Topics on Kähler Geometry and Hodge Theory

Course by Mario Garcia-Fernandez, notes by Søren Fuglede Jørgensen

Centre for Quantum Geometry of Moduli Spaces, Aarhus Universitet, 2011

Contents

Disclaimer

These are notes from a course given by Mario Garcia-Fernandez in 2011.[1] They have been written and TeX'ed during the lecture and some parts have not been completely proofread, so there's bound to be a number of typos and mistakes that should be attributed to me rather than the lecturer. Also, I've made these notes primarily to be able to look back on what happened with more ease, and to get experience with TeX'ing live. That being said, feel very free to send any comments and or corrections to fuglede@imf.au.dk.

The most recent version of these notes is available at http://home.imf.au.dk/pred.

1st lecture, August 24th 2011

1 Overview

Today we give an overview over and a plan for the course. Precise definitions are postponed to the next lecture. Let X be a complex manifold – i.e. a topological manifold with an atlas such that transition functions are holomorphic $\mathbb{C}^n \to \mathbb{C}^n$. Examples are $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$, $\mathbb{C}^n$, complex submanifolds $X \subseteq \mathbb{P}^n$ (projective manifolds), and $Z = \{z \in \mathbb{P}^n : f_j(z) = 0\}$ for homogeneous polynomials f_j (projective algebraic varieties). We want to address the following question: Which complex manifolds are projective?

Theorem 1.0.1 (Chow). *Any complex projective manifold is algebraic.*

So which complex manifolds can be embedded in $\mathbb{P}^n$? The answer turns out to be the following, known as Kodaira's embedding theorem: When there exists a holomorphic line bundle $L \to X$ (i.e. a complex manifold L with a map $L \to X$ whose fibers are isomorphic to $\mathbb{C}$), and a hermitian metric h on L such that the curvature F_h of h is such that iF_h is *positive*. Consider $X \subseteq \mathbb{P}^n$. The restriction of the natural metric g_{FS} on $\mathbb{P}^n$ to the manifold X has some nice properties. Namely, the Levi–Civita connection ∇ on (X, g_{FS}) preserves the complex structure on X. A metric with this property is called Kähler, so all projective manifolds are Kähler. The 2-form iF_h defines a Kähler metric on X.

So why is this true? Consider a line bundle $L \to X$ with a metric h such that iF_h is positive. Consider then the $N+1$-dimensional space $H^0(X, L)$ of holomorphic sections of L. This has a basis $\{s_i\}$ which can be used to embed $X \hookrightarrow \mathbb{P}^N$. The proof requires Hodge theory (and the theory of elliptic operators and Sobolev spaces) and sheaf theory.

Why should we care about this? If $\varphi : X \hookrightarrow \mathbb{P}^N$ is an embedding, then $\varphi^* g_{FS}$ has nice properties for suitable embeddings φ. For instance, if X admits a Kähler–Einstein metric g_0, one can approximate g_0 by $\varphi^* g_{FS}$. There is a similar story for hermitian metrics on bundles, which approach Hermite–Einstein metrics.

The course will follow [Wel08].

2nd lecture, August 25th 2011

2 Complex manifolds

2.1 Manifolds and morphisms

We begin with a discussion of complex manifolds and holomorphic vector bundles. A reference for the following is [Hör90]. Let $\mathbb{C}$ be the complex numbers, and let $U \subseteq \mathbb{C}^n$ be an open subset of $\mathbb{C}^n$,

[1] The course homepage is located at http://aula.au.dk/courses/QGME11TKG/ – that probably won't be true forever though.

and let $\mathcal{O}(U)$ denote the set of complex valued functions $f : U \to \mathbb{C}$ that are holomorphic. That is, we let $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$ with coordinates $z = (x, y)$, and holomorphicity of f means that

$$\overline{\partial} f := \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j = 0,$$

where we think of $d\overline{z}_j$ as formal variables and let

$$\frac{\partial f}{\partial \overline{z}_j} = \tfrac{1}{2}(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}).$$

Another way of describing holomorphicity is that near a point $z_0 \in U$, we have

$$f(z) = \sum_{\alpha_1,\ldots,\alpha_n=0}^{n} a_{\alpha_1,\ldots,\alpha_n}(z_1 - z_1^0)^{\alpha_1} \cdots (z_n - z_n^0)^{\alpha_n}.$$

Let M be a topological manifold (i.e. M is Hausdorff, its topology has a countable basis, and it is locally homeomorphic to $\mathbb{R}^{2n}$).

Definition 2.1.1. A *holomorphic structure* on M, denoted $\mathcal{O}_M$, is a family of $\mathbb{C}$-valued functions defined on open subsets U, $f : U \to \mathbb{C}$, satisfying the following two properties:

a) For all $p \in M$, there is an open neighbourhood U of P, and a homeomorphism $h : U \to U' \subseteq \mathbb{C}^n$, such that for all open $V \subseteq U$ we have $f \in \mathcal{O}_M(V)$ if and only if $f \circ h^{-1} \in \mathcal{O}(h(V))$.

b) If a function $f : U \to \mathbb{C}$ is in $\mathcal{O}_M(U)$, and $U = \bigcup_i U_i$, then $f \in \mathcal{O}_M(U_i)$ for all i.

Remark 2.1.2. We make the following general remarks.

- You can define smooth manifolds similarly.
- There are "more" holomorphic structures such that the corresponding smooth structures are the same.

Definition 2.1.3. For complex manifolds $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ define the following:

a) An *$\mathcal{O}$-morphism* (or a *holomorphic map*) $F : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ is a continuous map $F : M \to N$, such that $f \in \mathcal{O}_N$ if and only if $f \circ F \in \mathcal{O}_M$.

b) An *$\mathcal{O}$-isomorphism* (or a *biholomorphism*) F is an $\mathcal{O}$-morphism, which is also a homeomorphism with F^{-1} an $\mathcal{O}$-morphism.

Given two coordinate systems $h_j : U_j \to \mathbb{C}^n$ with $U_1 \cap U_2 \neq \emptyset$, then we have that $h_2 \circ h_1^{-1} : h_1(U_1 \cap U_2) \to h_2(U_1 \cap U_2)$ is a biholomorphism.

Conversely, having a family of functions with this property defines a complex structure: Let $\{U_\alpha\}_{\alpha\in\mathbb{A}}$ be an open cover of M, and $\{h_\alpha : U_\alpha \to U'_\alpha \subseteq \mathbb{C}^n\}$ a set of properties satisfying the above property, then we let $\mathcal{O}_M$ consist of functions $f : U \to \mathbb{C}$, where U is open on M, and $f \circ h_\alpha^{-1} \in \mathcal{O}(h_\alpha(U_\alpha \cap U))$.

Example 2.1.4. The following are examples of complex manifolds.

1. $\mathbb{C}^n$.
2. Subsets $U \subseteq (M, \mathcal{O}_M)$.
3. Let $k \leq n$, and let $M_{k,n}$ be $k \times n$ matrices with entries in $\mathbb{C}$. Let $M_{k,n}^m$ be those of rank m. We observe that $M_{k,n}^k$ is a holomorphic manifold, since $M_{k,n} \cong \mathbb{C}^{kn}$ has a holomorphic structure, and

$$M_{k,n}^k = \bigcup_{i=1}^{l} \{A \in M_{k,n} \mid \det A_i \neq 0\},$$

where $a_1, \ldots, A_l$ are the k-minors of A.

We endow $M_{k,n}^m$ with a holomorphic structure as follows: Let $T_0 \in M_{k,n}^m$, and define an open neighbourhood W of T_0 as follows. For permutation matrices P and Q,

$$PT_0Q = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix},$$

with A_0 a non-singular $m \times m$-matrix. Now, put

$$W = \{T \in M_{k,n} \mid PTQ = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \|A - A_0\| < \varepsilon\},$$

which is an open subset of $M_{k,n}$. Let $U = W \cap M_{k,n}^m$. We now need to define the map h. Note that $T \in U$ if and only if $D = CA^{-1}B$; this follows from the fact that

$$\begin{pmatrix} I_m & 0 \\ -CA^{-1} & I_{k-m} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}.$$

Now define $h : U \to \mathbb{C}^{m^2+(n-m)m+(k-m)m} = \mathbb{C}^{m(n+k-n)}$ by

$$h(T) = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix},$$

and note that h is a homeomorphism onto its image.

Definition 2.1.5. A closed subset $N \subseteq (M, \mathcal{O}_M)$ of a complex manifold is called a *submanifold* if for every $x_0 \in N$, there is a coordinate map $h : U \to U' \subseteq \mathbb{C}^n$ such that $h(U \cap N) = U' \cap \mathbb{C}^k$, $k \leq n$.

Definition 2.1.6. A holomorphic map $f : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ is an *embedding*, if it is a biholomorphism into a submanifold of $(N, \mathcal{O}_N)$.

Example 2.1.7. Whereas the previous examples were all non-compact, the following is an example of a compact complex manifold: Let the *Grassmannian* be the space

$$G_k(\mathbb{C}^n) = \{k\text{-dimensional linear subspaces of } \mathbb{C}^n\},$$

and define a map $\pi : M_{k,n}^k \to G_k(\mathbb{C}^n)$ by

$$A \mapsto \pi \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} = \{k\text{-dim subspace spanned by } a_1, \ldots, a_k\}.$$

Note that $\mathrm{GL}(\mathbb{C}^k)$ acts on $M_{k,n}^k$ by multiplication of matrices, $(g, A) \mapsto g \cdot A$. The quotient is

$$M_{k,n}^k / \mathrm{GL}(\mathbb{C}^k) \cong G_k(\mathbb{C}^n).$$

Thus, we endow $G_k(\mathbb{C}^n)$ with the quotient topology, so it is Hausdorff and its topology has countable basis. The map $\pi : \{A \mid A^T A = \mathrm{Id}\} \to G_k(\mathbb{C}^n)$ is surjective, so $G_k(\mathbb{C}^n)$ is compact. Coordinates are defined as follows: Let

$$\begin{aligned} U_\alpha &= \{s \in G_k(\mathbb{C}^n) \mid s = \pi(A), \det A_\alpha \neq 0\}, \\ h_\alpha &: U_\alpha \to \mathbb{C}^{k(n-k)}, \\ h_\alpha([A]) &= A_\alpha^{-1} \tilde{A}_\alpha, \quad A \cdot P_\alpha = [A_\alpha \tilde{A}_\alpha]. \end{aligned}$$

For $k = 1$, $G_1(\mathbb{C}^{n+1}) = \mathbb{P}^n$, and the map $U_\alpha \to \mathbb{C}^n$ is given by

$$[(x_0 : \cdots : x_n)] \mapsto (x_0/x_\alpha, \ldots, x_n/x_\alpha).$$

3rd lecture, August 30th 2011

2.2 Projective manifolds

Last time we defined complex manifolds and gave examples. Compact examples were projective space and Grassmannians. Today we begin with holomorphic bundles.

Definition 2.2.1 (Embedding)**.** A holomorphic map $f : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ is an *embedding* if it is a biholomorphism onto a submanifold of $(N, \mathcal{O}_N)$.

Definition 2.2.2 (Projective manifold)**.** A compact complex manifold which admits an embedding into $\mathbb{P}^n$ for some n is called *projective*.

Definition 2.2.3 (Projective algebraic manifold)**.** A manifold X is called *projective algebraic* if it is biholomorphic to the zeroes of a finite set of homogeneous polynomials in $\mathbb{P}^n$.

A projective algebraic manifold is

$$X = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : p_j(x) = 0, p_j \text{ hom. of degree } d_i\}.$$

Remark 2.2.4. Define $I(X) = \{p \in \mathbb{C}[x_0, \ldots, x_n] : p(x) = 0 \,\forall x \in X\}$. This is an ideal of $\mathbb{C}[x_0 : \cdots : x_n]$ generated (by the Hilbert basis theorem) by a finite set $\{p_1, \ldots, p_k\}$.

We have a map $\{\text{Sets in } \mathbb{P}^n\} \to \text{Ideals in } \mathbb{C}[x_0, \ldots, x_n]$ mapping $S \mapsto I(S)$. There is a sort of inverse map provided by the Nullstellensatz.

Example 2.2.5. Let $H = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : \sum_j a_j x_j = 0\}$ for $(a_0, \ldots, a_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. We claim that this is an $(n-1)$-dimensional submanifold of $\mathbb{P}^n$, so we need to find suitable coordinates. Let $U_0 = \{x \in \mathbb{P}^n : x_0 \neq 0\}$. Then $H \cap U_0$ are now given by those points satisfying $a_1 x_1/x_0 + \cdots + a_n x_n/x_0 = -a_n$. Using $x_1, \ldots, x_n$ this defines an affine subset of $\mathbb{C}^n$. By an affine transformation, we take $H \cap U_0$ to $\mathbb{C}^{n-1}$.

In general, if $X = \{x \in \mathbb{P}^n : p_j(x) = 0\}$ we need to apply the implicit function theorem to prove that X is a submanifold, i.e. that locally it is just $\mathbb{C}^k \subseteq \mathbb{C}^n$ for some coordinates.

Exercise 2.2.6. Check conditions on $p_1, \ldots, p_k$ to provide a submanifold.

Remark 2.2.7. By the theorem by Chow, every projective manifold is algebraic. By a theorem of Whitney, every smooth manifold can be embedded in $\mathbb{R}^{2n+1}$, but not every complex manifold can be embedded in $\mathbb{C}^N$. All examples must be non-compact: Suppose X is a complex, compact and connected manifold with an embedding $\varphi : X \hookrightarrow \mathbb{C}^N$. Taking the i'th coordinate x_i, we obtain a map $\varphi_i : X \hookrightarrow\to \mathbb{C}^n \overset{x_i}{\to} \mathbb{C}$. Since X is compact, $|\varphi_i|$ attains a maximum, so φ_i is locally constant by the maximum principle, amd so φ_i is constant, but this means that φ can not be an embedding unless $\dim_{\mathbb{C}} X = 0$.

We ask the following question: Which compact complex manifolds are projective? The above remarks tell us that to do extrinsic geometry, we can not use holomorphic functions on X, so instead one uses "twisted functions". These are something like locally defined functions which "patch together". For example, there are no non-constant holomorphic functions on $\mathbb{P}^1$, but $p \in \mathbb{C}_k[x_0, x_1]$ defines a map p/x_0^k on U_0 and p/x_1^k on U_1. Here $\mathbb{C}_k$ denotes homogeneous polynomials. More precisely, our patching together will corresponding to having global sections in a certain line bundle.

Remark 2.2.8. Every smooth manifold admits a real analytic structure, but not all smooth manifolds admit complex structures; for example, they have to be even-dimensional, but if it is not clear if there are obstructions to having complex structures on even-dimensional smooth manifolds. If there is one complex structure, then usually there are many, which leads to considering moduli spaces of such. For example, any generic polynomial p in $\mathbb{C}_4[x_0, x_1, x_2, x_3]$ determines a 2-dimensional projective manifold in $\mathbb{P}^3$. These are called $K3$-surfaces. It turns out that the manifold X_p obtained from p is diffeomorphic to a fixed smooth manifold M.

2.3 Vector bundles

Definition 2.3.1. A *complex vector bundle* is given by a continuous map $\pi : E \to X$ between Hausdorff (second countable) spaces if

a) The fiber E_p over $p \in X$ is a $\mathbb{C}$-vector space of dimension rank $r =: \mathrm{rk}(E)$ for all p.

b) For all $p \in X$ there exists an open neighbourhood $U \subseteq X$ of p and a homeomorphism $h : \pi^{-1}(U) \to U \times \mathbb{C}^r$ satisfying $H(E_p) \subseteq \{p\} \times \mathbb{C}^r$ and such that $h_p : E_p \xrightarrow{h} \{p\} \times \mathbb{C}^r \to \mathbb{C}^r$ is a vector space isomorphism.

In the definition of a vector bundle, E is called the *total space*, and X is called the *base space*. Given local trivializations $(U_\alpha, h_\alpha), (U_\beta, h_\beta)$, we can form homeomorphisms $h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$, and we view these as maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$, called *transition functions*. These will determine the patching together mentioned in the previous section.

The transitions functions satisfy the following properties:

$$\begin{aligned} g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} &= \mathrm{Id}, \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma, \\ g_{\alpha\alpha} &= \mathrm{Id}, \quad \text{on } U_\alpha. \end{aligned}$$

Definition 2.3.2. A *holomorphic vector bundle* over a complex manifold X is a complex vector bundle $\pi : E \to X$ such that

- E is a complex manifold,
- the map π is holomorphic,
- and the local trivializations are biholomorphisms.

Note that for a complex vector bundle $\pi : E \to X$ we have a family $\{(U_\alpha, h_\alpha)\}$ of trivializations and continuous transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$ satisfying the properties from before. We want to do a converse construction, and let $\{U_\alpha\}$ be a covering of X with a family $g_{\alpha\beta}$ as before satisfying the properties from before. Consider $\tilde{E} = \bigsqcup_\alpha U_\alpha \times \mathbb{C}^r$ with the product topology and form $E = \tilde{E}/\sim$, where $(x, v) \sim (y, w)$ if $x = y$ and $v = g_{\alpha\beta} = \beta$ for $(x, v) \in U_\alpha \times \mathbb{C}^r$ and $(y, w) \in U_\beta \times \mathbb{C}^r$.

Exercise 2.3.3. Endow E' with a structure of a complex vector bundle.

We turn now to the question of creating holomorphic bundles from transition functions. Again, let $\{U_\alpha\}$ be a covering of a complex manifold X, and assume that the maps $g_{\alpha\beta} : U\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{C})$ are holomorphic (noting that $\mathrm{GL}(r, \mathbb{C})$ is a complex manifold). Then $E' = \bigsqcup_\alpha U_\alpha \times \mathbb{C}^r/\sim$ has the structure of a holomorphic vector bundle (this is left as an exercise).

Example 2.3.4. The following are holomorphic vector bundles.

1. If X is a complex manifold, then the trivial bundle $\pi : X \times \mathbb{C}^r \to X$ is holomorphic.
2. If X is a complex n-dimensional manifold, with a family of local coordinates $h_\alpha : U_\alpha \to U'_\alpha \subseteq \mathbb{C}^n$ on $\{U_\alpha\}_{\alpha \in I}$. Now define $g_{\alpha\beta} = dh_\alpha \circ dh_\beta^{-1}$. This defines a holomorphic vector bundle TX called the *tangent bundle*.

Definition 2.3.5. Let $\pi : E \to X$ be a holomorphic vector bundle. Then for $U \subseteq X$ we define a bundle $E|_U \to U$ by restricting the bundle E to $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$.

Definition 2.3.6. For holomorphic vector bundles $E \to X$, $F \to X$, a holomorphic map $\varphi : E \to F$ is called a *homomorphism of holomorphic bundles* if

1. $\varphi(E_p) \subseteq F_p$ for all $p \in X$,
2. $\varphi_p : E_p \to F_p$ is a homomorphism of complex vector spaces.

The map is called an *isomorphism* if furthermore the φ_p are isomorphisms.

Remark 2.3.7. We will typically be interested in isomorphism classes of vector bundles. In particular, moduli constructions parametrize vector bundles up to isomorphism.

Example 2.3.8. We give another description of the tangent bundle TX. Let again X be an n-dimensional complex manifold. Let $\mathcal{O}_{X,p} = \varinjlim_{U \ni p} \mathcal{O}_X(U)$, which is an algebra over $\mathbb{C}$. In other words,

$$\mathcal{O}_{X,p} = \{f \in \mathcal{O}_X(U) : p \in U\}/\sim,$$

where $f_1 \sim f_2$ if there is an open neighbourhood $V \subseteq U_1 \cap U_2$, $p \in V$, such that $f_1|_V = f_2|_V$. The class $[f]$ is called the germ at p.

We define the fibers of the bundle TX as follows: For $p \in X$, let

$$T_pX = \{D : \mathcal{O}_{X,p} \to \mathbb{C} : D(fg) = fD(g) + gD(f), D \text{ is } \mathbb{C}\text{-linear}\}$$

be a $\mathbb{C}$-vector space. Then we let $TX = \bigsqcup_{p \in X} T_pX$ with projection $\pi : TX \to X$ mapping $D_p \mapsto p$. This defines a bundle with trivializations defined as follows: Let (U_α, h_α) be an atlas for X, so $h_\alpha : U_\alpha \to U'_\alpha \subseteq \mathbb{C}^r$. These induce isomorphisms $h^*_\alpha : \mathcal{O}_{\mathbb{C},h_\alpha(p)} \to \mathcal{O}_{X,p}$ for $p \in X$ defined by $[f] \mapsto [f \circ h_\alpha]$. We obtain dual isomorphisms $h_{\alpha*} : T_pX \to T_{h_\alpha(p)}\mathbb{C}^n$. We know that $T_{h_\alpha(p)}\mathbb{C}^n \cong \mathbb{C}^n$ with a basis given by partial derivatives $\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\}$, where $\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial y_j})$ for $z = x + iy$. Now the trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha$ are given by $D_p \mapsto (p, h_{\alpha*}(D_p))$, using the isomorphism above.

The topology on TX is defined such that $U \subseteq TX$ is open if $\psi_\alpha(U \cap \pi^{-1}(U_\alpha))$ is open, and one can check that a change of coordinates corresponds to the transition function description given above.

In this picture, the "twisted functions" from before correspond to sections of a holomorphic bundles.

Definition 2.3.9. Let $\pi : E \to X$ be a holomorphic bundles over X. A *holomorphic section* of E is a holomorphic map $s : U \to E$, $U \subseteq X$ an open subset, such that $\pi \circ s = \text{id}$. The space of holomorphic sections on U is denoted $\mathcal{O}_E(U)$, and for reasons that will be clear later, we write $H^0(X, E) = \mathcal{O}_E(X)$.

Example 2.3.10. Consider the trivial bundle $X \times \mathbb{C}^r$. Then $\mathcal{O}_{X \times \mathbb{C}^r}(U) = \mathcal{O}_X(U)$, the space of holomorphic functions on U. This is the reason for viewing sections as twisted functions. If X is compact and connected, $\mathcal{O}_{X \times \mathbb{C}^r}(X) = \mathcal{O}_X(X) = \mathbb{C}$.

Holomorphic sections of TX are called holomorphic vector fields.

Remark 2.3.11. A global section $s \in \mathcal{O}_E(X)$ is equivalent to the following data: A collection of holomorphic maps $s_\alpha : U_\alpha \to \mathbb{C}^r$, $\{U_\alpha\}_{\alpha \in I}$ an open cover of X with transition functions $g_{\alpha\beta}$, satisfying $g_{\alpha\beta} \cdot s_\beta = s_\alpha$ on $U_\alpha \cap U_\beta$.

Exercise 2.3.12. With a description of a section s of E as in the above remark, construct a section of $E' = (\bigsqcup_\alpha U_\alpha \times \mathbb{C}^r)/\sim$. Show that E' and E are isomorphic.

4th lecture, September 1st 2011

Recall from last time that for a holomorphic line bundle $\pi : E \to X$ (i.e. a complex manifold locally looking like $U \times \mathbb{C}^r$, $U \subset X$), we obtained holomorphic transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{GL}(\mathbb{C}^r)$, for open subsets $\{U_\alpha\}_{\alpha \in I}$. From these we could recover the line bundle through the construction $E' = (\bigsqcup_\alpha (U_\alpha \times \mathbb{C}^r))/\sim \cong E$. We considered global holomorphic sections $s : X \to E$ (i.e. holomorphic maps satisfying $\pi \circ s = \text{Id}$), which in the E' description were given by holomorphic maps $\{s_\alpha : U_\alpha \to \mathbb{C}^r\}$ such that $g_{\alpha\beta}s_\beta = s_\alpha$.

We consider now various bundle constructions. The idea is that bundles inherit properties from vector spaces. For vector spaces V, W over $\mathbb{C}$, we can construct new vector spaces $V \oplus W$, $V \otimes W$, V^*, $\wedge^k V$, $S^k V$ and so on. These have counterparts for holomorphic (or topological complex) bundles. I.e. for bundles E and F, we can form $E \oplus F$, $E \otimes F$, and so on.

The global description of these new bundles are as follows: For example,

$$E \oplus F := \bigsqcup_{p \in X} E_p \oplus F_p.$$

The projection to X is simply given by $\pi(v_p \oplus w_p) = p$. One can then use trivializations of E and F to construct the holomorphic structure. Namely, define

$$h_\alpha^{E \oplus F} : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r \times \mathbb{C}^k$$

by $h_\alpha^{E\oplus F} = h_\alpha^E \oplus h_\alpha^F$. Similarly, the local structure is given by the transitions functions $g_{\alpha\beta}^{E\oplus F} = g_{\alpha\beta}^E \oplus g_{\alpha\beta}^F$. Similarly, we could construct tensor products of bundles and so on.

