takes the first point of $C(\Lambda)$ hit by these (drawing).

The retraction maps Ω to $\partial C(\Lambda)$. If $K \subseteq \Omega$ is compact, then so is $r(K)$. Since Γ acts properly discontinuously, if $\gamma(K) \cap K \neq \emptyset$, then $\gamma(r(K)) \cap r(K) \neq \emptyset$, and the proper discontinuity of the action of Γ on Ω follows. $\square$

So, $\hat{M} = \mathbb{H}^3 \cup \Omega/\Gamma$ is topologically a manifold with boundary. The boundary $\partial\hat{M} = \Omega/\Gamma$ is called the *conformal boundary at infinity*; it is called "conformal" since Γ acts on Ω by Möbius maps that act conformally. One of the foundational results is the following.

Theorem 7.4.5 (Ahlfors Finiteness Theorem)**.** *If Γ is a finitely generated Kleinian group, then Ω/Γ is a finite area Riemann surface.*

Remark 7.4.6. The quotient Ω/Γ is in fact a surface with a complex projective structure (i.e. it has charts with maps to $\mathbb{C}P^1$ such that transition maps are Möbius transformations).

The quotient $N(\Lambda) = C(\Omega)/\Gamma \hookrightarrow \mathbb{H}^3/\Gamma = M$ is called the *convex core* (or *Nielsen core*) of M (drawing).

Facts 7.4.7.

1. *The convex core N contains all closed geodesics of M.*
2. *The fibers of the retraction are* rays*, so the complement of the thickening $N_\delta(\Gamma)$ is diffeomorphic to $\partial N \times [0, \infty)$. To understand the deformation of hyperbolic metrics on M^3 it turns out to be important to study these* ends *of M, and to do this, one studies surface group representations $\rho : \pi_1 S \to \mathrm{PSL}_2\mathbb{C}$.*

Definition 7.4.8. A hyperbolic 3-manifold M is called *geometrically finite* if its convex core has finite volume.

7.5 Quasi-Fuchsian manifolds

We turn now to quasi-Fuchsian representations (that form a subset of $\mathrm{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{C})/\mathrm{PSL}_2\mathbb{C} =: R(\pi_1 S)$). [A bunch of drawings illustrating Fuchsian manifolds, where the limit set is $\Lambda = S^1$.]

(Even more drawings explaining the following:) In the quasi-Fuchsian case, take this picture with an action of Γ and quasiconformal deformation $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ to something with an action of a Kleinian group Γ', $f\Gamma f^{-1} = \Gamma'$, where the limit set Λ is some Jordan curve splitting $\partial\mathbb{H}^3$ into $\Omega = \Omega^+ \cup \Omega^-$. The surfaces $X = \Omega^+/\Gamma$, $Y = \Omega^-/\Gamma$ are homeomorphic to the surface S that would arise in the Fuchsian case. The manifold $\hat{M}$ is homeomorphic to $S \times [0, 1]$, and the manifold $M^3 = \mathbb{H}^3/\Gamma'$ is *quasi-Fuchsian* (one precise definition of a quasi-Fuchsian manifold would be one obtained when the limit set is a Jordan curve). A *quasi-Fuchsian representation* $\rho : \pi_1 S \to \mathrm{PSL}_2\mathbb{C}$ is one whose limit set is a Jordan curve.

Theorem 7.5.1 (Simultaneous Uniformization Theorem (Bers))**.** *The set $QF(S) \subseteq R(\pi, S)$ of quasi-Fuchsian representations is homeomorphic to $\mathcal{T}_g \times \mathcal{T}_g$. In particular, for any pair (X, Y) there exists a quasi-Fuchsian Kleinian group $\Gamma = Q(X, Y)$ such that $\Omega^+/\Gamma = X$, $\Omega^-/\Gamma = Y$.*

Recall that a quasiconformal map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is one satisfying that

$$\|\frac{f_{\bar{z}}}{f_z}\|_\infty \leq K$$

for some K and that the Beltrami differential is $\mu(z) = \frac{f_{\bar{z}}}{f_z}(z)$ (defined almost everywhere; this is the *Beltrami equation*).