For the dual bundle E^* of E, we do the following: In the E' description, we simply let $g_{\alpha\beta}^{E^*} = ((g_{\alpha\beta}^E)^{-1})^T$. We know there is a pairing $E^* \times E \to \mathbb{C}$ mapping $(v_p, w_p) \mapsto \langle v_p, w_p \rangle$. Now, take a section $\{s_\alpha\}_{\alpha \in I}$ of E given in the local description, so $s_\alpha : U_\alpha \to \mathbb{C}^r$ and take a section $\{w_\alpha\}_{\alpha \in I}$ of E^*. Then

$$\langle w, s \rangle|_p = w_\alpha^T|_p \cdot s_\alpha|_p = ((g_{\alpha\beta}^T)^{-1} w_\beta)^T|_p \cdot (g_{\alpha\beta} s_\beta)|_p = w_\beta^T|_p \cdot s_\beta|_p.$$

We will now see how the bundle and its dual may be different in the holomorphic category.

Remark 2.3.13. On the tangent bundle TM, as a smooth bundle, with M a smooth manifold, any Riemannian metric g defines an isomorphism $TM \to T^*M = (TM)^*$. However, holomorphic bundles do not behave in this way. If E is a holomorphic bundle which has "many" global sections and is "very non-trivial", then E^* has no global sections, and so the bundles can not be isomorphic.

Let us make this precise.

Definition 2.3.14. A *subbundle* $F \subseteq E$ is a complex submanifold of E such that

a) the fibers $F \cap E_p$ is a vector subspace of E_p, and

b) the projection $\pi|_F : F \to X$ has the structure of a holomorphic bundle which is compatible with that on E in the following sense: For every point $p \in X$ there is a trivialization of E and F such that the diagram below is commutative.

$$\begin{array}{ccc} E|_U & \xrightarrow{\sim} & U \times \mathbb{C}^r \\ \uparrow & & \uparrow {\scriptstyle \mathrm{id} \times \mathrm{injection}} \\ F|_U & \xrightarrow{\sim} & U \times \mathbb{C}^k \end{array}$$

Now, we think of a very non-trivial bundle as one with no trivial subbundles of rank greater than 0, and the remark becomes the following.

Proposition 2.3.15. *Assume that X is compact. If E is generated by global sections (E is ggs, see below), and it has no trivial subbundles, then E^* has no non-zero global sections.*

Definition 2.3.16. A bundle E is *ggs* if there exist sections $\{s_j\}_{j=1}^N \subseteq \mathcal{O}_E(X)$ such that for all $p \in X$, the fiber E_p is generated by the $s_j|_p$.

Proof of Proposition. Let $s^* \in \mathcal{O}_{E^*}(X)$. Then since X is compact

$$\langle s^*, s_j \rangle \in \mathcal{O}_X(X) \cong \mathbb{C}.$$

Since E has no trivial subbundles, s_j must vanish somewhere on X (since otherwise, s_j would determine a rank 1 subbundle). This implies that $\langle s^*, s_j \rangle = 0$for all j, and so $0 = s^*|_p \in E_p^*$ for all $p \in X$, since the s_j generated E_p. Thus $s^* = 0$. ☐

The reason for being interested in questions as the above is that having a bundle being generated by global sections defines a map into a Grassmannian (and we are trying to find out when manifold embed into projective space). Namely, if E is ggs, then $\{s_j\}_{j=1}^N \subseteq \mathcal{O}_E(X)$ induces a holomorphic map $\varphi_s : X \to G_r(\mathbb{C}^N)$, where $r = \operatorname{rank} E$, constructed as follows:

First, the Grassmannian is $G_r(\mathbb{C}^N) = M_{r,n}^r / \sim$, where $\sim$ denotes the action of $\mathrm{GL}(\mathbb{C}^r)$. Now, for $x \in X$ define

$$\varphi_s(x) = [(s_{1\alpha}|_x, \dots, s_{N\alpha}|_x)],$$

where $s_{j\alpha} \in \mathbb{C}^r$ is a local expression of s_j on a trivializing set U_α. This is well-defined, which can also be seen as follows: Another description of the Grassmannian is given by the identification of $G_r(\mathbb{C}^N)$ with surjective maps $\varphi : \mathbb{C}^N \to \mathbb{C}^r$ modulo the action of $\mathrm{GL}(\mathbb{C}^r)$, i.e. modulo the choice of basis in $\mathbb{C}^r$. In this description, $\varphi_s(x)(v_1, \dots, v_n) = \sum v_j \cdot s_j|_x \in E_x$, which is clearly well-defined, and the map $\psi_s(x)$ is surjective since E is ggs. The first description tells us that φ_s is holomorphic.

Example 2.3.17. We consider the *universal bundle over* $G_r(\mathbb{C}^n)$. In the first description, this is

$$U_r = \{(v, [A]) : A \in M_{r,n}^r, v \in \langle a_1^T, \dots, a_r^T \rangle \subseteq \mathbb{C}^r \times G_r(\mathbb{C}^n)\}.$$

Here, the a_j are the rows of A. A trivialization of U_r is given on

$$U_\alpha = \{[A] : \det A_\alpha \neq 0\}$$

by $U_r|_{U_\alpha} \to U_\alpha \times \mathbb{C}^r$ mapping

$$(v, [A]) \mapsto ([A], A_\alpha^T (AA^T)^{-1} Av).$$

This means that if $v \in \langle a_1^T, \dots, a_n^T \rangle$, then v has the coordinate expression $v = A^T \lambda$ in terms of the basis $\{a_j\}$. Here, $\lambda = (AA^T)^{-1} Av$, and in order to obtain a well-defined element of $\mathbb{C}^r$, we multiply by A_α^T in the expression above.

In this example, the transition functions are given by

$$g_{\alpha\beta} = A_\alpha^T (A_\beta^T)^{-1}.$$

Example 2.3.18. Using the second description of $G_r(\mathbb{C}^n)$, we define the *universal quotient bundle* $Q_r \to G_r(\mathbb{C}^n)$ with fibers

$$Q_r|_{[\varphi]} = \mathbb{C}^n / \ker \varphi,$$

for a surjective map $\varphi : \mathbb{C}^n \to \mathbb{C}^r$. This is well-defined since $\ker \varphi$ does not depend on the basis. To obtain a trivialization, we choose bases to identify $\varphi = A$. We then define $Q_r|_{U_\alpha} \to U_\alpha \times \mathbb{C}^r$ by letting $([v]|_{[A]}) \mapsto ([A], A_\alpha^{-1} Av)$ (which is well-defined as before).

Here, transition functions are

$$g_{\alpha\beta} = A_\alpha^{-1} A_\beta.$$

So, by the previous description of transition functions of dual bundles, we see that $U_r \cong Q_r^*$. We have a pairing

$$\langle w|_{[A]}, (v, [A]) \rangle = w^T A^T (AA^T)^{-1} Av$$

for $w|_{[A]} \in Q_r|_{[A]}$, $(v, [A]) \in U_r|_{[A]}$. This defines a map $Q_r \to U_r^*$ mapping $w \mapsto \langle w, \cdot \rangle$ that turns out to be an isomorphism (this is left as an exercise).

Observe that Q_r is generated by global sections. To see this, consider the map $\mathbb{C}^n \to \mathcal{O}_{Q_r}(G_r(\mathbb{C}^n))$ mapping $w \mapsto (s : G_r(\mathbb{C}) \to Q_r)$, where s is given by $s([\varphi]) \to [w] \in \mathbb{C}^n / \ker \varphi$. For any choice of basis, $\{w_j\}$ of $\mathbb{C}^n$, we can generate $\mathbb{C}^n / \ker \varphi$ for all φ, which implies that Q_r is ggs.

Remark 2.3.19. If Q_r has no non-trivial subbundles, then it follows from the Proposition above that U_r has no non-zero global sections. Let us see that this is true for $r = 1$. Note that $G_1(\mathbb{C}^n) = \mathbb{P}^{n-1}$. Consider now $Q_1^{\otimes m}$. We use the following notation: $Q_1 =: \mathcal{O}_{\mathbb{P}^{n-1}}(1)$, $Q_1^{\otimes m} =: \mathcal{O}_{\mathbb{P}^{n-1}}(m)$ for $m \in \mathbb{Z}$ where by definition, $Q_1^{\otimes -m} = (Q_1^{\otimes m})^*$. Let $k = n-1$. We want to compute $\mathcal{O}_{Q_1^{\otimes m}}(\mathbb{P}^k)$. For $m > 0$, we have transition functions

$$g_{\alpha\beta}([x_0 : \cdots : x_k]) = x_\beta^m / x_\alpha^m,$$

where we use trivializations $U_\alpha = \{[x_0 : \cdots : x_k] : x_\alpha \neq 0\}$. This is because transition functions of tensor products are simply tensor products of transition functions, which correspond to products in the 1-dimensional case. Recall that a global section is $s_\alpha : U_\alpha = \{(x_0/x_\alpha, \ldots, x_k/x_\alpha) \to \mathbb{C}$ satisfying $g_{\alpha\beta} s_\beta = s_\alpha$. This is satisfied if and only if

$$x_\beta^m s_\beta(x_0/x_\beta, \ldots, x_k/x_\beta) = x_\alpha^m s_\alpha(x_0/x_\alpha, \ldots, x_k/x_\alpha).$$

Thus, $p_\alpha = s_\alpha(x_0/x_\alpha, \ldots, x_k/x_\alpha) x_\alpha^m$ is a homogeneous polynomial of degree m which does not depend on α by the relation above. Here, that p is *homogeneous* means that

$$p(\lambda x_0, \ldots, \lambda x_m) = \lambda^m p(x_0, \ldots, x_k).$$

Therefore we have an isomorphism

$$\mathcal{O}_{Q_1^{\otimes m}}(\mathbb{P}^k) \xrightarrow{\cong} \mathbb{C}_m[x_0, \ldots, x_k].$$

The case $m = 0$ is trivial, since the bundle $Q_1^{\otimes m}$ is the trivial bundle by definition, and global sections are identified with the complex numbers.

For $m < 0$, if one tries to copy the argument for $m > 0$, one finds that "global sections" have poles (so they are not globally defined).

Let us check that for $m > 0$, global sections of $Q_1^{\otimes m}$ induce a map in $G_1(\mathbb{C}^N)$ to $\mathbb{P}^M$ (where N is the number of global sections we choose to generate the fibers). Choose a basis of $\mathbb{C}_m[x_0, \ldots, x_k]$ given by $\{x_0^{m_0} \cdots x_k^{m_k} : \sum m_i = m\}$. As before, this induces a map $\varphi : \mathbb{P}^k \to \mathbb{P}^M$ given by

$$\varphi : [x_0 : \cdots : X_n] \to [x_0^m : x_0^{m-1} x_1 : X_0^{m-2} x_1^2 : \ldots].$$

We claim that this is an embedding. We have coordinates given by $x_0^{m_0} \cdots x_k^{m_k} = Z_{m_0 \ldots m_k}$. The image of φ is given by zeros of polynomials,

$$Z_{i_0 \ldots i_k} Z_{j_0 \ldots j_k} - Z_{m_0 \ldots z_k} Z_{l_0 \ldots l_k} = 0,$$

where $i_h + j_h = m_h + l_h$ for all h. That the image coincides with the zero locus is left as an exercise.

To see that φ is an embedding, define $V_i \subseteq \mathbb{P}^m$ by

$$V_i = \{[\cdots : Z_{m_0 \ldots z_k} : \cdots] : Z_{0 \ldots 0 m 0 \ldots 0} \neq 0\},$$

where the m is at the i'th place. Note now that $\varphi(U_i) \subseteq V_i$ and $\varphi(\mathbb{P}^k) \subseteq \bigcup_i V_i$. We can define an inverse of φ over V_i. On V_0, it maps

$$[\cdots : Z_{m_0 \ldots m_k} : \cdots] \mapsto [Z_{m0\ldots 0}, Z_{(m-1)10\ldots 0}, Z_{(m-1)010\ldots 0}] = [x_0^m, x_0^{m-1} x_1, x_0^{m-1} x_2].$$

The final thing to check is that these partial inverses can be glued together over the image of φ by the equations to give us a biholomorphism from $\mathbb{P}^k$ to its image under φ.

5th lecture, September 5th 2011

We begin today with a discussion of pullback bundles. Recall from last time that we discussed how bundles over complex manifolds generated by global sections give rise to maps from the manifold to the Grassmannian $G_r(\mathbb{C}^n)$.

We ask the following question: Given a holomorphic map $\varphi : X \to G_r(\mathbb{C}^N)$, can we recover the sections that induce φ?

Definition 2.3.20. Let $\check{f} : X \to Y$ be a complex map. A *holomorphic morphism* $f : E \to F$ of bundles $E \to X$ FX is a holomorphic map $f : E \to F$, such that fibers are taken isomorphically to fibers.

Note that such a map f induces $\check{f} : X \to Y$ by $\check{f}(\pi_E(e)) = \pi_F(f(e))$. Let $0_E : X \to E$ be the zero section. Then $\check{f}(x) = f(0_E(x))$. Identify $Y \cong 0_F(Y)$. Then $0_F : Y \to F$ is an embedding.

Proposition 2.3.21. *Given a holomorphic map $\check{f} : X \to Y$ and a holomorphic bundle $E \to Y$. Then there exists a holomorphic bundle $E' \to X$ with a holomorphic morphism $f : E' \to E$ so that $\pi_E \circ f = \check{f} \circ \pi_{E'}$.*

Furthermore, this bundle is unique up to isomorphism

Proof. Let $E' = \{(x, e) \in X \times E : \pi(e) = \check{f}(x)\}$. The morphism $g : E' \to E$ maps $(x, e) \mapsto e$ and the projection is $\pi_{E'} : E' \to X$ mapping $(x, e) \mapsto x$. Let (U, h) be a trivialization of E. Then

$$E'|_{\check{f}^{-1}(U)} \to \check{f}^{-1}(U) \times \mathbb{C}^r$$

mapping $(x, e) \mapsto (x, \pi_{\mathbb{C}^r}(h(e)))$ defines a trivialization, and we obtain transition functions on E', which we can check satisfies the properties of the Proposition. □

Remark 2.3.22. 1. All of this holds for topological and differentiable bundles as well.

2. We have the following theorem: If $E \to M$ is a real rank r differentiable bundle over a compact differentiable manifold, there is $N > 0$ and a embedding $\varphi : M \to G_r(\mathbb{R}^N)$ such that $\varphi^* Q_r \cong E$.

3. This is not true for holomorphic bundles. The reason is that pullbacks have the nice property of preserving the property of being generated by global sections, which we know not all bundles have. More precisely, given $E \to Y$ and $f : X \to Y$ as before, then we have a homomorphism of $\mathbb{C}$-vector spaces $f^* : \mathcal{O}_E(Y) \to \mathcal{O}_{f^*E}(X)$ which maps $s \mapsto f^*s$, where $f^*s : X \to F^*E$ maps $x \mapsto (x, s(f(x)))$. In general this may not be injective nor surjective, but we have the following:

Fact 2.3.23. *If E is ggs, then f^*E is ggs. Moreover, if $\{s_j\}_{j=1}^N$ generate E, then $\{f^*s_j\}_{j=1}^N$ generate f^*E.*

Proof. If $x \in X$, then $E_{f(x)} = \langle s_j|_{f(x)} \rangle$ if and only if $f^*E|_x = \langle f^*s_j|_x \rangle$. □

Example 2.3.24. For $m > 0$, $\mathcal{O}_{\mathbb{P}^n}(-m)$, as defined in the previous lecture, has no non-zero global zero sections. Thus it can not be the pullback of $\mathcal{O}_{\mathbb{P}^N}(1)$ over $\mathbb{P}^N$, since $\mathcal{O}_{\mathbb{P}^n}$ is ggs.

Proposition 2.3.25. *Let E be ggs and let $\varphi : X \to G_r(\mathbb{C}^N)$ be the map induced by sections $\{s_j\}_{j=1}^N \subseteq \mathcal{O}_E(X)$. Then $\varphi^* Q_r \cong E$.*

Proof. We do the case $r = 1$, so $\varphi : X \to \mathbb{P}(\mathbb{C}^N)$, which by definition is given by $x \mapsto [s_1(x) : \cdots : s_N(x)]$, where $[x_1 : \cdots : x_n]$ are the homogeneous coordinates on $\mathbb{P}(\mathbb{C}^N)$. Define $V_j = \{x \in X : s_j(x) \neq 0\}$. Then $X = \cup_j V_j$ (since if not, there is an $x \in X$ with $s_j(x) = 0$ for all j so E is not ggs). A trivialization of E is $h : E|_{V_j} \to V_j \times \mathbb{C}$ given by $v_x \mapsto (x, v_x/s_j(x))$. The transition functions are given by

$$g_{ji}(x)v = \frac{s_i(x)}{s_j(x)} v$$

To prove that $\varphi^* Q_r \cong E$, we just need to check that they have the same transition functions, so we compute those of $\varphi^* Q_1$. These are $g_{ij}^{\varphi^* Q_1} : \varphi^{-1}(U_i) \cap \varphi^{-1}(U_j) \to \mathbb{C}^*$. Note that by construction $\varphi^{-1}(U_i) = V_i$, and

$$g_{ji}^{\varphi^* Q_1} = \left(\frac{x_i}{x_j} \circ \varphi \right) = \frac{s_i(x)}{s_j(x)} \in \mathbb{C}^*.$$

□

Remark 2.3.26. The sections $\{s_j\}_{j=1}^N$ are pullbacks of the socalled *hyperplane sections* $\{x_j\}_{j=1}^N \subseteq \mathbb{C}[x_1,\ldots,x_N] \cong \mathcal{O}_{Q_1}(\mathbb{P}(\mathbb{C}^N))$. In conclusion, holomorphic maps $\varphi : X \to G_r(\mathbb{C}^N)$ correspond bijectively to bundles $E \to X$ globally generated by sections $\{s_j\}_{j=1}^N$.

Recall at this point that what we are looking for are holomorphic *embeddings* into $G_r(\mathbb{C}^N)$. In the vector bundle picture, these maps correspond to so-called ample line bundles.

Example 2.3.27 (The red carpet of embeddings 2011)**.** The following are examples of embeddings and the corresponding bundles.

1. The Veronese embedding $\mathbb{P}^N \hookrightarrow \mathbb{P}^{\binom{n+m}{n}-1}$ corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^n}(m)$, $m > 0$.
2. The Segre embedding of the first exercise $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \to \mathbb{P}^{n_1 n_2 + n_1 + n_2}$ corresponds to the line bundle $\pi_1^*\mathcal{O}_{\mathbb{P}^{n_1}}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^{n_2}}(1)$, where $\pi_i : \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \to \mathbb{P}^{n_1}$ are projections.
3. The *Plucker embedding* $\mathrm{Pl} : G_r(\mathbb{C}^n) \to \mathbb{P}^{\binom{n}{r}-1} = \mathbb{P}(\wedge^r\mathbb{C}^n)$ mapping
$$[\varphi : \mathbb{C}^n \twoheadrightarrow \mathbb{C}^r] \mapsto [\wedge^r\varphi : \wedge^r\mathbb{C}^n \twoheadrightarrow \wedge^r\mathbb{C}^r \cong \mathbb{C}].$$
Here, $\wedge^r\varphi$ maps $v_1 \wedge \cdots \wedge v_r \mapsto \varphi(v_1) \wedge \cdots \wedge \varphi(v_r)$. Recall that we have a map $\mathbb{C}^n \to \mathcal{O}_{Q_r}(G_r(\mathbb{C}^n))$ which maps $w \mapsto s_w = [w] \in \mathbb{C}^n/\ker\varphi$, where $G_r \ni [\varphi] \mapsto [w]$. Recall that Q_r is ggs. The image of the map $\wedge^r\mathbb{C}^n \to \mathcal{O}_{\wedge^r Q_r}(G_r(\mathbb{C}^n))$ mapping
$$w_1 \wedge \cdots \wedge w_n \mapsto s_{w_1\wedge\cdots\wedge w_r} = [w_1] \wedge \cdots \wedge [w_r]$$
generates $\det Q_r := \wedge^r Q_r$ and induces a map
$$\mathrm{Pl} : G_r(\mathbb{C}^n) \to \mathbb{P}(\wedge^r\mathbb{C}^n)$$
mapping a map $[\varphi]$ to the map $\wedge^r\mathbb{C}^n \to \det Q|_{[\varphi]}$ mapping $w \mapsto s_w|_{[\varphi]}$. This map Pl coincides with the map Pl given previously, which is therefore well-defined.

Exercise 2.3.28. Write up the last example in matrix notation; see the equations for the image of Pl given by [Mil11, p. 114].

3 Detour on GIT

GIT stands for *Geometric Invariant Theory* and involves the 15th problem off Hilberts problem list. It has to do with rings and invariants by representations. In modern algebraic geometry, rings can be viewed as algebraic varieties or schemes. To understand this, we give a brief introduction to Lie groups and actions and try to understand how to parametrize orbits on a projective manifold by the holomorphic action of a complex Lie group (see Fig. 1). The main reference for our discussion on Lie groups is [War83]. For a discussion of Lie group actions, see [Kna02], and for a discussion on GIT quotients, see [Tho05] (further references on geometric invariant theory are [Dol03] and [MFK94]).

3.1 From projective manifolds to rings and back

Our first goal is to understand relation between rings and algebraic varieties and schemes. Let $i : X \hookrightarrow \mathbb{P}(\mathbb{C}^N)$ be a projective manifold. Then X determines a ring R which is finitely generated, has no zero divisors and is graded. Define
$$\mathcal{O}_X(k) = i^*\mathcal{O}_{\mathbb{P}^N}(k) = i^*\mathcal{O}_{\mathbb{P}(\mathbb{C}^N)}(1)^{\otimes k},$$
$$R = \oplus_{k\geq 0}\mathcal{O}_{\mathcal{O}_X(k)}(X) = \oplus_{k\geq 0}H^0(X, \mathcal{O}_X(k)),$$
noting we have a multiplication $H^0(X,\mathcal{O}_X(k)) \otimes H^0(X,\mathcal{O}_X(m)) \to H^0(X,\mathcal{O}_X(k+m))$. This ring is graded by k, and it is finitely generated: Take, by Chow's theorem, polynomials $\{p_j\}_{j=1}^k$, $\deg p_j = d_j$ homogeneous polynomials (but see the first comment of Lecture 6), such that
$$X = \{x \in \mathbb{P}(\mathbb{C}^N) : p_j(x) = 0 \text{ for all } x \in X\}.$$

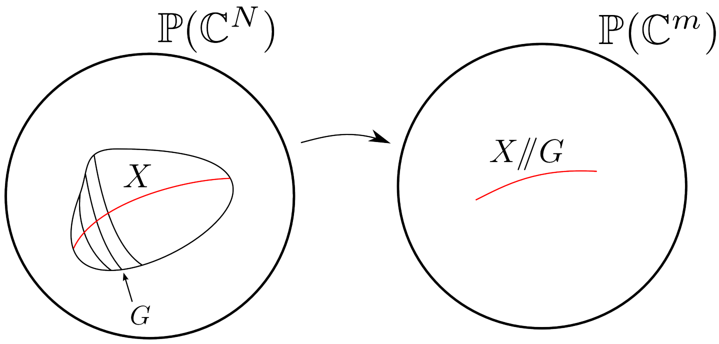

Figure 1: A projective manifold X in $\mathbb{P}(\mathbb{C}^N)$ with an action of a Lie group G – the red line symbolizes a parametrization of the orbits of the action.

Claim 3.1.1. *The ring is* $R = \mathbb{C}[x_1, \ldots, x_N]/\langle p_1, \ldots, p_k\rangle =: R'$.

If this is true, then clearly R is finitely generated, and it has no zero divisors (since if it had, X which would be singular – this is left as an exercise).

Proof of Claim. Let $[p] \in R'$ be homogeneous of degree k. Associate to this the section $s_p \in H^0(\mathbb{P}(\mathbb{C}^N), \mathcal{O}_{\mathbb{P}(\mathbb{C}^N)}(k))$. Pulling back, we get an element $i^* s_p \in H^0(X, \mathcal{O}_X(k))$ which depends only on the class of p.

Conversely, if $s \in H^0(X, \mathcal{O}_X(1))$, view s as a holomorphic map $s : \mathcal{O}_X(-1) \to \mathbb{C}$, linear on fibers, where

$$\mathcal{O}_X(-1) = \{(v, x) \in \mathbb{C}^N \times X : [v] = X\} \subseteq \mathbb{C}^N \times \mathbb{P}(\mathbb{C}^N).$$

By definition, s gives a holomorphic map $f : U \to \mathbb{C}$, $\mathcal{O}_X(-1) \subseteq U \subseteq \mathbb{C}^N \times \mathbb{P}(\mathbb{C}^N)$, such that $f|_{\mathcal{O}_X(-1)} = s$. Then for $x \in X$,

$$s(v, x) = s(\lambda v, x)/\lambda = \lim_{\lambda \to 0} \frac{f(\lambda x, x')}{\lambda} = df|_{(0,x)}(v) = \sum \lambda_i v_i,$$

where λ_i are holomorphic on the zero section, so by compactness of X, the λ_i are constant. Thus $s = \sum \lambda_i i^* X_i$, where the X_i are the hyperplane sections. Now, to s associate $[\lambda_i X_i] \in R'$, and extend this construction to higher degree pieces.

It is easy to check that these two associations induce an isomorphism of the full graded rings. □

So, to a projective manifold X, we have associated a ring R_X. To invert this map, we need something more general than projective manifolds; namely projective algebraic varieties. To give some idea about how this goes, let R be a finitely generated ring with no zero divisors. Pick generators $e_1, \ldots, e_N$ of R. Consider the map $\nu : \mathbb{C}[x_1, \ldots, x_N] \to R$ given by $x_j \mapsto e_j$. The kernel of this map $\ker \nu \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is a graded ideal. The Hilbert basis theorem implies that graded ideals of $\mathbb{C}[x_1, \ldots, x_n]$ are finitely generated, and we pick $p_1, \ldots, p_k$ of $\ker \nu$, which can be taken to be homogeneous, since the ideal is graded. Now, let

$$X_R = \{x \in \mathbb{P}(\mathbb{C}^N) : p_j(x) = 0,\ j = 1, \ldots k\}.$$

Then $R_{X_R} = R$. Note however that for general R, X_R is singular; an example could be $x_1^2 + x_2^2 + x_3^2 = 0$ on $\mathbb{P}(\mathbb{C}^4)$. I.e. X_R is not necessarily a manifold but a projective variety.

In working with GIT, we will only consider the rings R_X. Therefore, the theory holds for things more general than projective manifolds.

In the next lecture, we will be considering groups that further have a complex (or differentiable) structure, and we will consider representations of such, $\rho : G \to \mathrm{GL}(\mathbb{C}^N)$. Whenever we have such a thing, we also have an induced "action" of G on $\mathbb{P}(\mathbb{C}^N)$, i.e. a map $G \times \mathbb{P}(\mathbb{C}^N) \to \mathbb{P}(\mathbb{C}^N)$. This will induce "actions" on complex submanifolds of $\mathbb{P}(\mathbb{C}^N)$, and our goal is to parametrize the orbits of X to obtain the so-called GIT quotient $X /\!\!/ G$. The idea is to assocate to invariant elements R_X^G of R_X with a suitable action of G, to obtain a projective manifold $X_{R_X^G}$.

6th lecture, September 7th 2011

We begin with some comments about yesterday's lecture. Recall that for a projective manifold $X \subseteq \mathbb{P}(\mathbb{C}^N)$, we defined a ring $R = \oplus_{k \geq 0} H^0(X, \mathcal{O}_X(k))$, which turns out to be finitely generated, to have no zero divisors and to be graded. In the proof of this, we used that $R \cong R' = \mathbb{C}[x_1, \dots, x_N]/\langle p_1, \dots, p_k \rangle$. The p_j are homogeneous polynomials given by Chow's theorem, which tells us that

$$X = \{x \in \mathbb{P}(\mathbb{C}^N) : p_j(x) = 0\}.$$

We do in fact need to assume that the equations $p_j = 0$ to have maximal rank away from 0. This is what allows us to use the implicit function theorem. For example, if we changed $p_1 = 0, \dots, p_k = 0$ to $p_1^2 = 0, \dots, p_k^2 = 0$, we would not change the manifold, but it would change the ring to one with zero divisors. So, alternatively, we could have defined $I_X = \{p \in \mathbb{C}[x_1, \dots, x_N] : p(x) = 0, \forall x \in X\}$ and then defined $R = \mathbb{C}[x_1, \dots, x_n]/I_X$.

Remark 3.1.2. The I_X defined above is the radical $I_X = \operatorname{rad}\langle p_1, \dots, p_k \rangle$. Chow's theorem implies that $I_X \neq \emptyset$, and that we can recover the manifold X as $X = X_{I_X} := \{x \in \mathbb{P}(\mathbb{C}^N) : p(x) = 0, \forall p \in I_X\}$. A proof of this statement can be found in [Mil11].

3.2 Lie groups

We turn now to Lie groups and actions. We want to parametrize orbits on complex manifolds.

Definition 3.2.1. A *topological group* G is a group endowed with a topology such that the multiplication $(g, h) \mapsto gh$ and inversion $g \mapsto g^{-1}$ are continuous operations.

Definition 3.2.2. A differentiable resp. complex *Lie group* is a topological group endowed with a differentiable resp. holomorphic structure, such that multiplication and inverstion are differentiable resp. holomorphic maps.

Example 3.2.3. Examples of Lie gruops are the groups of matrices over either $\mathbb{R}$ or $\mathbb{C}$ with the topology induced from $\mathbb{C}^{n^2}$. For instance

$$\mathrm{GL}(n, \mathbb{C}) = \{A \in M_{n \times n}(\mathbb{C}) : \det A \neq 0\}$$

and similarly $\mathrm{GL}(n, \mathbb{R})$ are Lie groups. Similarly

$$\mathrm{SL}(n, \mathbb{C}) = \{A \in \mathrm{GL}(n, \mathbb{C}) : \det A = 1\}$$

is a complex Lie group. In these examples, the Lie bracket corresponds to $[A, B] = AB - BA$ for matrices A, B.

Denote by $L_g : G \to G$ the map $h \mapsto gh$ and by $R_g : G \to G$ the map $h \mapsto hg$. Let $\operatorname{Lie} G$ be the space of *left-invariant vector fields*,

$$\operatorname{Lie} G = \{y \text{ vector fields on } G : L_g^* y = y \forall g \in G\}.$$

This is a Lie algebra (see [War83]) over $\mathbb{R}$ or $\mathbb{C}$, inheriting a Lie algebra structure from the vector field Lie bracket on G.

Proposition 3.2.4. *We can identify* $\operatorname{Lie} G \cong T_e G$*, where* e *is the neutral element in* G*.*

Proof. Define a map $T_e G \to \operatorname{Lie} G$ by letting $\xi \mapsto (y_\xi : G \to TG, g \mapsto (L_g)_*)$. This has an inverse $y \mapsto y|_e$, and the maps are isomorphisms (exercise). $\square$

The *exponential map* $\exp : \operatorname{Lie} G \to G$ maps $y \mapsto \varphi_y^1(e)$, where φ_y^t is the flow of y. Remark that it is not obvious that φ_y^t exists for $t = 1$, since G may not be compact. The exponential is a diffeomorphism or biholomorphism on a neighbourhood of $e \in G$ which follows from the fact that $\mathrm{Id} = d\exp|_e : \operatorname{Lie} G \to \operatorname{Lie} G$.

3.3 Lie group actions

Lie group actions intuitively describe symmetries of objects. Let X be a differentiable or complex manifold.

Definition 3.3.1. An *action* of a topological group G in a topological space M is a continuous map $G \times M \to M$ mapping $(g, x) \mapsto g \circ x$ satisfying the following properties:

1. For all $g_1, g_2 \in G$, $p \in M$, we have $(g_1 g_2) \circ p = g_1 \circ (g_2 \circ p)$.
2. For all $p \in M$, $e \circ p = p$.

The idea of is roughly the following: An action is a parametrization by M of copies of G inside M by looking on the image of a group elements under the map $G \times \{x\} \to M$ for a point $x \in M$. In the following, we will write $g \cdot p = g \circ p$.

Definition 3.3.2. The *orbit* of p by G is the set $G \cdot p = \{g \cdot p : g \in G\}$.

Definition 3.3.3. The *isotropy group* of $p \in M$ is the subgroup $G_p \subseteq G$ given by

$$G_p = \{g \in G : g \cdot p = p\}.$$

The isotropy group is exactly the obstruction to the action parametrizing copies of G in M. More precisely, we can identify orbits as $G \cdot p \cong G/G_p$, so we are really parametrizing copies of G and quotients of G.

3.4 Quotients by actions

Let $G \times M \to M$ be an action. We define the *topological quotient* $G\backslash M$ to be the quotient topological space $M/\sim$ where $p_1 \sim p_2$ if and only if there is an element $g \in G$ such that $g \cdot p_1 = p_2$. In other words, if $p_1 \in G \cdot p_2$.

One finds various problems with this: One is that typically complex groups are non-compact, which means that orbits may have lower dimensional orbits on their closures, and the quotient $G\backslash M$ is not necessarily Hausdorff. Thus we need another notion of a quotient by an action; the GIT quotient is a machinery that tells us which points to remove from the space M to make sure the quotient spaces are in fact manifolds.

7th lecture, September 13th 2011

Last time, we introduced Lie groups (i.e. topological groups with either a differentiable or complex structure compatible with the group structure), and for a Lie group G, we defined the Lie algebra $\operatorname{Lie} G$ as the left-invariant vector fields on G. We introduced the exponential map $\exp : \operatorname{Lie} G \to G$, which is a diffeomorphism onto a neighbourhood of the identity. Finally, we introduced actions of topological groups on topological spaces.

Definition 3.4.1. A *Lie subgroup* $K \subseteq G$ of a (differentiable/complex) Lie group G, is a subgroup which further is a (differentiable/complex) submanifold.

Definition 3.4.2. A *Lie group homomorphism* $\rho : G_1 \to G_2$ between (differentiable/complex) Lie groups G_1 and G_2 is a group homomorphism, which is also a differentiable/holomorphic map.

Example 3.4.3. The only example we need is $\mathrm{SL}(n, \mathbb{C})$. Let

$$\mathrm{GL}(n, \mathbb{C}) = \{A \in M_{n\times n}(\mathbb{C}) : \det A \neq 1\}$$

which we have already seen is a complex manifold, and it is also a Lie group. Now

$$\mathrm{SL}(n, \mathbb{C}) = \{A \in \mathrm{GL}(n, \mathbb{C}) : \det A = 1\}$$

is a Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$. Their Lie algebras of $\mathrm{GL}(n, \mathbb{C})$ is

$$\mathrm{Lie}\,\mathrm{GL}(n, \mathbb{C}) = T_{\mathrm{Id}}\,\mathrm{GL}(n, \mathbb{C}) \equiv M_{n\times n}(\mathbb{C}).$$

To describe $\mathrm{Lie}(\mathrm{SL}(n, \mathbb{C}))$, take a curve A_t on $\mathrm{SL}(n, \mathbb{C})$ such that $A_0 = \mathrm{Id}$, $\det A_t = 1$ for all t. Then

$$0 = \frac{d}{dt}|_{t=0} \det(A_t) = \mathrm{tr}\left(\frac{d}{dt}|_{t=0} A_t\right) \in T_{\mathrm{Id}}\,\mathrm{SL}(n, \mathbb{C}),$$

and we conclude that

$$\mathrm{Lie}(\mathrm{SL}(n, \mathbb{C})) = \{A \in M_{n\times n}(\mathbb{C}) : \mathrm{tr}\, A = 0\}.$$

The exponential $\exp : \mathrm{Lie}(\mathrm{SL}(n, \mathbb{C})) \to \mathrm{SL}(n, \mathbb{C})$ is simply the restriction of the map $\exp : \mathrm{Lie}\,\mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ mapping

$$\exp(A) = \sum_{j=0}^{\infty} \frac{A^j}{j!},$$

which can be checked to be well-defined and defines an invertible matrix. For $n = 1$, $\mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$, and exp is the standard exponential.

Another important example is the differentiable (but *not* complex) Lie subgroup $\mathrm{SU}(n) \subseteq \mathrm{SL}(n, \mathbb{C})$ defined by

$$\mathrm{SU}(n) = \{A \in \mathrm{SL}(n, \mathbb{C}) : \overline{A}^T = A^{-1}\}.$$

This is not a complex subgroup, since conjugation is not holomorphic. The group has Lie algebra

$$\mathrm{Lie}\,\mathrm{SU}(n) = \{A \in \mathrm{Lie}\,\mathrm{SL}(n, \mathbb{C}) : \overline{A}^T = -A\},$$

which can be proved as above.

Definition 3.4.4. A *action* of a (differentiable/complex) Lie group G on a (differentiable/complex) manifold M is a topological action such that the map $G \times M \to M$ is differentiable/holomorphic. The action is *free* if

$$G_p = \{g \in G : gp = p\} = \{e\}$$

for all $p \in M$.

The algebraic analogue of an action is a G-module, if G were a ring and M an abelian group. Consider the differentiable action

$$\mathrm{SU}(n) \times \mathrm{SL}(n, \mathbb{C}) \to \mathrm{SL}(n, \mathbb{C})$$

maping $(A, B) \mapsto A\cdot B$, where $\cdot$ denotes matrix multiplication. The topological quotient $\mathrm{SU}(n)\backslash \mathrm{SL}(n, \mathbb{C})$ is a differentiable manifold. For this it is important that $\mathrm{SU}(n)$ is compact, as illustrated by the following proposition.

Proposition 3.4.5. *Let G be a compact differentiable Lie group with a free action $G \times M \to M$ with M a differentiable manifold. Then the topological quotient* $\mathrm{pr} : M \to G\backslash M$ *has a natural differentiable structure, making* pr *a differentiable map. We have a canonical identification*

$$T_{[p]}G\backslash M \cong T_p(G \cdot p)\backslash T_p M,$$

where $G \cdot p$ denotes the orbit of p under the action.

Idea of proof. Since G is compact, the map $G \times M \to M$ is *proper* (i.e. the pre-image of compact sets are compact), which implies that G-orbits are closed, which on the other hand implies that $G\backslash M$ is Hausdorff. We leave out the rest of the details. □

3.5 Complex actions and the GIT quotient

Assume now that G is a complex Lie group and X a complex manifold with an action $G \times X \to X$.

Example 3.5.1. Consider $\mathbb{C}^* = \mathrm{GL}(1, \mathbb{C})$, and consider the homomorphism $\rho : \mathbb{C}^* \to \mathrm{SL}(2, \mathbb{C})$ mapping

$$t \mapsto \rho(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

This induces an action $\mathbb{C}^* \times \mathbb{C}^2 \to \mathbb{C}^2$ mapping $(t, v) \mapsto \rho(t)v$. The picture is the following: Let $\mathbb{C}^2 = \{(x, y)\}$. We then have orbits $\{x \cdot y = \alpha \neq 0\}$, $\{x = 0, y \neq 0\}$, $\{x \neq 0, y \neq 0\}$, $\{x = 0, y = 0\}$. The topological quotient of this action is homeomorphic to $\mathbb{R}^2$ with two points attached to the origin.

In conclusion, since complex Lie groups are typically non-compact, we can not hope to endow the topological quotient with a holomorphic structure, and we need to come up with something better. The idea of the GIT quotient is that it makes choices in the orbits which are represented.

Definition 3.5.2. A complex Lie group G is *reductive* if there is a compact differentiable subgroup $K \subseteq G$ such that the inclusion $i : \mathrm{Lie}\, K \to \mathrm{Lie}\, G$ induces an isomorphism $i_{\mathbb{C}} : \mathrm{Lie}\, K \otimes \mathbb{C} \to \mathrm{Lie}\, G$. Here $\mathrm{Lie}\, K \otimes \mathbb{C}$ is the complex Lie algebra with the $\mathbb{C}$-linear extension of the Lie bracket.

Example 3.5.3. The group $\mathrm{SL}(n, \mathbb{C})$ is reductive, since $\mathrm{SU}(n) \subseteq \mathrm{SL}(n, \mathbb{C})$, and we have a natural decomposition

$$\mathrm{Lie}\, \mathrm{SL}(n, \mathbb{C}) = \mathrm{Lie}\, \mathrm{SU}(n) \oplus i\, \mathrm{Lie}\, \mathrm{SU}(n)$$

given by $A = (A - \overline{A}^T) + i(-i)(A + A^T)/2$ for $A \in \mathrm{Lie}\, \mathrm{SL}(n, \mathbb{C})$.

From now on, all the groups we consider will be assumed to be reductive. The complex actions that we are going to deal with are the following: Let G be a complex Lie group, and let $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ be a homomorphism of complex Lie groups (i.e. a *representation*). This induces a $\mathbb{C}$-linear action $G \times \mathbb{C}^n \to \mathbb{C}^n$ by $(g, v) \mapsto \rho(g)v$. From now on, we write $g \cdot v := \rho(g)v$. Linearity of the action means that we get an induced action on $G_r(\mathbb{C}^n)$ for all r and in particular on $\mathbb{P}(\mathbb{C}^n)$. Consider $X \subseteq \mathbb{P}(\mathbb{C}^n)$ which is preserved by the action; that is, for all $g \in G$ and $x \in X$, we have $g \cdot x \in X$, so we have an induced action $G \times X \to X$. We want to give meaning to $G\backslash X$. As a sketch, we consider the associations $X \to R_X \to R_X^G \to X_{R_X^G} = G\backslash X$.

Recall that to $X \subset \mathbb{P}(\mathbb{C}^n)$, we associate the finitely generated (as a $\mathbb{C}$-algebra) graded ring R_X with no zero divisors, defined to be

$$R_X = \bigoplus_{k \geq 0} H^0(X, \mathcal{O}_X(k)).$$

Whenever we have a finitely generated graded ring with no zero divisors, we can construct $X_R \subseteq \mathbb{P}(\mathbb{C}^m)$, by taking generators $e_1, \ldots, e_n$ of R, letting $\varphi : \mathbb{C}[x_1, \ldots, x_m] \to R$, $x_i \mapsto e_i$, taking the homogeneous ideal $\ker \varphi \subseteq \mathbb{C}[x_1, \ldots, x_m]$ and defining $X_R \subseteq \mathbb{P}(\mathbb{C}^m)$ to be the zeroes of polynomials in $\ker \varphi$.

Remark 3.5.4. Recall that X_R may be singular, and that for $X \subseteq \mathbb{P}(\mathbb{C}^n)$ we have $R_X \cong \mathbb{C}[x_1, \ldots, x_n]/I_X$, which, as a $\mathbb{C}$-algebra, is always generated by the degree 1 piece – this is not true for general R; this tells us that the right setup for this construction is really that of algebraic geometry, considering something more general than projective manifolds. Here $I_X = \{p \in \mathbb{C}[x_1, \ldots, x_n] : p(x) = 0 \forall x \in X\}$.

The embedding $i : X_R \to \mathbb{P}(\mathbb{C}^m)$ constructed above is not canonical; for example, we could always add generators to the ring and embed X_R into another projective space. However, $(X_R, i^*\mathcal{O}_{\mathbb{P}(\mathbb{C}^m)}(1))$ as an abstract complex manifold (possibly singular) without the embedding is canonical. For example, for generators $(e_1, \ldots, e_n)$ gives an embedding $X_R \subseteq \mathbb{P}(\mathbb{C}^m)$, and taking generators $(e_1, \ldots, e_n, e_1^2)$ would give an embedding $X_R \subseteq \mathbb{P}(\mathbb{C}^{m+1})$, but $X_R \subseteq \{x_{m+1} = 0\} \subseteq \mathbb{P}(\mathbb{C}^{m+1})$, and $\{x_{m+1} = 0\} \subseteq \mathbb{P}(\mathbb{C}^{m+1})$ is canonically identified with $\mathbb{P}(\mathbb{C}^n)$.

Definition 3.5.5. The *GIT quotient* $X \subseteq \mathbb{P}(\mathbb{C}^n)$ by $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ is

$$G \backslash\!\!\backslash X := X_{R_X^G}$$

endowed with the corresponding line bundle from the above remark.

Here, R_X^G is the ring of invariants of R_X: Write $R_X = \mathbb{C}[x_1, \ldots, x_n]/I_X$. Then G acts on $\mathbb{C}[x_1, \ldots, x_n]$: For $p \in \mathbb{C}[x_1, \ldots, x_n]$, we let $(g \cdots p)(x) = p(g^{-1}x)$ for $x = (x_1, \ldots, x_n)$. Here, by an action we mean that we have a map $G \times \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ such that for all $g \in G$, we have a homomorphism of $\mathbb{C}$-algebras (satisfying the usual properties of an action).

Because X is preserved by G, we get an induced action $G \times R_X \to R_X$. Whenever we have such an action, we can consider the *ring of invariants*

$$R_X^G := \{s \in R_x : g \cdot s = s \ \forall g \in G\}$$

Coming back to the definition of the GIT quotient, if we write $Y = X_{R_X^G}$, then there is a line bundle $L \to Y$, and $R_Y = \bigoplus_{k \geq 0} H^0(Y, L^{\otimes k})$ must coincide with $R_Y = R_X^G$.

Remark 3.5.6. The GIT quotient $G \backslash\!\!\backslash X$ may be horribly singular.

We need to check the following for the definition to make sense:

1. R_X^G has no zero divisors. This is easy to check.
2. R_X^G is finitely generated. This is less obvious.
3. Finally, R_X^G is graded; here is something to consider as well – R_X comes with a grading, but as we saw before, it would be generated by the degree 1 piece, which is not obvious here.

Example 3.5.7. Let $\rho : \mathbb{C}^* \to \mathrm{SL}(4, \mathbb{C})$ be the map

$$t \mapsto \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t^{-1} \end{pmatrix}$$

inducing an action $\mathbb{C}^* \times \mathbb{P}(\mathbb{C}^4)$. Note that $R_{\mathbb{P}(\mathbb{C}^4)} = \mathbb{C}[x_1, x_2, x_3, x_4]$. Recall that if $p \in \mathbb{C}[x_1, x_2, x_3, x_4]$, then $(t \cdot p)(x_1, x_2, x_3, x_4) := p(tx_1, t^{-1}x_2, tx_3, t^{-1}x_4)$. In particular, there are no invariant degree 1 polynomials. We find that

$$R_{\mathbb{P}(\mathbb{C}^4)}^{\mathbb{C}^*} = \langle x_1 \cdot x_2, x_1 \cdot x_4, x_2 \cdot x_3, x_3 \cdot x_4 \rangle.$$

Remark that this is generated by the degree 2 piece. This does not agree well with R_X being generated by the degree 1 piece for projective manifolds X. Let $z_{ij} = x_i x_j$. Consider $\mathbb{P}(\mathbb{C}^4)$ with (free) coordinates $[z_{12} : z_{14} : z_{23} : z_{34}]$, and let $Y \subseteq \mathbb{P}(\mathbb{C}^4)$ be the set

$$\{z \in \mathbb{P}(\mathbb{C}^4) : z_{12} \cdot z_{34} - z_{14} \cdot z_{23} = 0\},$$

which is just the image of the Segre embedding of $\mathbb{P}(\mathbb{C}^2) \times \mathbb{P}(\mathbb{C}^2)$ in $\mathbb{P}(\mathbb{C}^4)$. Now it is easy to check that $R_Y \cong R_{\mathbb{P}(\mathbb{C}^4)}^{\mathbb{C}^*}$ as $\mathbb{C}$-algebras (but not as graded $\mathbb{C}$-algebras).

8th lecture, September 14th 2011

Let G be a complex reductive Lie group, which we will write $G = K^C$, where K is the compact subgroup entering the definition of a reductive Lie group.

One question one could ask is how many subgroups K will work to give G the structure of a reductive group. The answer turns out to be the set $K \backslash G$ for some fixed K. In fact, K is unique up to conjugacy: For all $g \in G$, the subgroup $gKg^{-1} \subseteq G$ will work as well, but this is all that could happen.

Another question is when complex Lie groups are actually compact, and it turns out that the only compact ones are the tori T^{2n}.

We return to our discussion of GIT quotients. Let $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ be an action. This induces an action $G \times \mathbb{P}(\mathbb{C}^n) \to \mathbb{P}(\mathbb{C}^n)$, so that for G-invariant $X \subseteq \mathbb{P}(\mathbb{C}^n)$, we got an action of G on X. We defined the GIT quotient $G \backslash\!\backslash X = (Y, L)$ for a line bundle $L \to Y$. The construction was as follows:

1. We first take the ring of invariants R_X^G of the initial manifold X.
2. Then, choose generators $(e_1, \ldots, e_m)$ of R_X^G (and here was our first issue – is R_X^G in fact finitely generated as a $\mathbb{C}$-algebra?).
3. Consider the homomorphism $\mathbb{C}[x_1, \ldots, x_m] \to R_X^G$ mapping $x_i \mapsto e_i$ and let $Y = \{x \in \mathbb{P}(\mathbb{C}^m) : p(x) = 0 \, \forall p \in \ker \varphi\} \subseteq \mathbb{P}(\mathbb{C}^m)$. We noted that this is only a homomorphism of $\mathbb{C}$-algebras and not a homomorhpism of graded algebras – we saw one of this example last lecture.