Theorem 7.5.2 (Measurable Riemann Mapping Theorem (Morrey, Ahlfors, Bers))**.** *Given any measurable $\mu \in L^\infty(\hat{\mathbb{C}})$, $\|\mu\|_\infty < 1$, there exists a quasiconformal map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ whose Beltrami differential is μ. Moreover, f is unique up to post-composition by a Möbius map. Further, the solutions f depend analytically on μ.*

This Theorem is used in the proof of the Simultaneous Uniformization Theorem: Start with Γ_0 a Fuchsian group, $X = \Omega^+/\Gamma_0 = \Omega^-/\Gamma_0$. Suppose we want to deform this to something with $(\Omega')^-/\Gamma' = X$, $(\Omega')^+/\Gamma' = Y$.

21st Lecture, December 4th 2012

Today, we will finish the discussion of quasi-Fuchsian manifolds and talk about fibered 3-manifolds. Part of the discussion will be the important Double Limit Theorem.

Last time, we considered hyperbolic 3-manifolds $M^3 = \mathbb{H}^3/\Gamma$, which generally might have cusps of various kinds (drawing) and inside consisting of a convex core (drawing). We considered Kleinian groups coming from surface group representations $\rho : \pi_1 S \to \mathrm{PSL}_2\mathbb{C}$ (faithful with discrete image) (the space of these, up to conjugacy, we denote $R(\pi_1 S)$). We considered the quasi-Fuchsian space $QF(S) \subseteq R(\pi_1 S)$. In fact, $\mathbb{H}^3/\rho(\pi_1 S)$. If the group Γ is Fuchsian (i.e. the image is in $\mathrm{PSL}_2\mathbb{R}$), M^3 is homeomorphic to $\mathbb{R} \times (\Omega^+/\Gamma)$ (drawing). We will see how to deform this to $\Gamma' \subseteq \mathrm{PSL}_2\mathbb{C}$, where the resulting manifold X is quasi-Fuchsian and has two boundaries at infinity, X and Y, that will not be conformally equivalent.

Definition 7.5.3. A *quasi-Fuchsian group* Γ' is obtained from a *quasiconformal deformation* (i.e. there exists quasiconformal $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $f\Gamma f^{-1} = \Gamma'$, Γ Fuchsian) of a Fuchsian group. Equivalently, Γ is quasi-Fuchsian if and only if its limit set is a Jordan curve.

One of the ingredients in this theory is the Measurable Riemann Mapping Theorem: For any $\mu \in L^\infty(\hat{\mathbb{C}})$, $\|\mu\|_\infty < 1$, then there exists a quasiconformal mapping $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}$. Any other solution g to this solution is of the form $g = A \circ f$, where A is a Möbius map.

Now, start with Γ_0 a Fuchsian group, $\Omega^+/\Gamma_0 = \Omega^-/\Gamma_0 = X$; we want to deform it to Γ' so that the complements of the limit set consists of $Y = \Omega^+/\Gamma'$, $X = \Omega^-/\Gamma'$, for a given $Y \in \mathcal{T}_g$. Let $g : X \to Y$ be some quasiconformal map, lifting to $\tilde{g} : \Omega_+ \to \tilde{Y}$. In particular, this defines a Beltrami differential μ on Ω^+, which we can extend to $\hat{\mathbb{C}}$ by 0 on Ω^-. It then still satisfies the conditions of the Measurable Riemann Mapping Theorem, and so there exists a map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}$. By the Γ_0-invariance, $f \circ \gamma$ is also a solution, for any $\gamma \in \Gamma_0$. By the uniqueness part of the MRMT, there exists a Möbius transformation γ' such that $f \circ \gamma = \gamma' \circ f$. In fact, the map $\varphi : \Gamma_0 \to \Gamma' \subseteq \mathrm{PSL}_2\mathbb{C} : \gamma \mapsto \gamma'$ is an isomorphism of groups, and Γ' is the quasiconformal deformation of Γ_0.

(Drawing of a closer look at the boundary of the convex core. Keywords: it consists of *pleated surfaces* that are not smooth but piecewise totally geodesic and bent along a geodesic lamination; complex earthquakes). It is a fact (by Sullivan) that $d_\tau(\partial_1 C, X) < K$ for some universal constant (where C is is the convex core).