To fix the grading issue, for now we assume that the generators e_i all have some fixed degree d (which is not possible in general). Consider the line bundle $i^* \mathcal{O}_{\mathbb{P}(\mathbb{C}^m)}(1) = L \to Y$. By construction,

$$R_Y = \bigoplus_{k \geq 0} H^0(Y, L^{\otimes k}) \cong \ker \varphi \backslash \mathbb{C}[x_1, \ldots, x_m],$$

which is isomorphic to R_X^G up to some shift in the grading. Moreover, by this assumption on the degrees of the e_j, we have that i'th piece of R_Y is $(R_Y)_i = (R_X^G)_{d \cdot i}$.

Remark 3.5.8. Algebraic geometry solves this shift by defining an abstract object called a scheme, denoted $\mathrm{Proj}(R_X^G)$, and this has a canonical (abstract) line bundle, called Serre's twisting sheaf and denoted L_{abs}, which induces the right grading in the sense that $H^0(\mathrm{Proj}(R_X^G), L_{\mathrm{abs}}^{\otimes k}) = (R_X^G)_k$. In our differential geometric approach, we will lose this subtle information.

We ask the following question: Which points are represented in the quotient $G \backslash\!\backslash X$? Our choice of generators $\{e_i\} \subseteq H^0(X, \mathcal{O}_X(d))$ induces a map $p : X \dashrightarrow \mathbb{P}(\mathbb{C}^m)$, $x \mapsto [e_1(x) : \cdots : e_m(x)]$, where we use a dotted line, since the map is not necessarily defined over X. More precisely, it is not well-defined where all the invariant sections vanish. By definition, $\mathrm{Im}\, p \subseteq Y$ and in fact, one can show that $\mathrm{Im}\, p = Y$. More generally, we can take a basis $\{s_j\}_{j=1}^N$ of the finite dimensional vector space $H^0(X, \mathcal{O}_X(k))^G$ for $k \gg 0$ and consider the map $p_k : X \dashrightarrow \mathbb{P}(\mathbb{C}^{N_k})$ mapping $x \mapsto [s_1(x) : \cdots : s_N(x)]$. One can now check that the p_k is defined where p was defined, since the e_i were generators. Now, $\mathrm{Im}\, p_k$ is biholomorphic to Y, not necessarily in full generality, but at least for k a multiple a d, since in fact $\mathrm{Im}\, p_k$ can be identified with image of the embedding $\varphi_{L^{\otimes k}}$ given by the line bundle $L^{\otimes k} = (i^* \mathcal{O}_{\mathbb{P}(\mathbb{C}^m)}(1))^{\otimes k}$.

Now, we can assume the assumption on the degrees of the generators by defining the GIT quotient as above. Namely, for arbitrary degrees of generators e_i define our GIT quotient (which is not quite the algebraic geometric one) (Y, L') to be the image of p_k, for some $k \gg 0$, and $L' = i^* \mathcal{O}_{\mathbb{P}(\mathbb{C}^{N_k})}(1)$.

Remark 3.5.9. The abstract line bundle $L_{\mathrm{abs}}^{k'} \cong L'$. In our differential geometric approach, we only "see" a power of L_{abs}.

In practice, to identify the GIT quotient, one usually computes the maps p_k (for $k \gg 0$ large enough).

We now turn to the question of where the p_k are defined. The idea is that the points (or orbits) where p_k are defined are represented in the quotient, whereas the points where they are not defined are not represented in the quotient. This is captured by the following notion.

Definition 3.5.10. A point $x \in X$ is called *semistable* if there is a $k > 0$ and an invariant section $s \in H^0(X, \mathcal{O}_X(k))^G$ such that $s(x) \neq 0$. The set of semistable points $X^{\mathrm{ss}} \subset X$ is open and is called the *locus of semistable points*.

The points in $X \setminus X^{\mathrm{ss}}$ are called *unstable*.

Example 3.5.11. The representation $\mathbb{C}^* \to \mathrm{SL}(3, \mathbb{C})$ mapping $t \mapsto \mathrm{diag}(1, t, t^{-1})$ induces an action on $\mathbb{P}(\mathbb{C}^3)$. The ring of invariants is

$$R^{\mathbb{C}^*}_{\mathbb{P}(\mathbb{C}^3)} = \mathbb{C}[x_1, x_2, x_3]^{\mathbb{C}^*} = \langle x_1, x_2 \cdot x_3 \rangle.$$

Notice that this is an example where the generators have different degree. Here the fact that k was chosen $k \gg 0$ is important, since the basis of $(R^{\mathbb{C}^*}_{\mathbb{P}(\mathbb{C}^3)})_1$ is just x_1, and the map $p_1 : \mathbb{P}(\mathbb{C}^3) \to \mathbb{P}(\mathbb{C})$ simply contracts everything to a point, mapping $[x_1 : x_2 : x_3] \mapsto [x_1]$. However, $(R^{\mathbb{C}^*}_{\mathbb{P}(\mathbb{C}^3)})_2 = \langle x_1^2, x_2 \cdot x_3 \rangle$, and $p_2 : \mathbb{P}(\mathbb{C}^3) \to \mathbb{P}(\mathbb{C}^2)$ maps $[x_1 : x_2 : x_3] \mapsto [x_1^2 : x_2 x_3]$. The unstable points in $\mathbb{P}(\mathbb{C}^3)$ are $\{x_1 = 0\} \cap \{x_2 \cdot x_3\} = 0$.

Take the open set $\{x_1 \neq 0\}$ and write coordinates $x = x_2/x_1$, $y = x_3/x_1$. The $\mathbb{C}^*$-action induces an action $\mathbb{C}^* \to \mathrm{SL}(2, \mathbb{C})$ on $\mathbb{C}^2 = \{(x, y)\}$ which is given by $t \mapsto \mathrm{diag}(t, t^{-1})$, which is the first example we considered. Recall that the orbits are $\{xy = \alpha\}$, $\{x = 0, y \neq 0\}$, $\{x \neq 0, y = 0\}$, and $\{x = 0, y = 0\}$, and the three last sets are those identified in the GIT quotient, which unlike the topological quotient, is Hausdorff.

9th lecture, September 15th 2011

Let $X \subseteq \mathbb{P}(\mathbb{C}^n)$ be a projective manifold, and $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ be a representation of a complex reductive Lie group such that X is G-invariant, and we constructed the GIT quotient $G \backslash\!\backslash X$. The last thing we need for this to be well-defined is the following proposition.

Proposition 3.5.12. *The ring of invariants R^G_X is finitely generated as a $\mathbb{C}$-algebra.*

Proof. Here, recall that $R_X \cong \mathbb{C}[x_1, \ldots, x_n]/I_X$ which implies, since $\mathbb{C}[x_1, \ldots, x_n]$ is a Noetherian ring, that R_X is Noetherian (by the Hilbert basis theorem). Here, a *Noetherian* ring is one in which every ideal is finitely generated. Write $R = R_X$ and let

$$R^G_+ = \bigoplus_{k>0} R^G_k = \bigoplus_{k>0} H^0(X, \mathcal{O}_X(k))^G.$$

Since R is noetherian, the ideal $R \cdot R^G_+ \subseteq R$ is finitely generated. Take generators $\{s_i\} \subseteq R^G_+$. Now, given a section $s \in R^G_k = H^0(X, \mathcal{O}_X(k))^G$, we can write $s = \sum_i f_i s_i$, where $\deg f_i < \deg s = k$, $f_i \in R$.

We claim that it is enough to prove that we can take these f_i lying on R^G. In terms of algebra, this means that R^G_+ is finitely generated as a module over R^G. That it is also finitely generated as a $\mathbb{C}$-algebra follows by induction on k

This claim follows from Weyl's trick (which works for $G = K^C$ a reductive group, K compact). We use that s is the class of a polynomial f_s in $\mathbb{C}[x_1, \ldots, x_n]/I_X$. The trick is to average over the compact group K. In doing so, we need that there is a consistent way of integrating on manifolds (using a certain measure). It is a fact that the compact Lie group K carries an invariant measure $d\mu$ (given by the action of the group).

Averaging the f_s, we obtain an function $\mathrm{Av}(f_s) : \mathbb{C}^n \to \mathbb{C}$, more precisely given by

$$\mathrm{Av}(f_s)(v) = \int_K f_s(g^{-1} v)\, d\mu.$$

Since X is G-invariant (and therefore also K-invariant), this well-defines a K-invariant class in $\mathbb{C}[x_1, \ldots, x_n]/I_X$, and since f_s is invariant by G, we have

$$f_s = \mathrm{Av}(f_s) = \sum_i \mathrm{Av}(f_i)\,\mathrm{Av}(s_i) = \sum_i \mathrm{Av}(f_i) s_i.$$

Now the $\mathrm{Av}(f_i)$ are K-invariant. Finally, we prove the claim that since G is reductive, any polynomial which is K-invariant is also G-invariant. This follows from the polar decomposition $G = \exp(i \operatorname{Lie} K) \cdot K$ (for example, any $A \in \mathrm{SL}(n, \mathbb{C})$ can be written $A = \sqrt{A\overline{A}^T} U$ where $U \in \mathrm{SU}(n)$. If we let $B = A\overline{A}^T$ then $B = \overline{B}^T$, and in general, such a B can be written $B = \exp(B')$ and $\sqrt{B} = \exp(B'/2)$). To prove the claim, it is enough to prove that $f(e^{i\xi}x) = f(x)$ for $\xi \in \operatorname{Lie} K$ if f if f is K-invariant.

Consider the curve $t \mapsto f(e^{it\xi} \cdot x)$, $t \in \mathbb{R}$. It suffices to prove that this curve is constant. We find that

$$\begin{aligned} \frac{d}{dt}|_{t=t_0} f(e^{it\xi}) &= df|_{e^{it_0\xi}}(i\xi e^{it\xi} \cdot x) = i\, df|_{e^{it_0\xi}\cdot x}(\xi(e^{it_0\xi}) \cdot x) \\ &= \frac{d}{ds}|_{s=0} f(e^{s\xi} \cdot e^{it_0\xi} x) = 0, \end{aligned}$$

since $e^{s\xi} \in K$, and f is K-invariant. Here in the notation we use our representation $G \to \mathrm{GL}(n, \mathbb{C})$ and that $\operatorname{Lie} \mathrm{GL}(n, \mathbb{C}) = \mathrm{SL}(n, \mathbb{C})$. □

3.6 GIT quotients for linearized actions

Definition 3.6.1. For a complex manifold X, a line bundle $L \to X$ is called *very ample* if there is an embedding $X \to \mathbb{P}(\mathbb{C}^n)$ such that $L = \varphi^* \mathcal{O}_{\mathbb{P}(\mathbb{C}^n)}(1)$.

We say that L is *ample* if there is a $k > 0$ such that $L^{\otimes k}$ is very ample.

Definition 3.6.2. Given an action $G \times X \to X$ and a line bundle $L \to X$, a *linearization* of the action is an action on the total space such that the following diagram is commutative:

$$\begin{array}{ccc} G \times L & \longrightarrow & L \\ \downarrow & & \downarrow \\ G \times X & \longrightarrow & X \end{array}$$

Remark 3.6.3. If we have a linearization, then this induces a G-equivariant embedding of $X \hookrightarrow \mathbb{P}(\mathbb{C}^N)$. Namely, we take $H^0(X, L^{\otimes k})$ for $k \gg 0$. Now G acts on $H^0(X, L^{\otimes k})$ by $(g, s) \mapsto g \cdot s \circ g^{-1}$ for a section $s : X \to L$. Here $\cdot$ is the action on L, and $\circ$ is the action on X. In other words, $g \cdot s \circ g^{-1}$ maps $x \in X$ to $g \cdot s(g^{-1}x) \in L$.

The embedding produced by a choice of invariant basis is G-equivariant. With this more general notion in place, we can define the *GIT quotient* of (X, L) by G (which is linearized), which is just the GIT quotient of X embedded by one of these G-equivariant embeddings.

The manifold underlying this GIT quotient is the image under the under the map $p_k : X \to \mathbb{P}(\mathbb{C}^{N_k})$ mapping $x \mapsto [s_1(x) : \cdots : s_{N_k}(x)]$ where $\{s_j\}$ is a basis of invariant sections of $H^0(X, L^{\otimes k})$.

Remark 3.6.4. This GIT quotient depends heavily on the linearization. In fact, whenever we have a *character* χ of G, i.e. $\chi \in \mathrm{Hom}(G, \mathbb{C}^*)$, we can modify the linearization by multiplying by χ on the fibers. This corresponds to changing the action on $H^0(X, L^{\otimes k})$ from $g \cdot s \circ g^{-1}$ to $\chi(g) \cdot g \cdot s \circ g^{-1}$.

10th lecture, September 20th 2011

3.7 Moduli spaces

We end our discussion of GIT quotients by giving an application to moduli constructions.

Let X be a compact *Riemann surface* (i.e. a complex manifold of dimension $\dim_{\mathbb{C}} X = 1$). By the classification of such things, we can associate to X its genus g, which we assume to be $g \geq 2$.

Consider $E \to X$ a holomorphic bundle. To such a thing, we can associate two numbers. One is its rank r_E and the other one its degree $\deg(E) \geq 0$, which is a number depending only on the topology of E. The idea of degree is the following: For $g = 0$, $X = \mathbb{P}^1$ is a sphere, one considers the equator and counts how much transition functions twist around it. More precisely, one covers X by two charts, each of them missing a pole. The transition functions are maps from X without the poles, which we identify with $\mathbb{C} \setminus \{0\}$ to $\mathrm{GL}(n, \mathbb{C})$.

Definition 3.7.1. A holomorphic bundle $E \to X$ is called *semistable* if for every holomorphic subbundle $F \subseteq E$,

$$\frac{\deg F}{r_F} \leq \frac{\deg E}{r_E}.$$

The idea now is that we can parametrize all semistable bundles over X up to isomorphism using GIT quotients.

Fact 3.7.2. *Fix an ample line bundle $L \to X$; i.e. one giving rise to an embedding $X \hookrightarrow \mathbb{P}(\mathbb{C}^{N_k})$ for $k \gg 0$, where $N_k = \dim H^0(X, L^{\otimes k})$. This is always possible for Riemann surfaces. Given a bundle E, there exists an $m \gg 0$ such that $E \otimes L^{\otimes m}$ is generated by global sections, and we have a surjection*

$$H^0(E \otimes L^{\otimes m}) \otimes H^0(L^{\otimes}) \twoheadrightarrow H^0(E \otimes L^{\otimes(m+k)})$$

induced by the map on sections defined pointwise by $(e, s) \mapsto e \otimes s$.

Write $M_m = \dim H^0(E \otimes L^m)$. Choose a basis $\{s_j\}$ of $H^0(E \otimes L^k)$ such that $H^0(E \otimes L^m) \cong \mathbb{C}^{M_m}$. This induces a point in $P(E, \{s_j\}) \in G_{M_{m+k}}(\mathbb{C}^{M_m} \otimes H^0(L^k))$.

A point in $G_{M_{m+k}}(\mathbb{C}^{M_m} \otimes H^0(L^k))$ corresponds to a surjective map $\mathbb{C}^{M_m} \otimes H^0(L^k) \twoheadrightarrow H^0(E \otimes L^{m+k}) \cong \mathbb{C}^{M_{m+k}}$. The choice of basis in the latter isomorphism is not reflected in the Grassmannian, which therefore does not depend on the bundle E (but the particular point does).

Taking isomorphic bundles gives points in the same orbit of the action by $\mathrm{GL}(M_m, \mathbb{C})$. Morally, we want to parametrize orbits by this action.

Theorem 3.7.3 (Giesecker–Simpson)**.** *E is semistable if and only if $P(E, \{s_j\})$ is GIT semistable with respect to the action $\mathrm{SL}(M_m, \mathbb{C})$ on the Grassmannian with linearization given by the determinant of the universal quotient bundle, $\det Q_{M_{m+k}}$.*

Here, we use the following notation: If F is a rank r bundle, the *determinant* of F is $\det F = \wedge^r F$.

The next question is how to choose $m = m(E)$.

Theorem 3.7.4 (Maruyama)**.** *For fixed rank and degree of our bundles, m can be taken uniformly over all semistable bundles with the same properties.*

Putting all of this together, we obtain a correspondence

$$\mathcal{M} : \{E \text{ semistable with rank } r \text{ and degree } d\} \to G_{M_{m+k}}(W),$$

where W is the space from before. This induces a map

$$\{E \text{ semistable with rank } r \text{ and degree } d\}/\text{isomorphism} \to \text{GIT quotient by } \mathrm{SL}(m, \mathbb{C}).$$

The image $\mathcal{E} = \operatorname{Im} \mathcal{M}$ of $\mathcal{M}$ is a (closed subscheme) horribly subvariety of $G_{M_{m+k}}(W)$. The GIT quotient of $\mathcal{E}$ by $\mathrm{SL}(M_m, \mathbb{C})$ is the moduli space of semistable bundles over X with fixed rank and degree. Note that the composition of $\mathcal{M}$ with the quotient

$$\{E \text{ semistable with rank } r \text{ and degree } d\} \to \mathcal{E} \to \mathrm{SL}(M_m, \mathbb{C}) \backslash\!\backslash \mathcal{E}$$

is only surjective and not a bijection. We conclude that all semistable bundles with fixed rank and degree are represented in the moduli space, but $E \ncong E'$ may be identified in the quotient, namely E is identified with bundles E' which determine points in the closure of its orbit.

11th lecture, September 21st 2011

4 Complex geometry from a real point of view

Throughout this section, let M be a differentiable manifold.

4.1 Tensors

We will be considering differentiable vector bundles $E \to M$ – for a definition of a differentiable vector bundle, copy the definition for holomorphic bundles (if M were complex) and change holomorphic to differentiable.

Definition 4.1.1. A *tensor* τ on M is a differentiable section of

$$T_{r,s}(M) = TM \otimes \cdots \otimes TM \otimes T^*M \otimes \cdots \otimes T^*M,$$

where $T^*M = (TM)^*$, and we have r copies of TM and s copies of T^*M. Recall that a section of this bundle is simply a map $\tau : M \to T_{r,s}(M)$ satisfying $\pi \circ \tau = \mathrm{Id}$, where $\pi : T_{r,s}(M) \to M$ is the projection.

Remark 4.1.2. Let $\mathcal{T}_{r,s}$ be the space of tensors of type (r, s). This is a module over the space $C^\infty(M)$ of differentiable functions $M \to \mathbb{R}$: In general, if $E \to M$ is a differentiable vector bundle over M, and $C^\infty(E)$ is the space of sections of E, then for $s \in C^\infty(E), f \in C^\infty(M)$, we let $f \cdot s|_p = f(p) \cdot s|_p$. In coordinates, the space of tensors described as follows: For a point $p \in M$, let (ψ, U) be a chart on M around p, $\psi : U \to \mathbb{R}^n$. We use the canonical identification $T_v\mathbb{R}^n \cong \mathbb{R}^n$ for $v \in \mathbb{R}^n$. This identification is given mapping $w \in \mathbb{R}^n$ to the derivation $D_w \in T_v\mathbb{R}^n$, which maps $D_w[f] = \frac{d}{dt}|_{t=0} f(f + tw)$ for $f : V \to \mathbb{R}^n$, $v \in V$. Let $\tau \in \mathcal{T}_{r,s}$, and let $\{\frac{\partial}{\partial x_i}|_v\}$ be the canonical basis of $T_v\mathbb{R}^n \cong \mathbb{R}^n$. In the notation from before, $\frac{\partial}{\partial x_i} = D_{(0,\dots,0,1,0,\dots,0)}$, where the 1 is at the i'th place. Let $\{dx_i|_v\}$ be the dual basis on $T_v^*\mathbb{R}^n$. They induce local sections of $\mathcal{T}_{r,s}\mathbb{R}^n \cong T_{r,s}M|_U$ given by

$$\frac{\partial}{\partial x_{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{j_r}} \otimes dx_{i_1} \otimes \cdots \otimes dx_{i_s}.$$

Using the identifications above, we extract a trivialization $T_{r,s}M|_U$. So given τ a tensor, we can write

$$\tau|_U = \sum \tau_{j_1,\dots,j_r,i_1,\dots,i_s} \frac{\partial}{\partial x_{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{j_r}} \otimes dx_{i_1} \otimes \cdots \otimes dx_{i_s},$$

where the $\tau_{j_1,\dots,j_r,i_1,\dots,i_s}$ are smooth functions on U.

Fact 4.1.3. *Alternatively, $\mathcal{T}_{r,s}$ can be viewed as the space of $C^\infty(M)$-multilinear maps*

$$T : C^\infty(T^*M) \times \cdots \times C^\infty(T^*M) \times C^\infty(TM) \times \cdots \times C^\infty(TM) \to C^\infty(M),$$

where as before we have r copies of $C^\infty(T^*M)$ and s copies of $C^\infty(TM)$. Recall that $C^\infty(TM)$ is the space of vector fields on M, and $\Omega^1(M) = C^\infty(T^*M)$ is the space of 1-forms. By $C^\infty(M)$-multiplinear, we mean that

$$T(\dots, f_1\alpha_j^1 + f_2\alpha_j^2, \dots) = f_1 T(\dots, \alpha_j^1, \dots) + f_2 T(\dots, \alpha_j^2, \dots)$$

for 1-forms α_j^1, α_j^2 and functions $f_1, f_2 \in C^\infty(M)$, and similarly for the vector fields.

Proof. For $\tau \in \mathcal{T}_{r,s}$ define T_τ by

$$T_\tau(\dots, \alpha_j, \dots, v_i, \dots)|_p = \tau|_p(\dots, \alpha_j|_p, \dots, v_i|_p, \dots).$$

It is easy to check that T_τ is differentiable. Conversely, given T_τ a $C^\infty(M)$-multilinear map as above, we construct a tensor τ_T as follows: Given 1-forms α_j and vector fields v_i, we note that $T(\dots,\alpha_j,\dots,v_i,\dots)|_p$ only depends on the values of the α_j and v_i at p; the idea of this is the following: For example, for $r=0$, $s=1$, it is enough to check for all $p \in M$ that $T(v_1)|_p = 0$ if $v_1|_p = 0$. To see this, take coordinates around $p \in M$ defined on $U \subseteq M$. Take $\varphi \in C^\infty(M)$ which takes the value 1 in a neighbourhood of p and is 0 on a neighbourhood of $M \setminus U$ – here it is essential that we are in the smooth and not holomorphic category. Write

$$v_1 = \sum_j a_j \varphi \frac{\partial}{\partial x_j} + (1-\varphi) v_1$$

for $a_j \in C^\infty(M)$. Then we obtain

$$T(v_1)|_p = \left(\sum a_j T\left(\varphi \frac{\partial}{\partial x_j}\right)\right)|_p = \sum a_j|_p T\left(\varphi \frac{\partial}{\partial x_j}\right)|_p,$$

which shows what we wanted.

Now, $T(\dots,\alpha_j,\dots,v_i,\dots)|_p$ defines an element in $T_{r,s}(M)|_p$ by

$$\tau_T|_p(\tilde{\alpha}_1|_p,\dots,\tilde{\alpha}_r|_p,\tilde{v}_1|_p,\dots,\tilde{v}_s|_p) = T(\tilde{\tilde{\alpha}}_1,\dots,\tilde{\tilde{v}}_1,\dots)|_p,$$

where $\tilde{\tilde{\alpha}}_j, \tilde{\tilde{v}}_i$ are smooth extensions of the elements of the fiber by 1-forms and vector fields respectively, constructed using the bump function φ (where, again, it is important that we are in the smooth category). This is well-defined by our above observation. □

Let $f : M \to N$ be a diffeomorphism of C^∞-manifolds, and let $\tau \in \mathcal{T}_{r,s}(N)$. From this we want to construct a tensor $f^*\tau \in \mathcal{T}_{r,s}(M)$. First, we define the *push-forward* of 1-forms and vector fields. For $v \in C^\infty(TM)$, we define $f_*(v) \in C^\infty(TN)$ by $f_*v|_p = df(v)|_{f^{-1}(p)}$. Similarly, for a 1-form $\alpha \in \Omega^1(M)$, we define $f_*\alpha \in \Omega^1(N)$ by $f_*\alpha|_p(v) := \alpha|_{f^{-1}(p)} \circ df^{-1}(v)$. To define the *pull-back* tensor $f^*\tau$, we consider

$$T_{f^*\tau}(\alpha_1,\dots,\alpha_r,v_1,\dots,v_s)|_p = T_\tau(f_*\alpha_1,\dots,f_*\alpha_r,f_*v_1,\dots,f_*v_s)|_{f(p)}.$$

Similarly, we can define the push-foward of a general tensor $\tau \in \mathcal{T}_{r,s}(M)$ by changing f to f^{-1}.

Example 4.1.4. Let $f : M \to M$ be a diffeomorphism. For $\tau \in \mathcal{T}_{1,1}(M) \cong C^\infty(\mathrm{End}TM)$, we obtain

$$f^*\tau(v)|_p = df^{-1}|_{f(p)} \tau|_{f(p)}(df\, v|_{f^{-1}f(p)})$$

for $v \in C^\infty(TM)$.

For a diffeomorphism $f : M \to N$ and $\tau \in \Omega^1(N)$, we have

$$(f^*\tau)(v)|_p = \tau|_{f(p)}(df\, v|_{f^{-1}f(p)}) = \tau|_{f(p)}(df\, v|_p).$$

Notice that in this expression, we are not taking inverses of every thing and so we can make sense of pull-backs of 1-forms for arbitrary smooth maps, not necessarily diffeomorphisms.

4.2 Differential forms and the Lie derivative

For more details on the following, see [War83].

Definition 4.2.1. Let $v \in C^\infty(TM)$ be a vector field on M. The *flow* of v is the differentiable map $\varphi_v^T = \varphi_v(t,\cdots)$, where $\varphi_v : U \to M$ is a differentiable map, where $\{0\} \times M \subseteq U \subseteq \mathbb{R} \times M$, and φ_v satisfies

$$\frac{d}{dt}\varphi_v^t(p) = v|_{\varphi^t(p)}$$

for $p \in M$.

Definition 4.2.2. For $\tau \in \mathcal{T}_{r,s}(M)$, the *Lie derivative* of τ with respect to $v \in C^\infty(TM)$ is the element $L_v\tau \in \mathcal{T}_{r,s}(M)$ defined by

$$L_v\tau = \frac{d}{dt}|_{t=0}(\varphi_v^t)^*\tau$$

Definition 4.2.3. For vector fields $v_1, v_2 \in C^\infty(TM)$, we define the *Lie bracket of vector fields* by

$$[v_1, v_2] := L_{v_1}v_2.$$

Now $(C^\infty(TM), [\,,\,])$ is an infinite-dimensional Lie algebra. That is, it satisfies that $[v,w] = -[w,v]$, it satisfies the Jacobi identity, and for $f, g \in C^\infty(M)$,

$$[fv, gw] = fg[v,w] + f(v(g))w + g(w(f))v.$$

In coordinates $v = \sum_j v^j \frac{\partial}{\partial x_j}$, $w = \sum_j w^j \frac{\partial}{\partial x_j}$, then one checks from the definition that

$$[v,w] = \sum_{j,k} v^j \frac{\partial}{\partial x_j}(w^k)\frac{\partial}{\partial x_k} - w^j\frac{\partial}{\partial x_j}(v^k)\frac{\partial}{\partial x_k}.$$

Example 4.2.4. For $\tau \in C^\infty(\mathrm{End}TM)$ and vector fields $v, w \in C^\infty(TM)$, one finds

$$\begin{aligned}(L_v\tau)(w) &= \frac{d}{dt}|_{t=0}((\varphi_v^t)^*\tau)w = \frac{d}{dt}|_{t=0}((\varphi_v^t)^*(\tau(\varphi_v^t)_*w)) \\ &= [v, \tau w] - \tau[v,w]\end{aligned}$$

Definition 4.2.5. A *differential form* of degree k is a differentiable section $\wedge^k M = \wedge^k T^*M$. Alternatively, we can see it as a special type of $(0,k)$-tensor, regarding $\wedge^k T_p^*M \subseteq (T_p^*M)^{\otimes k}$. So, it can be seen as a map

$$\alpha : C^\infty(TM) \times \cdots \times C^\infty(TM) \to C^\infty(M),$$

which is alternating in the sense that $\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \operatorname{sgn}\sigma\alpha(v_1, \ldots, v_k)$, where σ is a permutation of k elements. In coordinates, we can write

$$\alpha = \sum_{i_1 < \cdots < i_k} \alpha_{i_1\ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

for maps $\alpha_{i_1\ldots i_k} \in C^\infty(U)$ on a coordinate chart U.

The fiberwise wedge product $\alpha_q \wedge \alpha_k|_p = \alpha_q|_p \wedge \alpha_k|_p$ induces a structure of a ring on $C^\infty(\wedge^* M)$, where $\wedge^* M := \bigoplus_{k=0}^n \wedge^k M$, and $n = \dim M$. Here, $\wedge^0 M$ is the trivial bundle of rank 1, $C^\infty(\wedge^0 M) = C^\infty(M)$.

Definition 4.2.6. The *exterior differential* is the map $d : C^\infty(\wedge^k M) \to C^\infty(\wedge^{k+1}M)$ mapping $\alpha \mapsto d\alpha$ defined locally by

$$d\alpha = \sum_j \sum_{i_1 < \cdots < i_k} \frac{\partial}{\partial x_j}(\alpha_{i_1\ldots i_k}) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

The $k+1$-form $d\alpha$ is defined globally by the above formula (exercise) and satisfies

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg\alpha}\alpha \wedge (d\beta).$$

In particular, in this abstract picture, we can view $df \in \Omega^1(M)$ for $f \in C^\infty(M)$.

For a vector field $v \in C^\infty(TM)$, we define the *contraction* by

$$v \lrcorner : C^\infty(\wedge^k M) \to C^\infty(\wedge^{k+1}M)$$

defined by the properties that $v \lrcorner f = 0$ for $f \in C^\infty(M)$, and $v \lrcorner \alpha(v)$ for $\alpha \in \Omega^1(M)$. The contraction satisfies

$$v \lrcorner(\alpha \wedge \beta) = (v \lrcorner \alpha) \wedge \beta + (-1)^{\deg\alpha}\alpha \wedge (v \lrcorner \beta).$$

For a proof of the following [War83].

Theorem 4.2.7 (Cartan's formula for the Lie derivative)**.** *Let* $\alpha \in C^\infty(\wedge^k M)$*,* $v \in C^\infty(TM)$*. Then*

$$L_v\alpha = d(v\lrcorner\alpha) + v\lrcorner(d\alpha).$$

Two other important formulae are

$$(L_{v_0}\alpha)(v_1,\ldots,v_k) = L_{v_0}(\alpha(v_1,\ldots,v_k)) - \sum_{i=1}^{k}\alpha(v_1,\ldots,v_i,L_{v_0}v_i,v_{i+1},\ldots,v_k)$$

$$d\alpha(v_0,\ldots,v_k) = \sum_{i=0}^{k}(-1)^i L_{v_i}(\alpha(v_0,\ldots,\hat{v_i},\ldots,v_k)) + \sum_{i<j}(-1)^{i+j}\alpha(L_{v_i}v_j,v_0,\ldots,\hat{v_i},\ldots,\hat{v_j},\ldots,v_k),$$

where $\hat{v_i}$ means v_i removed from the expression.

11th lecture, September 27th 2011

To unite some of the concepts considered previously we make a dictionary of global and local concepts[2].

Global

A vector field $v \in C^\infty(TM)$.
The flow of

$$\varphi_v : U \to M,$$

$\{0\} \times M \subseteq U \subseteq \mathbb{R} \times M$. Write $\varphi_v^t = \varphi_v(t,\cdot)$. This satisfies

$$\frac{d}{dt}\varphi_v^t(p) = V \circ \varphi_v^t(p)$$

and $\varphi_v^0(p) = p$ for $p \in M$.
A tensor $\tau \in \mathcal{T}_{r,s}(M)$.
Starting with $\tau \in \mathcal{T}_{r,s}(M)$, we want to produce $\tau' \in \mathcal{T}_{r,s}(M)$, which is the derivative with respect to a vector field. We need to identify different fibers of a vector bundle, analagously to in the local picture. We do this below.

Local

A map $v : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ mapping $p \mapsto (p, f_v(p))$ for $f_v : \mathbb{R}^n \to \mathbb{R}^n$.
A system of ODEs

$$(\frac{\partial}{\partial t}y_v)(t,p) = f_v(y_v(t,p))$$

with solution $\varphi_v(0,p) = p$ for all $p \in \mathbb{R}^n$.
A solution $\varphi_v : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$.
A differentiable map $f_\tau : \mathbb{R}^n \to \mathbb{R}^m$.
The differential $df_\tau : T\mathbb{R}^n \to T\mathbb{R}^m$. For v a vector field on $\mathbb{R}^n$, we consider $df_\tau(v) : \mathbb{R}^n \to \mathbb{R}^m$. Here, we simply let $df_\tau(v)|_p = \lim_{t\to 0}(f_\tau(p + tf_v(p)) - f_\tau(p)) \cdot t^{-1}$. Here,

$$f_\tau(p + tf_v(p)) \in \mathbb{R}^n \times \mathbb{R}^m|_{f_v(p)t+p}$$
$$f_\tau(p) \in \mathbb{R}^n \times \mathbb{R}^m|_p,$$

and we identify these two fibres of $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.

Note that for a diffeomorphism $f : M \to M$, we defined the pullback tensor

$$\begin{aligned} f^*\tau(\ldots,\alpha_j,\ldots,v_i,\ldots)|_p &= \tau|_{f(p)}(\ldots,(f_*\alpha_j)|_{f(p)},\ldots,(f_*v_i)|_{f(p)},\ldots) \\ &= \tau_{f(p)}(\ldots,\alpha_j|_p(df^{-1}\cdot),\ldots,dfv_i|_p,\ldots), \end{aligned}$$

where $\tau_{f(p)} \in T_{r,s}(M)|_{f(p)} = TM \otimes \cdots \otimes TM \otimes T^*M \otimes \cdots \otimes T^*M$. Lie's idea was to use the flow of the vector field v to identify different fibres (see Fig. 2) and define

$$L_v\tau|_p = \frac{d}{dt}|_{t=0}(\varphi_v^t)^*|_p\tau = \lim_{t\to 0}\frac{(\varphi_v^t)^*\tau|_p - \tau|_p}{t}.$$

[2]Okay, this one is slightly messed up and should probably be fixed

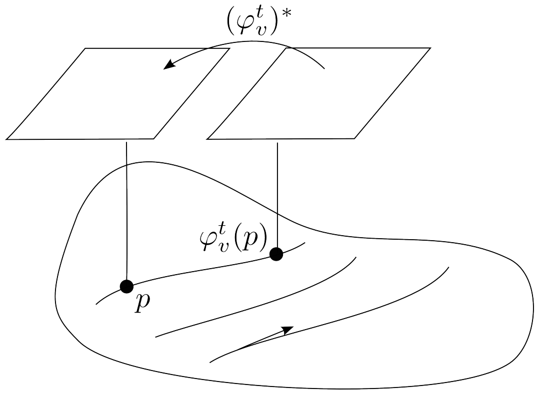

Figure 2: Integral curves of a vector field v on a manifold M, and two different fibres sitting above points p and $\varphi_v^t(p)$ being identified using $(\varphi_v^t)^*$.

In the local picture, for vector fields v_1, v_2, $v_j = \sum_{k=1}^n v_k^j \frac{\partial}{\partial x_k}$, and $f_{v_j} = (v_j^1, \ldots, v_j^n)$, we had

$$L_{v_1} v_2|_p = \sum_{k,m} \left(v_1^k \frac{\partial(v_2^m)}{\partial x_k} - v_2^k \frac{\partial(v_1^m)}{\partial x_k} \right) \frac{\partial}{\partial x_m}.$$

Note that the price we pay with this definition is that we need to involve derivatives of the vector field v in $L_v \tau$, which a priori was not was we were looking for. So we ask: What does $L_v \tau$ measure? And what about $L_{v_1} v_2$? Let $p \in \mathbb{R}^n$ and let $g : \mathbb{R} \to \mathbb{R}^n$ be the function

$$g(t) = \varphi_{v_1}^t \circ \varphi_{v_2}^t(p) - \varphi_{v_2}^t \circ \varphi_{v_1}^t(p).$$

That is, g measures how flowing in different directions of two vector fields in different orders give

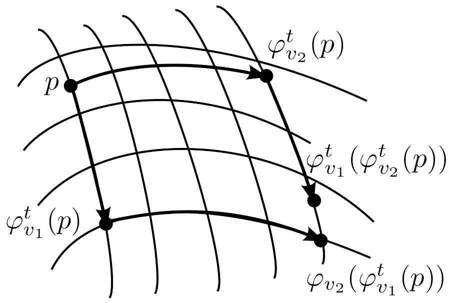

Figure 3: Flowing along two different vector fields in different orders.

different results (see Fig. 3). Note that

$$g(0) = p - p = 0,$$
$$g'(0) = f_{v_1}(p) + f_{v_2}(p) - f_{v_2}(p) - f_{v_1}(p) = 0,$$
$$g''(0) = L_{v_1} v_2|_p.$$

We conclude from this that the Lie derivative $L_{v_1} v_2$ measures a difference of accelerations; namely the accelerations of integral curves of v_1 along integral curves of v_2 minus the acceletations of integral curves of v_2 along integral curves of v_1.

Now, this picture does not depend on the local patch we have been considering, so this gives a way of thinking of the Lie derivative in general. We return to the dictionary.

Global

Differential forms $\alpha \in \Omega^k(M) = C^\infty(\wedge^k M)$ with exterior derivative

$$d : \Omega^k(M) \to \Omega^{k+1}(M).$$

For $v \in C^\infty(TM)$ we have the contraction $v \lrcorner d : \Omega^k(M) \to \Omega^k(M)$. For a function $f \in C^\infty(M)$, $(fv)\lrcorner d\alpha = f(v \lrcorner d\alpha)$, so in this expression, we are not taking derivatives of v. Corresponding to the formula in the local case, we have Cartan's magic formula

$$L_v \alpha = v \lrcorner d\alpha + d(v \lrcorner \alpha).$$

In general, to take derivatives of a tensor of arbitrary type with respect to a vector field, we need to somehow connect the fibres somehow: This can also be obtained by using a connection; we return to these later.

We have $d^2 = 0$ on forms of arbitrary degree, and we have a complex

$$\begin{aligned} 0 \to \Omega^0(M) &\xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \\ &\xrightarrow{d} \Omega^n(M) \to 0, \end{aligned}$$

where $\Omega^0(M) = C^\infty(M)$. The cohomology of this complex is called the the deRham cohomolgy of M. The idea is that the cohomology groups

$$H^k(M;\mathbb{R}) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

measure some global topological information of the manifold. For example, $H^0(M, \mathbb{R})$ is given by the set of $f : M \to \mathbb{R}$ with $df = 0$. This is the set of locally constant functions on M, and we identify $H^0(M, \mathbb{R}) \cong \mathbb{R}^d$, where d is the number of connected components of M.

Local

Take $\alpha = \sum_j \alpha^j dx_j$ a 1-form. Then we let $d\alpha = \sum_{k,j} \frac{\partial \alpha^j}{\partial x_k} dx_k \wedge dx_j$, which we can rewrite as

$$\sum_{k<j} \left(\frac{\partial \alpha^j}{\partial x_k} - \frac{\partial \alpha^k}{\partial x_j} \right) dx_k \wedge dx_j.$$

We have a contraction

$$\frac{\partial}{\partial x_m} \lrcorner d\alpha = \sum_{j=1}^n \frac{\partial \alpha^j}{\partial x_m} dx_j - \frac{\partial \alpha^m}{\partial x_j} dx_j.$$

If we write $\alpha = (\alpha^1, \ldots, \alpha^n)$, the first expression $\frac{\partial \alpha^j}{\partial x_m} dx_j$ is really the partial derivative of α with respect to the vector field $\frac{\partial}{\partial x_m}$, which is really what we were looking for. The sum is in fact equal to $L_{\partial/\partial x_m} \alpha - d(\alpha(\frac{\partial}{\partial x_m}))$. Thus, the price of taking our global definition is that we need to involve some funny combination of derivatives of the form.

For $\alpha = df$ for a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, it follows from the above formula that $d(df) = 0$.

4.3 Integration on manifolds

We begin with some linear algebra: Let V be a real n-dimensional vector space. Then $\wedge^n V$ is 1-dimensional real vector space, and $\wedge^n V \setminus \{0\}$ has two components.

Definition 4.3.1. An *orientation* of V is a choice of component of $\wedge^n V \setminus \{0\}$

Let M be a connected differentiable manifold.

Definition 4.3.2. The n-manifold M is called *orientable* if $\wedge^n T^* M \setminus \{0\}$ has two components.

Note that if M is connected, then $\wedge^n T^* M \setminus \{0\}$ has at most two components.

Example 4.3.3. The Möbius strip (see Fig. 4) is non-orientable.

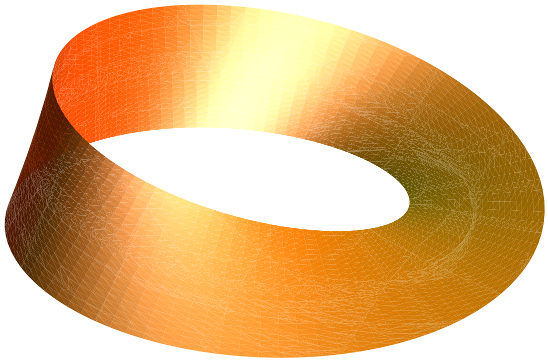

Figure 4: The Möbius strip

Definition 4.3.4. A non-connected manifold is *orientable* if every component is orientable.

Definition 4.3.5. If M is orientable, an *orientation* is a choice of connected component of $\wedge^n T^* M_i$ on each connected component M_i of M.

Definition 4.3.6. Let M, N be oriented n-manifolds, and let $f : M \to N$ be a diffeomorphism. We say that f *preserves orientation* if the map $f^* : \wedge^n T^* N \to \wedge^n T^* M$ given by $\alpha|_x \mapsto \alpha|_x \circ \wedge^n df|_{f^{-1}(x)}$ takes the components of the orientation on N to the components of the orientation on M.

For an arbitrary smooth map $f : M \to N$ we say that f *preserves orientation*, if for all $x \in M$, the map $f_x^* : \wedge^n T^* N|_{f(x)} \to \wedge^n T^* T^* M|_x$ mapping $\alpha|_{f(x)} \mapsto \alpha|_{f(x)} \circ \wedge^n df|_x$ preserves orientation.

Definition 4.3.7. Let M be a smooth n-manifold. A *volume form* is a form $\sigma \in \Omega^n(M)$ such that $\sigma_x \neq 0$ for all $x \in M$.

The following statement is from [War83].

Proposition 4.3.8. *The following are equivalent:*

(a) *M is orientable.*

(b) *There exists a collection of coordinate systems $\{(V_\alpha, \psi_\alpha)\}_{\alpha \in I}$, $\psi_\alpha : V_\alpha \to \mathbb{R}^n$ such that $M = \bigcup_\alpha V_\alpha$ and $\det(\psi_\alpha \circ \psi_\beta^{-1}) > 0$ on $V_\alpha \cap V_\beta$ for all $\alpha, \beta \in I$.*

(c) *There exists a volume form σ on M.*

Proof. Assume that M is connected (the general case follows easily).

To see that (a) implies (b), choose an orientation $\Lambda \subseteq (\wedge^n T^* M \setminus \{0\})$. Choose the family of charts (V, ψ), $\psi : V \to \mathbb{R}^n = \{(x_1, \ldots, x_n)\}$ such that $\psi^* dx_1 \wedge \cdots \wedge dx_n \in \Lambda$. If (V', ψ'), $\psi' : V' \varnothing \mathbb{R}^n = \{(x_1', \ldots, x_n')\}$ has the same property, then

$$dx_1' \wedge \cdots \wedge dx_n' = \det(d\psi' (d\psi)^{-1}) dx_1 \wedge \cdots \wedge dx_n.$$

This implies that $\det(d\psi' \circ d\psi^{-1}) > 0$, since otherwise, the components of $\wedge^n T^* M \setminus \{0\}$ would be interchanged.

To see that (b) implies (c), take a partition of unity subordinate to the cover in (b), i.e. take a family $\{\varphi_\alpha : M \to \mathbb{R}_+ \text{ diff.}\}_{\alpha \in I}$ such that $\sum_{\alpha \in I} \varphi_\alpha(p) = 1$ for $p \in M$ and such that $\varphi_\alpha|_{M \setminus V_\alpha} = 0$.

Here, we assume that the cover is locally finite (that is, for each point on M, only finitely many φ_α are non-zero), and that the cover is countable. Define

$$\sigma = \sum_{\alpha \in I} \varphi_\alpha \psi_\alpha^* dx_1 \wedge \cdots \wedge dx_n.$$

This is defined on V_α a priori, but σ extends to a form $\sigma \in \Omega^n(M)$. In coordinates V_β, we have

$$\begin{aligned} (\psi_\beta)_* \sigma &= \sum_{\{\alpha \in I : V_\alpha \cap V_\beta \neq \emptyset\}} (\varphi_\alpha \circ \psi_\beta^{-1})(\psi_\beta)_* \psi_\alpha^* dx_1 \wedge \cdots \wedge dx_n \\ &= \sum_{\{\alpha \in I : V_\alpha \cap V_\beta \neq \emptyset\}} (\varphi_\alpha \circ \psi_\beta^{-1}) \det(d\psi_\alpha \circ d\psi_\beta^{-1}) dx_1 \wedge \cdots \wedge dx_n. \end{aligned}$$

Now since $\det(d\psi_\alpha \circ d\psi_\beta^{-1}) > 0$, and since $\sum_\alpha \varphi_\alpha(p) = 1$, we have $(\psi_\beta)_* \sigma \neq 0$ on V_β, so $\sigma \neq 0$ on M.

To see that (c) implies (a), let $\sigma \in \Omega^n(M)$ be a volume form. Then the sets

$$\begin{aligned} \Lambda^+ &= \{\lambda \sigma|_p : \lambda \in \mathbb{R}_{>0}, p \in M\} \subseteq \wedge^n T^* M \setminus \{0\}, \\ \Lambda^- &= \{\lambda \sigma|_p : \lambda \in \mathbb{R}_{<0}, p \in M\} \subseteq \wedge^n T^* M \setminus \{0\}, \end{aligned}$$

define the orientability of M. □

We now want to integrate a top form $\tau \in \Omega^n(M)$ on M oriented. Take a family of coordinates of M given by the previous proposition. Define the *integral* of τ over M by

$$\int_M \tau = \sum_{\alpha \in I} \int_{\mathbb{R}^n} (\psi_\alpha)_* (\varphi_\alpha \cdot \tau).$$

Here, recall that we can write

$$(\psi_\alpha)_* (\varphi_\alpha \cdot \tau) = f_\alpha dx_1 \wedge \cdots \wedge dx_n$$

on $\mathbb{R}^n$ for $\psi_\alpha : V_\alpha \to \mathbb{R}^n$. Therefore, the integral above is simply

$$\int_{\mathbb{R}^n} (\psi_\alpha)_* (\varphi_\alpha \cdot \tau) := \int_{\mathbb{R}^n} f_\alpha \, d\mu,$$

where $d\mu$ is the Lebesgue measure.

Exercise 4.3.9. This integral is well-defined (i.e. it depends only on the orientation).

Remark 4.3.10. If M is compact, $\int_M \tau \in]-\infty, \infty[$, but if M is non-compact, $\int_M \tau \in [-\infty, \infty]$.

Example 4.3.11. Let G be a Lie group. Take $\sigma_e \in \Omega^n(\operatorname{Lie} G)^*$, $\sigma_e \neq 0$, and recall here that $\operatorname{Lie} G \cong T_e G$. Let $\sigma|_g = dL_g|_e \sigma_e \in \wedge^n (T^* G)|_g$. Since $\sigma_e = 0$, left-invariance tells us that $\sigma|_g \neq 0$ for all $g \in G$, and so σ is a volume form. Thus, any Lie group is orientable, and we can construct an integral on G.

Example 4.3.12. If M is oriented, any metric g on M determines a volume form σ_g defined locally by

$$\sigma_g = \sqrt{|\det g|} dx_1 \wedge \cdots \wedge dx_n.$$

We have the following change of variable formula: Given a differentiable map $f : M \to N$ of oriented manifolds, $\tau \in \Omega^n(N)$, we have

$$\int_M f^* \tau = \pm \int_{f(M)} \tau,$$

where the sign is positive if f is orientation preserving, and negative if f is not.

12th lecture, September 28th 2011

4.4 Complex geometry from a real point of view

Let V be a real vector space.

Definition 4.4.1. A *complex structure* on V is an $\mathbb{R}$-linear endomorphism $J : V \to V$ such that $J^2 = -\operatorname{Id}$.

Given a vector space with a complex structure, (V, J), we give V the structure of a complex vector space by letting $(\alpha + i\beta)v = \alpha v + \beta Jv$ for $v \in V$, $\alpha, \beta \in \mathbb{R}$.