Fact 7.5.4. *$QF(S)$ is an open set in $R(\pi_1 S)$, where $R(\pi_1 S)$ has the algebraic topology, i.e. $\rho_n \to \rho$ if $\rho_n(g) \to \rho_n(g)$ for all $g \in \pi_1 S$.*

Recall that for a closed hyperbolic manifold, there is no deformation: Mostow Rigidity implies that $R(\pi_1 M) = \{\mathrm{pt}\}$. Generalizing the idea of considering limit sets, one has *Sullivan rigidity*: If $\Lambda = \hat{\mathbb{C}}$, then any quasiconformal map $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that conjugates Γ to Γ' is *conformal*.

We can consider the *Bers slice*

$$B_X = \{Q(X,Y) \in R(\pi_1 S) \mid Y \in \mathcal{T}_g\}.$$

Fact 7.5.5. *The Bers slice is pre-compact in $R(\pi_1 S)$.*

In particular, for any $M \in B_X$, we have $l_M(\gamma) \leq 2l_X(\gamma)$ for any simple closed curve γ on S.

Remark 7.5.6. This leads to *Bers Embedding*: Fix $X \in \mathcal{T}_g$. Then we have an embedding $B_X : \mathcal{T}_g \to R(\pi_1 S) : Y \mapsto Q(X,Y)$. Via this embedding, $\mathcal{T}_X$ gets a complex structure.

Degenerations in the Bers slice: Fix again $X \in \mathcal{T}_g$. We can ask what happens to $Q(X,Y)$ as $Y \to \infty$ in $\mathcal{T}_g$. For example, what happens to $Q(X, \mathrm{tw}_\gamma^n X)$? In general, such things will not converge (but in this particular example, it will).

A special case of the Double Limit Theorem is the following:

Theorem 7.5.7 (Double Limit Theorem (Thurston))**.** *For φ a pseudo-Anosov, $Q(\varphi^{-i}X, \varphi^i X)$ is a bounded sequence (i.e. has a convergent subsequence).*

Definition 7.5.8. Two laminations (λ, μ) are binding if

1. They have no leaves in common, and
2. $S \setminus (\lambda \cup \mu)$ is a union of polygons, polygons with cusps.

The general Double Limit Theorem is the following:

Theorem 7.5.9. *Let $\rho_i = Q(X_i, Y_i)$ be quasi-Fuchsian such that $X_i \to [\lambda] \in PM\mathcal{L}, Y_i \to [\mu] \in PM\mathcal{L}$. If λ, μ* bind *the surface, then ρ_i converge to ρ after passing to a subsequence.*

7.6 Fibered 3-manifold

A *fibered 3-manifold M_f* is of the form $M_f = S \times [0,1]/(p,0) \sim (f(p),1)$ for some homeomorphism $f : S \to S$. The topology of M depends only on the mapping class of f. Its fundamental group has a presentation

$$\pi_1 M_f = \langle \pi_1(S), t \mid t\gamma t^{-1} = f_*(\gamma)\, \forall \gamma \in \pi_1 S \rangle$$

Assume from now on that $S_{g,n}$ satisfies $2g - 2 + n > 0$.

Theorem 7.6.1 (Hyperbolization Theorem (Thurston))**.** *The fibered 3-manifold M_f admits a complete hyperbolic metric of finite volume if and only if f is pseudo-Anosov.*

Remark 7.6.2. One direction is clear: If f is not pseudo-Anosov, there exists an incompressible torus $\pi_1 T \hookrightarrow \pi_1 M_f$, that is an obstruction to existence of a hyperbolic metric.

Over M_f we have the infinite cyclic cover $S \times \mathbb{R} \to M_f$. This cover has a deck translation ψ acting by an isometry in the homotopy class of f, i.e. $\psi : S \times \{0\} \to S \times \{1\}$ is in the homotopy class of f. It was surprising that such manifolds would have hyperbolic structures because of this periodicity.

Fact 7.6.3. *For any x, the closure of the universal cover $\widetilde{S \times \{x\}}$ in $\overline{\mathbb{H}^3}$ is the entire sphere.*

This fact follows from the following Lemma.

Lemma 7.6.4. *If $\mathrm{Id} \neq \Gamma' \triangleleft \Gamma$, then $\Lambda(\Gamma') = \Lambda(\Gamma)$.*

22nd Lecture, December 7th 2012

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