Remark 4.4.2. If V is finite dimensional, then (V, J) is finite dimensional. If $\{v_1, \dots, v_m\}$ is a basis of V over $\mathbb{R}$, then $\{v_1, \dots, v_m\}$ generate (V, J) over $\mathbb{C}$, and we can use these to construct a basis $\{v_1, \dots, v_k\}$ over $\mathbb{C}$, and it turns out that $\{v_1, \dots, v_k, Jv_1, \dots, JV_k\}$ is a basis for V over $\mathbb{R}$. In particular, V is even-dimensional, $\dim V = 2k$.

Given V a complex vector space, let V_0 be the underlying vector space over $\mathbb{R}$, and define $J : V_0 \to V_0$ by $v \mapsto J(V) := iv$. This defines a complex structure on V_0.

Example 4.4.3. Consider $\mathbb{C}_0^n \cong \mathbb{R}^{2n}$ with the identification $(z_1, \dots, z_n) \mapsto (x_1, y_1, \dots, x_n, y_n)$, where $z_j = x_j + i_j$. Then the map $J(x_1, y_1, \dots, x_n, y_n) = (-y_1, x_1, \dots, -y_n, x_n)$ is the *canonical complex structure* on $\mathbb{R}^{2n}$. In the basis $(x_1, \dots, x_n, y_1, \dots, y_n)$, we have

$$J = \begin{pmatrix} 0 & -\operatorname{Id} \\ \operatorname{Id} & 0 \end{pmatrix}.$$

We turn now to the global picture. Let M be a differentiable manifold.

Definition 4.4.4. An *almost complex structure* on M is $J \in C^\infty(\operatorname{End}(TM))$ such that $J^2 = -\operatorname{Id}$. A pair (M, J) is called a *almost complex manifold*.

Note that on an almost complex manifold, $(T_xM, J|_x)$ has a natural structure of a complex vector space.

The notion of an almost complex structure tells us how to multiply by i, infinitesimally rather than in neighbourhoods of points.

Proposition 4.4.5. *Any complex manifold X induces a canonical almost complex structure on the underlying differentiable manifold X_0.*

Here, we mean the following by the underlying differentiable manifold: As a topological space X_0 is just X, and to define a differentiable structure, take a holomorphic atlas $\{(V_\alpha, \psi_\alpha)\}$ of X where $\psi_\alpha : V_\alpha \to \mathbb{C}^n$ and define ψ_α^0 as the composition $V_\alpha \overset{\psi_\alpha}{\to} \mathbb{C}^n \to \mathbb{R}^{2n}$, where the last map is the identification from Example 4.4.3.

Similarly, a holomorphic bundle $E \to X$ gives rise to an underlying differentiable vector bundle $E_0 \to X_0$. As above, as a topological space, E_0 is just E. To define the differentiable structure, take a covering of holomorphic trivializations $\{(V_\alpha, \tilde{\psi_\alpha})\}$, $\tilde{\psi_\alpha} : E|_{V_\alpha} \to V_\alpha \times \mathbb{C}^r$ and define ψ_α^0 to be the composition $E|_{V_\alpha} \overset{\tilde{\psi_\alpha}}{\to} V_\alpha \times \mathbb{C}^r \to V_\alpha \times \mathbb{R}^{2r}$, where the latter map acts identically on V_α.

Proof of Proposition. Consider the differentiable vector bundle $(TX)_0 \to X_0$. Multiplication by i induces a section $J \in C^\infty(\operatorname{End}(TX)_0)$ such that $J^2 = -\operatorname{Id}$, but we really need a smooth section of $\operatorname{End}(T(X_0))$. Define $\Phi : T(X_0) \to (TX)_0$ which for $p \in X_0$ maps

$$\Phi(D_p) = (D_p)_c,$$

where $(D_p)_c$ is the $\mathbb{C}$-linear extension of the derivation D_p: Let $\mathcal{O}_{X,p}$ be the algebra of germs of holomorphic functions on X at p. Then there is a map $\mathcal{O}_{X,p} \hookrightarrow C^\infty_{X_0,p} \otimes \mathbb{C}$ to the algebra of germs of differentiable (complex valued) functions on X_0 at at p such that

$$(D_p)_c([f]_{\mathcal{O}_{x,p}}) = (D_p)_c([f]_{C^\infty_{X_0,p} \otimes \mathbb{C}}),$$

where for $f = u + iv$, $u, v \in C^\infty(X_0)$, $(D_p)_c$ maps

$$(D_p)_c([f]) = D_p[u] + iD_p[v].$$

We claim now that Φ is an isomorphism. Then we are done, since we simply define $J = (\Phi)^{-1}(J\Phi(\cdot))$. It is enough to check this claim locally.

Let $\psi : V \to \mathbb{C}^n$ be holomorphic coordinates for X. This defines trivializations of both $T(X_0)$ and $(TX)_0$: Recall that we have a trivialization of TX, $\tilde{\psi} : TX|_V \to V \times \mathbb{C}^n$ which maps $D_q \mapsto (q, (v_1, \dots, v_n))$ where $d\psi|_q(D_q) = \sum_j v_j \frac{\partial}{\partial z_j}$, and here $d\psi|_q(D_q)[f] = D_q[f \circ \psi]$. From this we get $\tilde{\psi}^0 : (TX)_0|_V \to V \times \mathbb{R}^{2n}$ mapping

$$\tilde{\psi}^0(D_q) = (q, (x_1, y_1, \dots, x_n, y_n)), \quad v_j = x_j + ix_j,$$

as well as a map $\tilde{\tilde{\psi}} : T(X_0)|_V \to V \times \mathbb{R}^{2n}$ which maps

$$\begin{aligned} \tilde{\tilde{\psi}} &= (q, (x_1', y_1', \dots, x_n', y_n')), \\ D_q' = d\psi^0|_q(D_q) &:= \sum_j x_j' \frac{\partial}{\partial x_j} + y_j' \frac{\partial}{\partial y_j}. \end{aligned}$$

We want to compute the map Φ locally, i.e. consider $\Phi \equiv \tilde{\psi}^0 \Phi \tilde{\tilde{\psi}}^{-1}$ and claim that $\Phi = \mathrm{Id}$. We find

$$\begin{aligned} \Phi(D_q') &= \sum_{k=1}^n \Phi(D_q')(z_k) \frac{\partial}{\partial z_k}|_q = \sum_{k=1}^n (x_k' + iy_k') \frac{\partial}{\partial z_k}|_q \\ &= (q, (x_1' + iy_1', \dots, x_n' + iy_n')) = (q, (x_1', y_1', \dots, x_n', y_n')). \end{aligned}$$

□

The canonical J of a complex manifold X knows everything about X:

Proposition 4.4.6. *Let X_1, X_2 be complex manifolds and let $f : (X_1)_0 \to (X_2)_0$ be differentiable. Then $f : X_1 \to X_2$ is holomorphic if and only if $df J_1 = J_2 df$.*

Now given X, we can take (X_0, J) and define

$$\mathcal{O}_X' = \{f : X_0 \to \mathbb{C} \text{ differentiable} : df J = idf\},$$

and it follows from the Proposition that in fact $\mathcal{O}_X' = \mathcal{O}_X$.

Proof of Propostion. The statement is local, so it is enough to prove it for a differentiable map $f : \mathbb{C}^n \to \mathbb{C}^m$. We need to prove that f is holomorphic if and only if $df J_0 = J_0 df$, where for $(z_1, \dots, z_n) \equiv (x_1, \dots, x_n, y_1, \dots, y_n)$,

$$J_0 = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix},$$

and we write $f = u + iv$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, and $f = f(x, y) = u(x, y) + iv(x, y)$. Then

$$df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix}, \ \frac{\partial u}{\partial x} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_1}{\partial x_n} & \cdots & \frac{\partial u_m}{\partial x_n} \end{pmatrix}, \dots.$$

$$df J_0 = \begin{pmatrix} \frac{\partial v}{\partial x} & -\frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} & -\frac{\partial u}{\partial y} \end{pmatrix}, J_0 df = \begin{pmatrix} -\frac{\partial u}{\partial y} & -\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \end{pmatrix}.$$

This $df J_0 = J_0 df$ if and only if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, which just means that for all $j = 1, \dots, m$, $k = 1, \dots, n$, we have

$$\frac{\partial v^j}{\partial x_k} = -\frac{\partial v^j}{\partial y_k}, \quad \frac{\partial v^j}{\partial y_k} = \frac{\partial v^j}{\partial x_k},$$

so if we write $f_j = u_j + iv_j$, then these are holomorphic for all $j = 1, \dots, n$ if and only if f is holomorphic. □

The question we ask now is: Which almost complex structures arise from holomorphic structures? The answer turns out to be no, and we now describe the condition for it to be true. Let (M, J) be an almost complex manifold.

Definition 4.4.7. An almost complex structure is called *integrable* if and only if for all pairs $v_1, v_2 \in C^\infty(TM)$ we have

$$N_J(v_1, v_2) := (L_{Jv_1} J)v_2 - J(L_{v_1} J)v_2 = 0.$$

Claim 4.4.8. *N_J is a tensor of type $(1, 2)$. This follows from properties of the Lie derivative. Recall that $(L_v J)w = [v, Jw] - J[v, w]$. From this we obtain the expression*

$$N_J(v_1, v_2) = [Jv_1, Jv_2] - J[Jv_1, v_2] - [J[v_1, Jv_2]] - [v_1, v_2].$$

Noting this, N_J is called the Nijenhuis tensor*.*

Theorem 4.4.9 (Newlander–Niremberg)**.** *The almost complex structure on an almost complex manifold (M, J) arises from a holomorphic structure on M with caononical almost complex structure J if and only if $N_J = 0$.*

We prove only that holomorphicity implies integrability. The other direction is hard.

We need to introduce so called complexifications, and begin with some linear algebra: Let V be a real vector space with complex structure J. We define the *complexification* $V_c := V \otimes_{\mathbb{R}} \mathbb{C}$ of V with $\dim_{\mathbb{C}} V_c = \dim_{\mathbb{R}} V$. We have an inclusion $I : V \to V_c$ mapping $v \mapsto v \otimes 1$. Similarly, we have a conjugation $\ : V_c \to V_c$ which maps $v \otimes \alpha \to \overline{v \otimes \alpha} := v \otimes \overline{\alpha}$. Note that $\operatorname{Im} I = \{w \in V_c : \overline{w} = w\}$. Consider the $\mathbb{C}$-linear extension of $J : V \to V$, also denoted $J : V_c \to V_c$ which maps $v \otimes \alpha \mapsto J(v \otimes \alpha) := Jv \otimes \alpha$. Then $J^2 = -\operatorname{Id}$ on V_c, which therefore splits into the eigenspaces $V_c = V^{1,0} \oplus V^{0,1}$ for i and $-i$ respectively. It is easy to see that $\overline{V^{1,0}} = V^{0,1}$. We have a $\mathbb{C}$-linear isomorphism $(V, J) \to V^{1,0}$ which maps $v \mapsto \frac{1}{2}(v \otimes 1 - Jv \otimes i)$. This isomorphism is just $\pi_{1,0} \circ I$, where $\pi_{1,0} : V_c \to V^{1,0}$ maps $w \mapsto \frac{1}{2}(w - iJw) =: w^{1,0}$. We also have a projection $\pi_{0,1} : V_c \to V^{0,1}$ which maps $w \mapsto \frac{1}{2}(w + iJw) =: w^{0,1}$. Finally, if V is complex, then $(V_0, J) \cong V_0^{1,0}$ by the isomorphism $\pi_{1,0} \circ I$.

Example 4.4.10. The following example is fundamental and provides all the tools necessary for dealing with almost complex manifolds. Consider $T_0\mathbb{R}^{2n}$, the algebra of germs of smooth functions $C^\infty_{\mathbb{R}^{2n},0}$. The complexified space $(T_0\mathbb{R}^{2n})_c$ is the space $\mathcal{C}^\infty_{\mathbb{R}^{2n},0} \otimes \mathbb{C}$ of germs of complex valued smooth functions, $D \otimes \alpha([f]) = \alpha(D(u) + iD(v))$ where $f = u + iv$ for u, v real-valued.

Consider the map $\mathbb{R}^{2n} \to \mathbb{C}^n$, $(x_1, y_1, \dots, x_n, y_n) \mapsto (x_1 + iy_1, \dots, x_n + iy_n)$. This induces a map $\mathcal{O}_{\mathbb{C}^n,0} \hookrightarrow C^\infty_{\mathbb{R}^{2n},0} \otimes \mathbb{C}$.

We have a map $h : (T_0\mathbb{R}^{2n})_c \to T_0\mathbb{C}^n$ which maps $D \mapsto h(D)([f]_{\mathcal{O}}) = D([f]_{C^\infty \otimes \mathbb{C}})$ with $\dim \ker h = n$. We want to describe this kernel. We use the canonical complex structure $T_0\mathbb{R}^{2n}$ to clarify this. Take coordinates $(x_1, y_1, \dots, x_n, y_n)$ and the associated basis $\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\}$. Then $J\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}$ and $J\frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}$. As before, we have a decomposition $(T_0\mathbb{R}^{2n})_c = (T_0\mathbb{R}^{2n})^{1,0} \oplus (T_0\mathbb{R}^{2n})^{0,1}$. We claim that $\ker h = (T_0\mathbb{R}^{2n})^{0,1}$. To prove this, note that $(T_0\mathbb{R}^{2n})^{0,1} = \pi_{0,1} I T_0\mathbb{R}^{2n}$, so we have a basis

$$(T_0\mathbb{R}^{2n})^{0,1} = \left\langle \pi_{0,1}\left(\frac{\partial}{\partial x_j} \otimes 1\right)\right\rangle = \left\langle \frac{1}{2}\left(\frac{\partial}{\partial x_j} \otimes 1 + \frac{\partial}{\partial y_j} \otimes i\right)\right\rangle.$$

Therefore we need to prove that the latter derivations kill all holomorphic functions. As before, write $f = f(x,y) = u(x,y) + iv(x,y)$. The derivation of f at 0 is

$$\left(\frac{\partial}{\partial x_j} \otimes 1 + \frac{\partial}{\partial y_j} \otimes i\right)(u(x,y) + iv(x,y))$$
$$= \left(\frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial y_j}\right)|_0 + i\left(\frac{\partial v}{\partial x_j} + \frac{\partial u}{\partial y_j}\right)|_0 = 0$$

by the Cauchy–Riemann equation (freezing all variables except (x_j, y_j)).

Thus, we have an identification $(T_0\mathbb{R}^{2n})^{1,0} \overset{h}{\cong} T_0\mathbb{C}^n$. Finally

$$h\left(\frac{1}{2}\left(\frac{\partial}{\partial x_j} \otimes 1 - \frac{\partial}{\partial y_j} \otimes i\right)\right) = \frac{\partial}{\partial z_j}, \tag{1}$$

so from now on, we simply write

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right).$$

To prove (1), let $f(z) = u(x,y) + iv(x,y) \in \mathcal{O}_{\mathbb{C}^n,0}$ and write $z_j = x_j + iy_j$. Then

$$\begin{aligned}
\frac{\partial}{\partial z_j} f &= \lim_{t\to 0} \frac{f(z_j)}{z_j} \\
&= \lim_{x_j,y_j\to 0} \frac{(u(x_j,y_j) + iv(x_j,y_j))(x_j - iy_j)}{|x_j|^2 + |y_j|^2} \\
&= \lim_{t\to 0} \frac{(u(t,t) + iv(t,t))(1-i)}{2t} \\
&= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x_j} + \frac{\partial u}{\partial y_j}\right) + i\left(\frac{\partial v}{\partial x_j} + \frac{\partial u}{\partial y_j}\right)\right]|_0(1-i) \\
&= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x_j} + \frac{\partial v}{\partial y_j}\right) + i\left(\frac{\partial v}{\partial x_j} - \frac{\partial u}{\partial y_j}\right)\right]|_0(1-i) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x_j} \otimes 1 - \frac{\partial}{\partial y_j} \otimes i\right)(f).
\end{aligned}$$

Here, we have taken the derivative along $x_j = y_j = t$ and used the Cauchy–Riemann equations.

In conclusion, derivations $\frac{\partial}{\partial z_j}$ of germs of holomorphic functions can be identified with their extensions to derivations $\frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$ of germs of smooth complex-valued functions. We define

$$\frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} \otimes 1 + \frac{\partial}{\partial y_j} \otimes 1\right).$$

These generate $(T_0\mathbb{R}^{2n})^{0,1}$ and a smooth differentiable function $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}_j} = 0$ for all j.

13th lecture, October 4th 2011

The question we are going to address is the following: Which almost complex structures on a differentiable manifold M arise from holomorphic structures? First, we characterize the condition of integrability of complex structures in terms of something related to complexifications.

Let (M, J) be an almost complex manifold with complexified tangent bundle $TM_c := TM \otimes_{\mathbb{R}} \mathbb{C}$, which is a smooth complex vector bundle over M.

The almost complex structure J extends to $J : TM_c \to TM_c$ by $\mathbb{C}$-linearity, $J \in \operatorname{End}(TM_c)$, $J(v \otimes \alpha) = Jv \otimes \alpha$.

To this J we have so called the eigenbundles, $TM = T^{1,0}M \oplus T^{0,1}M$, where $T^{1,0}M$ and $T^{0,1}M$ are the subbundles where $J = i$ and $J = -i$ respectively.

Remark 4.4.11. Consider the endomorphism $J \pm i\,\mathrm{Id}$ of TM_c which has constant rank. Then $\ker(J \pm i\,\mathrm{Id})$ is a subbundle.

We have projections $\pi^{1,0} : TM_c \to T^{1,0}M$, which maps $w \mapsto (w - iJw)/2 =: w^{1,0}$, and $\pi^{0,1} : TM_c \to T^{0,1}M$, which maps $w \mapsto (w + iJw)/w =: w^{0,1}$. We have a canonical inclusion $TM \to TM_c$, which maps $v \mapsto v \otimes 1$. We are going to endow TM_c with a Lie bracket which will be helpful in determining the integrability condition.

The Lie Bracket $C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$ extends in a unique way $\mathbb{C}$-linearly to

$$[\cdot,\cdot] : C^\infty(TM_c) \times C^\infty(TM_c) \to C^\infty(TM_c) \cong C^\infty(TM) \otimes \mathbb{C},$$

defined by

$$[v_1 \otimes \alpha_1, V_2\alpha_2] = [v_1, v_2] \otimes \alpha_1 \cdot \alpha_2.$$

for $v_1, v_2 \in C^\infty(TM)$, $\alpha_1, \alpha_2 \in \mathbb{C}$. This defines the bracket on all of $C^\infty(TM)$.

Proposition 4.4.12. *The following are equivalent:*

1. *The almost complex structure J is integrable.*
2. *The Lie bracket preserves $T^{1,0}$, i.e. $[v_1, v_2] \in C^\infty(T^{1,0})$ for $v_1, v_2 \in C^\infty(T^{1,0})$.*
3. *The Lie bracket preserves $T^{0,1}$.*

Proof. Recall that conjugation defines an isomorphism $\overline{T^{1,0}M} \cong T^{0,1}M$. Now (2) and (3) are equivalent by conjugation, since e.g. $\overline{[v_1^{0,1}, v_2^{0,1}]} = [\overline{v_1^{0,1}}, \overline{v_2^{0,1}}] \in C^\infty(T^{1,0}M)$.

To see that (1) and (2) are equivalent, note that $C^\infty(TM)$ generates $C^\infty(T^{1,0}M)$, $\pi_{1,0}I(C^\infty(TM)) = C^\infty(T^{1,0}M)$. For $V \in C^\infty(TM)$, note that $2\pi_{1,0}IV = V - iJV$. Define $Z = [V_1 - iJV_1, V_2 - iJV_2]$, and computations show that

$$iZ - JZ = J(N_J(V_1, V_2) - iN_J(V_1, V_2)),$$

so $iZ - JZ$ vanishes if and only if the Nijenhuis tensor does. □

Proposition 4.4.13. *The almost complex structure of a complex manifold X is integrable.*

Proof. In holomorphic coordinates, V of type $(1,0)$ can be written $V = \sum_k V^k \frac{\partial}{\partial z_k}$. Here, we use coordinates $\pi : V \to \mathbb{C}^n \to \mathbb{R}^{2n}$ and $V^k \in C^\infty(\mathbb{R}^{2n}) \otimes \mathbb{C}$. Identify $T^{1,0}X_0|_V \cong T^{1,0}\mathbb{R}^{2n}$, where X_0 is the smooth manifold underlying X. For V_1, V_2 of type $(1,0)$,

$$[V_1, V_2] = \sum_{j,k} V_1^k \frac{\partial V_2^j}{\partial z_k}\frac{\partial}{\partial z_j} - V_2^k \frac{\partial V_1^j}{\partial z_k}\frac{\partial}{\partial z_j},$$

which is of type $(1,0)$ again. □

The proof seems almost tautological but it was essential that we were able to use holomorphic coordinates. Our goal is to prove the converse, the Newlander–Niremberg theorem, in the case of real analytic manifolds.

We first want to characterize holomorphic vector fields on X a complex manifold. Recall that we have maps $T_0\mathbb{R}^{2n} \to (T_0\mathbb{R}^{2n})_c \to T_0\mathbb{C}^n$ which globally gives rise to morphisms $TX_0 \to (TX_0)_c \to (TX)_0$ of bundles over X_0. A vector field $V \in C^\infty(TX_0)$ induces a section $V^{1,0} \in C^\infty(TX)_0$. On the latter space, we have holomorphic coordinates which is what we want to relate to the almost complex structure on TX_0.

Proposition 4.4.14. *The section $V^{1,0}$ is holomorphic if and only if $L_V J = 0$.*

Proof. Note that $\tau \in C^\infty(\mathrm{End}TX_0)$ gives a section $\tau \in C^\infty(\mathrm{End}(TX_0)_c)$. We want to evaluate $L_V J$ on elements of $(TX_0)_c$. Identifying $\mathbb{C}^n \cong \mathbb{R}^{2n} = (x_1, y_1, \ldots, x_n, y_n)$, we can write V in holomorphic coordinates as

$$\begin{aligned} V &= \sum_j V_1^j \frac{\partial}{\partial x_j} + V_2^j \frac{\partial}{\partial y_j} \\ &= \sum_j V^j \frac{\partial}{\partial z_j} + \sum_j \overline{V^j} \frac{\partial}{\partial \overline{z}_j} = V^{1,0} + V^{0,1}, \end{aligned}$$

where $V^j = V_1^j + iV_2^j$. We evaluate $L_V J$ on $\frac{\partial}{\partial \overline{z}_m}$ and find

$$\begin{aligned} L_V J\left(\frac{\partial}{\partial \overline{z}_m}\right) &= \left[V, J\frac{\partial}{\partial \overline{z}_m}\right] - J\left[V, \frac{\partial}{\partial \overline{z}_m}\right] \\ &= -i\left[V^{1,0} + V^{0,1}, \frac{\partial}{\partial \overline{z}_m}\right] + J\left[V^{1,0} + V^{0,1}, \frac{\partial}{\partial \overline{z}_m}\right] \\ &= -2i\left[V^{1,0}, \frac{\partial}{\partial \overline{z}_m}\right] = -2i\sum_j \frac{\partial V^j}{\partial \overline{z}_m}\frac{\partial}{\partial z_j}. \end{aligned}$$

Thus $L_V J(\frac{\partial}{\partial \overline{z}_m}) = 0$ for all m if and only if every V^j is holomorphic. Now, $L_V J, V, J$ are real (i.e. invariant under conjugation), so we can take conjugates and find

$$\overline{(L_V J)\left(\frac{\partial}{\partial \overline{z}_m}\right)} = L_V J\left(\frac{\partial}{\partial z_m}\right),$$

and we can obtain the Proposition also for elements of type $(1,0)$. ☐

4.4.1 Complex valued forms

We begin with some linear algebra. Let (V, J) be a real $2n$-dimensional vector space with a complex structure J. Decompose as before $V = V^{1,0} \oplus V^{0,1}$, and consider the exterior algebras $\wedge V_c = \bigoplus_{k>0} \wedge^k V_c$, and similarly $\wedge V^{1,0}$, $\wedge V^{0,1}$. Let $V^{p,q}$ be the subspace of $\wedge V_c$ generated by elements of the form $u \wedge v$, where $u \in \wedge^p V^{1,0}$, $v \in \wedge^q V^{0,1}$ using the inclusions $\wedge V^{1,0} \to \wedge V_c$, $\wedge V^{0,1} \to \wedge V_c$. Then

$$\wedge V_c = \bigoplus_{r=0}^{2n} \bigoplus_{p+q=r} V^{p,q}.$$

In the global picture, let (M, J) be an almost complex manifold, and let $T^*M_c = T^*M \otimes \mathbb{C}$ be the complexified cotangent bundle. Let $\Omega^k(M)_c = \Omega^k(M) \otimes \mathbb{C} \cong C^\infty(\wedge^k T^*M_c)$. The exterior derivative

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \to \cdots$$

extends by $\mathbb{C}$-linearity to give a complex

$$0 \to \Omega^0(M)_c \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{2n}(M)_c \to 0,$$

where $2n = \dim M$, and d is $\mathbb{C}$-linear and satisfies $d^2 = 0$. Now, ysing J, endow T^*M with an almost complex structure (i.e. an endomorphism $J : T^*M \to T^*M$ such that $J^2 = -\mathrm{Id}$) given by $J \cdot \alpha_x := \alpha_x(J\cdot)$. Then we obtain a decomposition into $\pm i$-eigenbundles,

$$T^*M_c = T^*M^{1,0} \oplus T^*M^{0,1}.$$

By definition, we have an isomorphism $T^*M^{1,0} \cong (T^{1,0}M)^*$ of smooth complex bundles over M. Defining $T^*M^{p,q}$ fibrewise, the exterior algebra bundle decomposes as

$$\wedge T^*M_c = \bigoplus_{r=0}^{2n} \bigoplus_{p+q} T^*M^{p,q}.$$

Finally, the degree k complexified forms decompose as

$$\Omega^k(M)_c = \bigoplus_{p+q=k} \Omega^{p,q}(M),$$

where $\Omega^{p,q}(M) = C^\infty(T^*M^{p,q})$. We give a local expression for these.

Definition 4.4.15. Given a differentiable resp. holomorphic bundle $E \to X$ over X a differentiable resp. complex manifold, a *frame* of E on an open subset $U \subseteq X$ is a family $\{s_1, \dots, s_r\}$, $r = \operatorname{rank} E$ of differentiable resp. holomorphic sections of $E|_U$ such that $\{s_1|_x, \dots, s_r|_x\}$ is a basis of E_x for all $x \in U$.

Remark 4.4.16. Local frames are the same as trivializations. To see this, note that to a local frame $\{s_1, \dots, s_r\}$, we associate the isomorphism $U \times \mathbb{C}^r \to E|_U$ given by $(x, (v_1, \dots, v^r)) \mapsto \sum v^j s_j|_x$. Conversely, if $\psi : E|_U \to U \times \mathbb{C}^r$ is an isomorphism, define a frame by $s_j|_x = \psi^{-1}(x, (0, \dots, 1, \dots, 0))$ where the 1 is in the j'th place.

Take $\{v_1, \dots, v_n\}$ to be a differentiable local frame of $T^*M^{1,0}$ (e.g. by taking a trivialization). Then $\{\overline{v_1}, \dots, \overline{v_n}\}$ is a local frame of $T^*M^{1,0}$. Then a local frame for $T^*M^{p,q}$ is $\{v^I \wedge \overline{v}^J\}$, where $v^I = v_1^{i_1} \wedge \dots \wedge v_n^{i_n}$ for $I = (i_1, \dots, i_p)$ and similarly for $\overline{v}^J$. Here $|I| = \sum_j i_j = p$ and $|J| = q$. Then a form $\alpha \in \Omega^{p,q}(M)$ can be written

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{I,J} v^I \wedge \overline{v}^J,$$

where $\alpha_{I,J} \in C^\infty(M) \otimes \mathbb{C}$. The exterior derivative becomes

$$d\alpha = \sum_{|I|=p, |J|=q} d(\alpha_{I,J}) \wedge v^I \wedge \overline{v}^J + \alpha_{I,J} d(v^I \wedge \overline{v}^J).$$

14th lecture, October 5th 2011

Let (M, J) be an almost complex manifold with complexified cotangent bundle $(T^*M)_c = T^*M^{1,0} \oplus T^*M^{0,1}$. Recall that we had

$$\Omega^k(M)_c = \bigoplus_{p+q=k} \Omega^{p,q}(M),$$

where $\Omega^{p,q}(M) = C^\infty(T^*M^{p,q})$. The exterior derivative d extended to $\Omega^*(M)_c = \bigoplus_{r \geq 0} \Omega^r(M)_c$. Consider the projections $\pi_{p,q} : \Omega^k(M)_c \to \Omega^{p,q}(M)_c$ where $p + q = k$. Define

$$\partial = \pi_{p+1,q} \circ d : \Omega^{p,q}(M)_c \to \Omega^{p+1,q}(M)$$
$$\overline{\partial} = \pi_{q,p+1} \circ d : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M).$$

The magic of these operators are that they allow us to characterize integrability in a new way.

Proposition 4.4.17. *The almost complex structure J is integrable if and only if (the extended) d satisfies $d = \partial + \overline{\partial}$, i.e. that*

$$d(\Omega^{p,q}(M)) \subseteq \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M).$$

Proof. Let $\alpha \in \Omega^k(M)_c$. We can write

$$d\alpha(v_0, v_1, \ldots, v_k) = \sum_{i=0}^{k}(-1)^i v_i(\alpha(v_0, \ldots, \hat{v_i}, \ldots, v_k)) + \sum_{i<j}(-1)^{i+j}\alpha([v_i, v_j], \ldots, \hat{v_i}, \ldots, \hat{v_j}, \ldots, v_k).$$

Assume that $d = \partial + \overline{\partial}$. Note that

$$C^\infty(T^{1,0}(M)) = \{v \in C^\infty(TM_c) \mid \alpha(v) = 0\, \forall \alpha \in \Omega^{0,1}(M)\}.$$

Let $v_0, v_1 \in C^\infty(T^{1,0}M)$. We need to show that $[v_0, v_1] \in C^\infty(T^{1,0}M)$. Let $\alpha \in \Omega^{0,1}(M)$. Then

$$\alpha([v_0, v_1]) = -d\alpha(v_0, v_1) - v_0(\alpha(v_1)) + v_1(\alpha(v_0)) = -d\alpha(v_0, v_1) = 0,$$

since $d\alpha \in \Omega^{1,1} \oplus \Omega^{0,2}$ by the assumption on d.

Assume now that J is integrable. Take $\alpha \in \Omega^{p,q}(M)$ and let us prove that

$$d\alpha(v_1^{1,0}, \ldots, v_j^{1,0}, v_1^{0,1}, \ldots, v_l^{0,1}) = 0,$$

when (j, l) is neither $(p+1, q)$ nor $(p, q+1)$. Applying the formula from before, we find that $d\alpha(v_1^{1,0}, \ldots, v_j^{1,0}, v_1^{0,1}, \ldots, v_l^{0,1})$ consists of terms of the forms

$$\begin{aligned} v_i^{1,0}(\alpha(v_1^{1,0}, \ldots, \hat{v_i^{1,0}}, \ldots, v_j^{1,0}, v_1^{0,1}, \ldots, v_l^{0,1})), \\ v_i^{0,1}(\alpha(v_1^{1,0}, \ldots, v_j^{1,0}, v_1^{0,1}, \ldots, \hat{v_i^{0,1}}, \ldots, v_l^{0,1})), \\ \alpha([v_i^{1,0}, v_m^{1,0}], \ldots), \\ \alpha([v_i^{1,0}, v_m^{0,1}], \ldots), \alpha([v_i^{0,1}, v_m^{0,1}], \ldots). \end{aligned}$$

Terms of the first form vanish since we have $j-1$ vector fields of type $(1,0)$ and l of type $(0,1)$, and $(j-1, l) \neq (p, q)$. Terms of the second form vanish for the same reason. The ones of third form vanish since by integrability $[v_i^{1,0}, v_m^{1,0}]$ is again of type $(1,0)$, and we can use the same argument as for the first two cases. The same thing goes for those of the fifth form. Finally, those of the forth form vanish since we are removing terms of both type $(1,0)$ and $(0,1)$ and add at most one of them again.

Here is an alternative local proof: Let (M, J) be integrable and take holomorphic coordinates $\psi : V \to \mathbb{C}^n \equiv \mathbb{R}^{2n}$. Then we have frames

$$\begin{aligned} \left\{\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)\right\} \subseteq C^\infty(T^{1,0}\mathbb{R}^{2n}), \\ \left\{\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)\right\} \subseteq C^\infty(T^{0,1}\mathbb{R}^{2n}). \end{aligned}$$

We have dual frames

$$\begin{aligned} \{dz_j = dx_j + i dy_j\} \subseteq \Omega^{1,0}(\mathbb{R}^{2n}), \\ \{d\overline{z}_j = dx_j - i dy_j\} \subseteq \Omega^{0,1}(\mathbb{R}^{2n}). \end{aligned}$$

Note here that we really have $d(\overline{z}_j) = d(\overline{x_j + iy_j})$ where $z_j = x_j + iy_j$. For a function $f \in C^\infty(\mathbb{R}^{2n}) \otimes \mathbb{C}$, we can write

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j, \quad \overline{\partial} f = \sum_{j=1} n \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j.$$

In general, for a form $s \in \Omega^{p,q}\mathbb{R}^{2n}$, we write $s = \sum_{I,J} a_{I,J} dz^I \wedge d\overline{z}^J$, and the operators act on s by

$$\partial s = \sum_{j=1}^{n} \sum_{I,J} \frac{\partial a_{I,J}}{\partial z_j} dz_j \wedge dz^I \wedge d\overline{z}^J,$$

and similarly for $\overline{\partial}$. Then it is an exercise to see that if J is integrable, then $d = \partial + \overline{\partial}$. □

4.4.2 Proof of the Newlander–Niremberg Theorem

We turn now to the Newlander–Niremberg Theorem in the real analytic case. Let (M, J) be an almost complex manifold of dimension $\dim_{\mathbb{R}} M = 2n$.

Lemma 4.4.18. *If every point $p \in M$ has a neighbourhood U and n complex valued functions $f_1, \ldots, f_n$ such that if $df_j \in \Omega^{1,0}(U)$ for all j and such that $\{df_j\}$ are linearly independent (i.e. such that $\{df_j\}_{j=1}^n$ is a frame for $\Omega^{1,0}(M)$). Then J arises from a holomorphic structure on M.*

Proof. Consider $f : U \to \mathbb{C}^n$ mapping $x \mapsto (f^1(x), \ldots, f^n(x))$. Then

$$df = (df^1, \ldots, df^n) = ((df^1)^{1,0}, \ldots, (df^n)^{1,0}),$$

where we identify $df : TU \to T\mathbb{C}^n$ with $df : (TU)_c \to (T\mathbb{R}^{2n})_c$. We use $\equiv$ when we want to make this identification explicit. The map $f : U \to \mathbb{C}^n \cong \mathbb{R}^{2n}$ is a diffeomorphism, and we find that

$$df(J\,\cdot) \equiv (df^1(J\,\cdot), \ldots, df^n(J\,\cdot)) = i(df^1(\cdot), \ldots, df^n(\cdot)) \equiv J_0\, df, .$$

where J_0 is the standard complex structure on $\mathbb{C}^n$. Take f, f' as above. Then $f \circ f' : f(U) \to \mathbb{C}^n$ is holomorphic and these define holomorphic coordinates defining a holomorphic structure on M with almost complex structure J. ☐

Our goal now is the following: Given (M, J) with integrable almost complex structure, we want it to satisfy the properties of the Lemma. Take $p \in M$ and $U \subseteq M$ a real coordinate chart, identified with $U \subseteq \mathbb{R}^{2n} \cong \{(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n})\}$ such that $p \equiv 0$. Define a frame for $(T^*U)^{1,0}$ by $\{\omega^j = dx_j + iJdx_J \in \Omega^{1,0}(U), j = 1, \ldots, n\}$. We now want to find functions f_j such that $df_j = \omega^j$.

Define $\omega^{\bar{j}} = \overline{\omega^j} \in \Omega^{0,1}(U)$. These form a frame for $(T^*U)^{0,1}$.

The idea is the following: We want to extend $\mathbb{R}^{2n}$ to $\mathbb{C}^{2n}$. We write $\mathbb{R}^{2n} \equiv (x_1, \ldots, x_{2n})$ and $\mathbb{C}^{2n} \equiv (z_1, \ldots, z_{2n})$ where $z_j = x_i + iy_j$, so we consider the neighbourhood U as sitting inside the real part of $\mathbb{C}^{2n}$. We can identify $(T\mathbb{R}^{2n})_c \cong T\mathbb{C}^{2n}|_{\mathbb{R}^{2n}}$ and $(T^*\mathbb{R}^{2n})_c \cong T^*\mathbb{C}^{2n}|_{\mathbb{R}^{2n}}$. We will try to extend the $\{\omega^j\}$ to holomorphic 1-forms Ω^j on $U^* \subseteq \mathbb{C}^{2n}$ a neighbourhood of the origin (such that $\operatorname{Re} U^* \equiv U$). We then want to apply the holomorphic Frobenius Theorem to obtain holomorphic functions f_j such that $df_j = \partial f_j = \Omega^j$.

15th lecture, October 6th 2011

Consider (M, J) almost complex, and consider a coordinate patch $U \subseteq M$, with coordinates $U = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n})$. We consider the frame of 1-forms $\omega^j = dx_j + iJdx_j$, $j = 1, \ldots, n$ and $\omega^{\bar{j}} = \overline{\omega^j}$.

Lemma 4.4.19. *If J is integrable, then*

1. *we can write $d\omega^j = \sum_{h<k} a^j_{hk} \omega^h \wedge \omega^k + \sum_{h,k} b^j_{hk} \omega^h \wedge \omega^{\bar{k}}$,*
2. *and we c an write $d\omega^{\bar{j}}$ as the conjugate of the above.*

Proof. This follows from our characterization of integrability, since the ω^j form a frame of $\Omega^{1,0}(M)$ and the $\omega^{\bar{j}}$ one of $\Omega^{0,1}(M)$, since $d\Omega^{p,q} \subseteq \Omega^{p+1,q} \oplus \Omega^{p,q+1}$, and since the expression in the first part of the Lemma is a sum of forms of type $(2, 0)$ and $(1, 1)$. ☐

Now we use the assumption that J is real analytic. Take a real analytic structure on M (such a thing always exists). Then J is real analytic in real analytic coordinates on M, and we suppose that U is one of these real analytic coordinate charts. Then ω^j and $\omega^{\bar{j}}$ are real analytic. This means that if we write

$$\omega^j = \sum_{k=1}^{2n} f^j_k(x) dx_k, \quad \omega^{\bar{j}} = \sum_{k=1}^{2n} \overline{f^j_k} dx_k,$$

then $f_k^j, \overline{f_k^j} : U \to \mathbb{R}$ are real analytic. Now, take holomorphic extensions $F_k^j : U^* \to \mathbb{C}$ and $F_k^{\bar{j}} : U^* \to \mathbb{C}$, where here $0 \in U^* \subseteq \mathbb{C}^{2n} = (z_1, \ldots, z_{2n})$ such that $F_k^j|_U = f_j^k$, $F_k^{\bar{j}}|_U = \overline{f_j^k}$. Note that in general, we do not have $F_k^{\bar{j}} \neq \overline{F_k^j}$.

Define holomorphic $(1,0)$-forms on U^* by

$$\Omega^j = \sum_{k=1}^{2n} F_k^j dz_k, \quad \Omega^{\bar{j}} = \sum_{k=1}^{2n} F_k^{\bar{j}} dz_k.$$

Since $\{\omega^j, \omega^{\bar{j}}\}$ are linearly independent at $0 \in U$, we have that $\{\Omega^j, \Omega^{\bar{j}}\}$ are linearly independent at $0 \in U^*$, and so they are on a neighbourhood that we will also denote U^*. Here we identify $(T^*\mathbb{R}^{2n})_c = T^*\mathbb{C}^{2n}|_{\mathbb{R}^{2n}}$.

So, we have a frame $\{\Omega^j, \Omega^{\bar{j}} \mid j = 1, \ldots, n\}$ for $\Omega^{1,0}(U^*)$. Moreover, since Ω^j, $\Omega^{\bar{j}}$ are holomorphic, we have $\bar{\partial}\Omega^j = \bar{\partial}\Omega^{\bar{j}} = 0$, so $d\Omega^j$ and $d\Omega^{\bar{j}}$ are of type $(2,0)$, and we can write

$$d\Omega^j = \sum_{h<k} A_{hk}^j \Omega^h \wedge \Omega^k + \sum_{h,k} B_{hk}^j \Omega^h \wedge \Omega^{\bar{k}} + \sum_{h<k} C_{hk}^j \Omega^{\bar{h}} \wedge \Omega^{\bar{k}},$$

where all $A_{hk}^j, B_{hk}^j, C_{hk}^j$ are holomorphic functions. Since $d\Omega^j|_{\mathbb{R}^{2n}} = \omega^j$, we have that $C_{hk}^j|_{\mathbb{R}^{2n}} = 0$, since $dz_k|_{\mathbb{R}^{2n}} = dx_k$. Since the C_{hk}^j are holomorphic, we find $C_{hk}^j \equiv 0$ on $\mathbb{C}^{2n}$. Hence, $d\Omega^j$ can be expressed as wedge products of Ω^k's with other forms and so, we have the hypotheses of the following theorem satisfied.

Theorem 4.4.20 (Holomorphic Frobenius Theorem)**.** *Let $\varphi_1, \ldots, \varphi_r$ be holomorphic 1-forms on a neighbourhood V of $0 \in \mathbb{C}^m$. If $\{\varphi_j\}$ is linearly independent at each point, and moreover $d\varphi_j = \sum_{k=1}^r \psi_k^j \wedge \varphi_k$ for holomorphic 1-forms ψ_k^j, then there exist functions $g_1, \ldots, g_r : U^* \to \mathbb{C}$ such that*

$$\varphi_j = \sum_{k=1}^r p_k^j dg_k$$

with all p_k^j holomorphic.

Applying this to our system $\{\Omega^j\}$, we find $F_j : U^* \subseteq \mathbb{C}^{2n} \to \mathbb{C}$ holomorphic, $j = 1, \ldots, n$, such that $\Omega^j = \sum_k P_k^j dF_k$ with P_j^k holomorphic. We write $f_k = F_k|_U$. Then

$$\omega^j = \sum_{k=1}^n p_k^j df_k, \quad p_k^j = P_k^j|_U.$$

In particular, $\{df_k\}_{k=1}^n$ are a frame of $\Omega^{1,0}(U)$. That they are of type $(1,0)$ follows from the fact that $df_k = dF_k|_U = \partial F_k|_U$, where we use that the F_k are holomorphic. Thus the df_k satisfy the hypothesis of Lemma 4.4.18. This proves the Newlander–Niremberg theorem in the case of real analytic almost complex structures.

We give now the flavour of the proof of Theorem 4.4.20: Note that $\{\varphi^1, \ldots, \varphi^r\}$ generate an ideal

$$I \subseteq \Omega_{\text{hol}}^{*,0}(V) = \bigoplus_{k \geq q0} \Omega_{\text{hol}}^{k,0}(V) \equiv = \bigoplus_{k \geq 0} \{\alpha \in \Omega^{k,0}(V) : \partial\alpha = 0\}.$$

The ideal is given by

$$I = \{\alpha \in \Omega_{\text{hol}}^{*,0}(V) : \alpha = \sum_j \psi^j \wedge \varphi_i \text{ where } \psi^j \text{ are hol. } (k,0)\text{-forms}\},$$

and moreover, by the second assumption of the Theorem, $dI \subseteq I$. An ideal with this property is called a *differential ideal.* Define

$$D_I = \{v \in TV : v \lrcorner \varphi_i = 0 \, i = 1, \ldots, r\},$$

so $D_I|_p \subseteq TV|_p$ for all $p \in V$.

Proposition 4.4.21. *D_I is a* holomorphic distribution*, i.e. for all $p \in V$, we can find a neighbourhood U, $p \in U \subseteq V$ and holomorphic vector fields $\{X_{r+1}, \dots, X_m\}$ defined on U such that $D_I|_p = \langle x_{r+1}|_p, \dots, x_m|_p \rangle$ for all $p \in U$.*

Definition 4.4.22. A holomorphic distribution D is called *integrable* or *involutive* if $[x_1, x_2] \in D$ for all $x_1, x_2 \in D$. Here, we identify $TV \cong T^{1,0}V_0$ and use that V is a complex manifold.

Proposition 4.4.23. *D_I is involutive.*

Definition 4.4.24. A complex submanifold $N \subseteq V$ *integrates* D_I if $T_pN = (D_I)|_p \subseteq T_pV$ for all $p \in N$.

Note that (†) if for all $p \in V$ there is an integrating submanifold passing through $p \in V$, then D_I is involutive, since the restriction of brackets is the bracket of the restriction.

Theorem 4.4.25 (Frobenius Theorem, version 2)**.** *If D is involutive, then (†) holds. Moreover, there exist coordinates around each point such that $\{z : z_I = const, \forall i = 1, \dots r\}$.*

The link with the previous formulation of the Frobenius theorem is given by $z_i = g_i$ for $i = 1, \dots, r$.

To prove Theorem 4.4.25, one uses complex flows of holomorphic vector fields: For a holomorphic vector field v, one can construct $\varphi_v^z : \tilde{U} \to V$, $\tilde{U} \subseteq \mathbb{C} \times V$ satisfying

$$\frac{d}{dz}\varphi_v^z = v \circ \varphi_v^z, \quad \varphi_v^z(p) = p \,\forall p \in V.$$

Since D is involutive, $D = \langle v_{r+1}, \dots, v_n \rangle$, for v_i holomorphic with $[v_i, v_j = 0]$. We find that $\varphi_j^z \circ \varphi_i^{z'} = \varphi_i^{z'} \circ \varphi_j^z$ if and only if $[v_i, v_j] = 0$. Here $\varphi_j^z = \varphi_{v_j}^z$. Assume that $v_j|_0 = \frac{\partial}{\partial z_j}|_0$, $j = r+1, \dots, n$. Define $\Phi : \mathbb{C}^n \to \mathbb{C}^n$

$$\Phi(w_1, \dots, w_n) = \varphi_{r+1}^{w_{r+1}} \circ \cdots \varphi_n^{w_n}(w_1, \dots, w_r, 0, \dots, 0).$$

In these new coordinates, the v_j are the $\frac{\partial}{\partial w_j}$. and $\{w_j = \text{const.}\, j = 1, \dots, r\}$ are the *leaves* of the distribution.

An example of something which is not involutive is $v_1 = \frac{\partial}{\partial z_2}$, $v_2 = \frac{\partial}{\partial z_3} + z_2 \frac{\partial}{\partial z_1}$, since $[v_1, v_2] = \frac{\partial}{\partial z_1}$.

16th lecture, October 31th 2011

5 Hermitian metrics, connections, and curvature

5.1 Hermitian metrics

Let $E \to X$ be a smooth complex vector bundle over a smooth manifold M. Let $s \in C^\infty(U, E)$ be a local section, $U \subseteq M$ open, and let $\psi : E|_U \to U \times \mathbb{C}^r$ be trivializations. From this we get a frame $f = (e_1, \dots, e_r)$, where $e_i(x) = \psi^{-1}((x, (0, \dots, 1, \dots, 0)))$ with the 1 in the i'th place. Write $s = \sum_j s_j e_j$, and

$$s(f) = \begin{pmatrix} s_1 \\ \vdots \\ s_r \end{pmatrix}$$

with smooth functions $s_j : U \to \mathbb{C}$. Given $g \in C^\infty(U, \mathrm{GL}(\mathbb{C}^r))$, we can construct a new frame $f \cdot g = (e_1, \dots, e_r) \cdot g = (e_1', \dots, e_r')$, where $e_j' = \sum_{l=1}^r e_l g_l j$. In this new frame, the local section becomes $s(fg) = g^{-1}s(f)$.

Definition 5.1.1. A *hermitian metric* h on E is an assignment of a hermitian product on each fiber,

$$h_x(s_1|_x, s_2|_x), \quad s_1|_x, s_2|_x \in E_x,$$

such that for every open subset $U \subset M$ and local sections $s_1, s_2 \in C^\infty(U, E)$, the map

$$x \mapsto h_x(s_1|_x, s_2|_x)$$

is in $C^\infty(U) \otimes \mathbb{C}$. The pair (E, h) is called a smooth *hermitian vector bundle.*

Note that we use the convention that $h(v_1, \lambda v_2) = \overline{\lambda} h(v_1, v_2)$.

Given a frame $f = (e_1, \ldots, e_r)$, we define a matrix of functions by

$$h(f)_{jk} = h(e_k, e_j) =: \langle e_k, e_j \rangle.$$

This then defines a $M_{r\times r}(\mathbb{C})$-valued function $h(f) = (h(f)_{jk})$ on U such that $\overline{h(f)}^T = h(f)$.

Note that $h(s_1, s_2) = \overline{s_2(f)}^T h(f) s_1(f)$ for $s_j \in C^\infty(U, E)$.

Proposition 5.1.2. *Every smooth complex vector bundle E admits a hermitian metric.*

Proof. Choose a locally finite trivializing covering $\{U_\alpha\}_{\alpha \in I}$ such that $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^r$ and a partition of unity $\{\varphi_\alpha\}_{\alpha\in I}$ subordinate to this cover. Define

$$h^\alpha(s_1|_x, s_2|_x) = \overline{s_2(f : \alpha)}^T |_x s_1(f_\alpha),$$

where f_α is the frame on $E|_{U_\alpha}$. Now, define a hermitian metric by

$$h(s_1|_x, s_2|_x) = \sum_{\alpha \in I} \varphi_\alpha(x) h^\alpha(s_1|_x, s_2|_x).$$

□

Example 5.1.3 (Fubini–Study metric on the universal quotient bundle)**.** Take (V, h) a complex hermitian vector space and consider the Grassmannian

$$G_r(V) := \mathrm{GL}(\mathbb{C}^r)\backslash\{\varphi : V \twoheadrightarrow \mathbb{C}^r \ \mathbb{C}\text{-linear}\},$$

viewed as a smooth manifold. Over this we have the universal quotient bundle $Q_r \to G_r(V)$, which has fibers

$$Q_r|_{[\varphi]} = V/\ker\varphi \cong \mathbb{C}^r.$$

Consider the orthogonal projection $\Pi_{[\varphi]} : V \to \ker\varphi^\perp \subseteq V$ with respect to h. The *Fubini–Study metric* on Q_r is given by

$$h_{\mathrm{FS}}([v_1], [v_2])_{[\varphi]} = h(\Pi_{[\varphi]} v_1, \Pi_{[\varphi]} v_2) = h(\Pi_{[\varphi]} v_1, v_2).$$

This is well-defined, since the map $\Pi_{[\varphi]}$ defines an isomorphism $V/\ker\varphi \cong \ker\varphi^\perp$. Now, choose a basis $V \cong \mathbb{C}^n$, and denote by $H \in M_{r\times r}(\mathbb{C})$ the matrix $H \equiv h$, such that $h(v_1, v_2) = \overline{v_2}^T H V_1$ in the chosen basis. Then

$$G_r(V) = \mathrm{GL}(\mathbb{C}^r)\backslash\{A \in M_{r\times n}(\mathbb{C}) : A \text{ maximal rank}\}.$$

The orthogonal projection $\Pi_{[A]} : \mathbb{C} \to \ker A^\perp$ maps

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \mapsto A^*(AA^*)^{-1} A v,$$

where $A^* : \mathbb{C}^r \to \mathbb{C}^n$ is the adjoint matrix for H and the standard metric on $\mathbb{C}^r$, i.e. $\overline{A^* w}^T Hv = \overline{w}^T Av$, so $A^* = H^{-1}\overline{A}^T$. Thus the Fubini–Study metric is

$$h_{\mathrm{FS}}([v_1],[v_2])|_{[A]} = \overline{v_2}^T HA^*(AA^*)^{-1}Av_1.$$

For $r = 1$, recall that the Grassmannian is $\mathbb{P}(\mathbb{C}^n)$, and the hermitian metric on the universal quotient bundle is $1/|z|^2$, $H = \mathrm{Id}$.

5.2 Connections

Let $\Omega^p(E)$ be the space of *E-valued differential forms* of degree p on M, i.e.

$$\Omega^p(E) = C^\infty(M, \wedge^p T^*M \otimes_{\mathbb{C}} E).$$

Definition 5.2.1. A *connection* D on E is a $\mathbb{C}$-linear map $D : C^\infty(E) = \Omega^0(E) \to \Omega^1(E)$ satisfying the Leibniz rule $D(fs) = fDs + df \cdot s$, where $f \in C^\infty(M)$, $s \in C^\infty(E)$, and $\cdot$ in $df \cdot s$ means the image of the map $\Omega^p(M) \times C^\infty(E) \to \Omega^p(E)$ which maps $(\omega, s) \mapsto (x \mapsto (\omega_x \otimes_{\mathbb{C}} s_x))$.

Remark 5.2.2. The idea of connection is that for a vector field $v \in C^\infty(TM)$, the connection tells us how to differentiate sections with respect to vector fields via the assignment $s \mapsto v \lrcorner Ds = D_v s$. That it defines a differentiations follows from the Leibniz rule.

The process of differentiating a section is done by "connecting" different fibers through parallel transport. Before describing how, let us note that connections always exist. Note that if the bundle is $E = M \times \mathbb{C}$, then the exterior differential $d : C^\infty(M) \otimes \mathbb{C} \to \Omega^1(M) \otimes \mathbb{C}$ defines a connection. Generally, in a local trivialization $U \times \mathbb{C}^r$ we always have the trivial connection $d \times \cdots \times d$. Let $\{U_\alpha\}_{\alpha \in I}$ be a trivializing open cover of M. Then on every piece $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^r$, we have the trivial connection. For a partition of unity $\{\varphi_\alpha\}_{\alpha \in I}$, a section $s \in C^\infty(M, E)$, and a frame $f_\alpha = (e_1^\alpha, \ldots, e_r^\alpha)$ on U_α given by the trivialization, we define

$$\nabla s = \sum_\alpha \varphi_\alpha \cdot (f_\alpha \cdot d(s(f))) = \sum_\alpha \varphi_\alpha (\sum_j e_j^\alpha ds_j^\alpha),$$

This defines a connection ∇.

So what are the other connections? If D and D' connections on E, it turns out that $D - D' \in \Omega^1(\mathrm{End}E)$, where $(D - D') \cdot s = (Ds - D's)$. So, given one connection, all others are obtained by adding elements $\Omega^1(\mathrm{End}E)$, since for $B \in \Omega^1(\mathrm{End}E)$, we can define a connection $\nabla + B$ by $(\nabla + B)s = \nabla s + Bs$.

It follows that $\mathcal{A}_E$, the space of all connections, is an affine space modelled on $\Omega^1(\mathrm{End}E)$.

Definition 5.2.3. Given a local section $s \in C^\infty(U, E)$, we say that s is *parallel* with respect to a connection D, if $Ds = 0$.

Let $\gamma : I \to M$ be a smooth curve and s a *section along the curve*, i.e. a section $s \in C^\infty(I, \gamma^* E)$ or, equivalently, a curve $s : I \to E$ such that $ps(t) = \gamma(t)$ for all t, where $p : E \to M$ is projection. We say that s is *parallel along* γ if the pull-back of the connection D to $\gamma^* E$, also denoted D, satisfies $(D_{\frac{\partial}{\partial t}} s)|_t = 0$. In the other picture, this is equivalent to saying that for an arbitrary smooth extension of $s : I \to E$ to a local section of E, we have $D_{\dot\gamma(t)} s = 0$.

Claim 5.2.4. *Given γ with $\gamma(0) = x$ and $s_x \in E_x$, then there exists a unique $s_t \in E_{\gamma(t)}$ such that $D_{\dot\gamma(t)} s_t = 0$ and $s_0 = s_x$. That is, we can uniquely lift a curve in M to a curve $\tilde\gamma$ in E that is parallel with respect to D. This curve is called the* horizontal lift *of γ with respect to D.*

This property defines an isomorphism $\Gamma_{0,t}^\gamma : E_{\gamma(0)} \to E_{\gamma(t)}$ mapping $s_0 \mapsto s_t$ called parallel transport*.*

To prove this, we need to find a local expression for D. Pick a frame $f = (e_1, \dots, e_r)$ of local sections e_i on $U \subseteq M$. Define the *connection matrix*

$$\theta = \theta(D, f) \in \Omega^1(U \times M_{r\times r}(\mathbb{C})),$$
$$De_j = \sum_{k=1}^{r} \theta_{kj} e_k = (f \cdot \theta)_j.$$

For $s = \sum s_j e_j = f \cdot s(f) \in C^\infty(U, E)$, the Leibniz rule tells us that

$$\begin{aligned} Ds = D(f \cdot s(f)) &= \sum_j (ds_j) \cdot e_j + s_j De_j \\ &= f d(s(f)) + f \cdot \theta(f) \cdot s(f) = f(ds(f) + \theta s(f)) \end{aligned}$$

where in $d(s(f))$, we take the differential in every component of $s(f)$.

With this formula, the equation for parallel transport is very simple. For a local section $s : I \to E|_U \cong U \times \mathbb{C}^r$ along a curve γ, the equation is

$$D_{\dot\gamma(t)} s = \dot\gamma(t) \lrcorner Ds = \frac{d}{dt} s(t) + \dot\gamma(t) \lrcorner \theta \cdot s_t = 0.$$

This is a linear system of ordinary differential equations which admits a unique solution, once we prescribe the initial condition $s(0)$, which proves the claim.

An important idea is that parallel transport recovers the connection. We claim that

$$D_v s|_x = \frac{d}{dt}|_{t=0} (\Gamma_{0,t}^{\varphi_v^t(p)})^{-1} s(\varphi_v^t(x))$$

for a vector field $v \in C^\infty(TM)$, which defines the flow φ_v^t of v. To see this, write define $\gamma(t) = \varphi_v^t(x)$ and note that, locally,

$$\begin{aligned} \frac{d}{dt}|_{t=0} (\Gamma_{0,t}^{\gamma})^{-1}(s_{\gamma(t)}) &= ds(\dot\gamma(0)) + \frac{d}{t}|_{t=0} (\Gamma_{0,t}^{\gamma})^{-1} s|_{\gamma(0)} \\ &= ds(v) + \frac{d}{dt}|_{t=0} s_{-t} = ds(v) - \frac{ds_t}{dt}|_{t=0} = ds(v) + v \lrcorner \theta \cdot s = D_v s. \end{aligned}$$

The reason for calling the lift of Claim 5.2.4 *horizontal* will be discussed in the next lecture.

17th lecture, November 1st 2011

We begin today with another description of connections – a reference for the following is [BGV04]. Let D be a connection on $E \to M$. Note that we have a natural inclusion $M \subseteq E$ as the zero section of E. For a general $v \in E$, there is no natural way to embed M in E passing through v. A connection gives a way of doing this infinitesimally. What we will do it to define a "horizontal" distribution H on E using D.

Recall from the previous lecture that a section along a curve $\gamma \subseteq M$ can be identified with a curve on the total space of E.

Definition 5.2.5. A curve $\tilde\gamma \subseteq E$ such that $p\tilde\gamma = \gamma$ is a *horizontal lift* of γ with respect to D if and only if $\tilde\gamma$ is D-parallel; i.e. $D_{\dot\gamma(t)}\tilde\gamma(t) = 0$ for all t.

Define $H_v \subseteq T_v E$ for $v \in E$ by

$$H_v = \{w \in T_v E : w = \dot{\tilde\gamma} = 0 \text{ for some horizontal lift } \tilde\gamma \text{ of } \gamma \text{ in } M\}.$$

We will give a definition of H that is easier to work with.

Note that we have a short exact sequence of vector bundles over E,

$$0 \to VE \to TE \to p^*TM \to 0.$$

That the sequence is exact means that for all $v \in E$, the induced sequence of fibers is exact. Here

$$VE = \{w \in TE : dp(w) = 0\}$$

is the *vertical bundle*. We will define a 1-form on E with values in VE denoted $\theta_D \in \Omega^1_E(VE)$ such that the horizontal subspace becomes

$$H = \mathrm{Ann}\theta_D = \{w \in TE : \theta_D(w) = 0\}.$$

To define θ_D, pick a frame $f = (e_1, \dots, e_r)$ on E and use the local expression $D = d + \theta$. Here, $\theta = \theta(f)$ is defined by $D(f) = f \cdot \theta(f)$. Changing the frame by $g \in C^\infty(U, \mathrm{GL}(\mathbb{C}^r))$, we have

$$\theta(fg) = g^{-1}dg + g^{-1}\theta(f)g.$$

To see this, note that $D(fg) = fg \cdot \theta(fg)$ and by the Leibniz rule,

$$D(fg) = D(f) \cdot g + f \cdot dg = f(\theta(f) \cdot g + dg),$$

and the formula for $\theta(fg)$ follows. Associated to f, we have an identification $E|_U \cong U \times \mathbb{C}^r = \{(x, s)\}$ and a local trivialization of the tangent bundle $TE|_U \cong TU \times \mathbb{C}^r \times \mathbb{C}^r$. Note that we have no holomorphic structure on the bundle but we write $\mathbb{C}^r$ rather than $\mathbb{R}^{2r}$ to simplify notation.

Note also that a change of frame $f \to fg^{-1}$ induces a change of trivialization $U \times \mathbb{C}^r \to U \times \mathbb{C}^r$ which maps $(x, s) \mapsto (x, g(x)s)$. Taking differentials, we obtain a map

$$\begin{aligned} TU \times \mathbb{C}^r \times \mathbb{C}^r &\to TU \times \mathbb{C}^r \times \mathbb{C}^r \\ (v_x, s, w) &\mapsto (v_x, g(x)s, dg(v_x) \cdot s + g(x)w). \end{aligned}$$

Define, locally,

$$\theta_D(v_x, s, w) = (0_x, s, w + v_x \lrcorner \theta(f) \cdot s)$$

To see that this is independent of the choice of frame, note that in the frame fg^{-1},

$$\theta_D(v_x, g(x) \cdot s, dg(v_x)s + g(x)w) = (0_x, g(v_x)s, dg(v_x)s + g(x)w + v_x \lrcorner \theta(fg^{-1})g(x)s).$$

Since $d(g^{-1}) = -g^{-1}dgg^{-1}$, we obtain

$$\begin{aligned} dg(v_x)s + g(x)w + v_x \lrcorner \theta(fg^{-1})g(x) &= dg(v_x)s + g(x)w + v_x \lrcorner (gd(g^{-1}) + g\theta(f)g^{-1}) \\ &= g(x)w + v_x \lrcorner g(x)\theta(f) \cdot s \\ &= g(x)(w + v_x \lrcorner \theta(f) \cdot s), \end{aligned}$$

and it follows that θ_D is the local expression of a well-defined form in $\Omega^1_E(VE)$. Note that for $v \in C^\infty(VE)$, we have $\theta_D(v) = v$. We can now define

$$H = \mathrm{Ann}\theta_D = \{w \in TE : \theta_D(w) = 0\}.$$

We now want to see how to recover D from θ_D. The projection $p : E \to M$ induces an isomorphism

$$\begin{aligned} \Phi : p^*E &\to VE \\ (v, w) &\mapsto \frac{d}{dt}|_{t=0}(v + tw), \end{aligned}$$

recalling here that

$$p^*E = \{(v, w) \in E \times E : p(v) = p(w)\}.$$

Claim 5.2.6. *For $s \in C^\infty(E), v \in C^\infty(TM)$,*

$$\Phi(s, D_v s) = \theta_D(ds(v))$$

To see this, we note that locally, $TE|_U \cong TU \times \mathbb{C}^r \times \mathbb{C}^r$ and in this trivialization, we identify

$$ds(v) \equiv (v, s, ds(v)),$$

where on the right hand side, we view ds as a map $\mathbb{C}^r \to \mathbb{C}^r$. We find that

$$\theta_D(ds(v))|_x = (0_x, s, ds(v) + v \lrcorner \theta s) = (0_x, s, D_v s),$$

and the left hand side of the claim is

$$\Phi(s, D_v(s)) = (0_x, s, \frac{d}{dt}|_{t=0}(s + tD_v(s))) = (0_x, s, D_v(s)).$$

Note that $\dim H_v = \dim M$ since $\theta_D|_{VE} = \mathrm{Id}$, and $dp : TE \to p^*TM$ restricts to an isomorphism of bundles $dp|_H : H \to p^*TM$. Inverting this isomorphism, we obtain a *splitting* $A_D : p^*TM \to TE$ of

$$0 \to VE \to TE \to p^*TM \to 0,$$

that is, A_d satisfies $dp \circ A_D = \mathrm{Id}$. Not all splittings arise from connections in this way from connections on E, since our splittings always have the property that for $w \in C^\infty(TE)$, we have

$$\theta_D(w) = w - A_D dp(w),$$

and locally,

$$A_D((x, s), v_x) = (v_x, s, -v_x \lrcorner \theta \cdot s)$$

under the identification $TE|_U \cong TU \times \mathbb{C}^r \times \mathbb{C}^r$, so the dependence on s is linear, which is not true for general splittings. More generally, one is lead to consider principal bundles where connections are defined as certain splittings.

5.3 Curvature of connections

As motivation, suppose (M, g) is a Riemannian manifold. Associated to g we have the Levi–Civita connection D_g. Note that $p^*TM \cong V(TM)$, which allows us to define a metric $\hat{g}'$ on $V(TM)$; namely, we define

$$\hat{g}'((v_1, w_1), (v_2, w_2)) = g(w_1, w_2).$$

Consider $(TM, \hat{\hat{g}} = \hat{g} + p^*g)$, where $\hat{g}$ is defined by

$$\hat{g}(t_1, t_2) = \hat{g}'(\theta_D t_1, \theta_D t_2)$$

for $t_1, t_2 \in C^\infty(TTM)$. Then $\hat{\hat{g}}$ is a metric on TM as a manifold.

We turn to the concept of curvature. The horizontal distribution H_D may not be integrable (in the sense of Frobenius). By the Frobenius theorem, H_D is integrable, if and only if H_D is preserved by the bracket. I.e. if $w_1, w_2 \in C^\infty(TE)$, and $w_1, w_2 \in H_D$, then $[w_1, w_2] \in H_D$.

The *curvature* of D captures the non-integrability of H_D. It defines an element

$$F_D \in \Omega^2_E(VE),$$
$$F_D(w_1, w_2) = -\theta_D[A_D dp w_1, A_D dp w_2]$$

for $w_1, w_2 \in C^\infty(TE)$.

Exercise 5.3.1. Check that F_D is a tensor.

Note that F_D kills all vertical vector fields, i.e. if $v \in C^\infty(VE)$, then $v \lrcorner F_D = 0$. This allows us to define $F \in \Omega^2_M(\mathrm{End}E)$, also referred to as the *curvature*, satisfying

$$\Phi(s, F(u,v)s) = F_D(A_D u, A_D v)|_s$$

where $\Phi : p^*E \cong VE$ is the isomorphism from the previous section, $s \in C^\infty(E), u, v \in C^\infty(TM)$, and $A_D u, A_D v$ is understood as follows: For $u \in C^\infty(TM)$, we define $u \in C^\infty(p^*TM)$ defined by $u(s) = (s, u(p(s)))$ for $s \in E$, so that $A_D u, A_D v$ make sense.

The form $F \in \Omega^2(\mathrm{End}E)$ is given by

$$F(u,v) \cdot s = D_u(D_v s) - D_v(D_u s) - D_{[u,v]}s.$$

Locally, in a frame f, $D = d + \theta$, we find

$$\begin{aligned} F(u,v)s &= (d+\theta)_u((d+\theta)_v s) - (d+\theta)_v((d+\theta)_u s) - (d+\theta)_{[u,v]}s \\ &= (d+\theta)_u(v(s) + \theta(v)\cdot s) - (d+\theta)_v(u(s) + \theta(u)s) - [u,v](s) - \theta([u,v])s \\ &= u(v(s)) + u(\theta(v)s) + \theta(u)\cdot v(s) + \theta(u)\theta(v)s \\ &\quad - v(u(s)) - v(\theta(u)s) - \theta(v)u(s) - \theta(v)\theta(u)s \\ &\quad - [u,v](s) - \theta([u,v])s \\ &= u(\theta(v))\cdot s - v(\theta(u))\cdot s - \theta([u,v])s + \theta(u)\theta(v)s - \theta(v)\theta(u)s \\ &= d\theta(u,v)s + (\theta\wedge\theta)(u,v)s \\ &= (d\theta + \theta\wedge\theta)(u,v)s \end{aligned}$$

Here, $\theta \wedge \theta$ denotes multiplication of matrices where in components we take wedge products. Thus, we have the local expression

$$F = d\theta + \theta \wedge \theta$$

for the curvature. Next time, we relate F to F_D via Φ.

18th lecture, November 7th 2011

We begin by recalling the different versions of connections. Let (E, D) be a complex vector bundle with a connection. We defined this to be either

1. a map $D : \Omega^0(E) \to \Omega^1(E)$ which is $\mathbb{C}$-linear function satisfying the Leibniz rule,
2. defined using parallel transport,
3. defined through a form $\theta_D \in \Omega^1_E(VE)$,
4. defined by the splitting $A_D : p^*TM \to TE$ of $0 \to VE \to TE \xrightarrow{dp} p^*TM \to 0$, or
5. defined by H a horizontal distribution.

We go from (1) to (2) by solving an ordinary differential equation and from (2) to (1) by considering $\frac{d}{dt}|_{t=0}\Gamma^\gamma_{0,t}$. We go from (3) to (1) by considering $\Theta : p^*E \cong VE$ and $\theta_D \circ d = \Phi(\cdot, D\cdot)$ and from (1) to (3) by defining locally $\theta_D = (0_x, \mathrm{Id} + \theta)$. We go between (3) and (4) by letting $\theta_D = \mathrm{Id} - A_D dp$. Finally, we go between (3) and (5) by letting $H = \mathrm{Ann}\theta_D$.

We saw that the obstruction for H to be integrable is the curvature

$$F_D(v_1, v_2) = -\theta_D[A_D dp v_1, A_D dp v_2],$$

for $v_1, v_2 \in C^\infty(TE)$. We also defined $F \in \Omega^2(\mathrm{End}E)$ by

$$F(u,v)\cdot s = D_u D_v s - D_v D_u s - D_{[u,v]}s.$$

We saw that, locally, $F = d\theta + \theta \wedge \theta$.

Claim 5.3.2. *The two curvatures are related by* $\Phi(s, F(u,v)s) = F_D(A_D u, A_D v)$ *for* $s \in C^\infty(E)$. *Here,* $v, u \in C^\infty(TM)$, *and we view* u *and* v *as sections of* p^*TM *on which we can apply the splitting* A_D.

Proof. Locally, on a frame, we have $E|_U \cong U \times \mathbb{C}^r$ and $TE|_U \cong TU \times \mathbb{C}^r \times \mathbb{C}^r$. Recall that

$$\Phi(s_1, s_2) = \frac{d}{dt}|_{t=0}(s_1 + ts_2),$$

so locally,

$$\begin{aligned}\Phi(s, F(u,v)s)|_{x,s} &= \left(0_x, s, \frac{d}{dt}|_{t=0}(s + tF(u,v)\cdot s)\right) \\ &= \left(0_x, s, (d\theta + \theta\wedge\theta)_{(u,v)}\cdot s\right).\end{aligned}$$

Note that F_D kills vertical vector fields by definition. So, it is enough to compute F_D on elements of the form $A_D u, A_D v$ as before; in other words, F_D is determined by elements of this form.

Locally,

$$A_D u|_{x,s} = (u_x, s, -u_x \lrcorner \theta \cdot s) \equiv (u_x, -u_x \lrcorner \theta s),$$

i.e. we view the vector field as a map $U \times \mathbb{C}^r \to \mathbb{R}^n \times \mathbb{C}^r$.

On a manifold N, locally we can compute $[v_1, v_2]$, $v_1, v_2 \in C^\infty(TN) \equiv C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, as

$$[v_1, v_2] = v_1 \lrcorner dv_2 - v_2 \lrcorner dv_1 = \sum_k v_1^k \frac{\partial v_2}{\partial x_k} - \sum_k v_2^k \frac{\partial v_1}{\partial x_k}.$$

In our case,

$$\begin{aligned}[A_d u, A_d v]|_{(x,s)} &= (u_x, -u_x \lrcorner \theta \cdot s) \lrcorner d_{x,s}(v, -v \lrcorner \theta s) - (v_x, -v_x \lrcorner \theta \cdot s) \lrcorner d_{x,s}(u, -u \lrcorner \theta s) \\ &= (u_x, -u_x \lrcorner \theta \cdot s) \lrcorner (d_x v, d_{x,s}(-v \lrcorner \theta s)) - (v_x, -v_x \lrcorner \theta \cdot s) \lrcorner (d_x u, d_{x,s}(-u \lrcorner \theta s)) \\ &= (u_x \lrcorner d_x - v_x \lrcorner d_x u, W) = ([u,v], W)\end{aligned}$$

for some W. To compute W, note that

$$\begin{aligned}d_{x,s}(-v \lrcorner \theta s) &= -(d_x v) \lrcorner \theta s - v_x \lrcorner (d_x \theta) s - v_x \lrcorner \theta d_s s \\ &= -d_x(v \lrcorner \theta) s - v_x \lrcorner \theta d_s s,\end{aligned}$$

and

$$\begin{aligned}&(u_x, -u_x \lrcorner \theta s) \lrcorner d_{x,s}(-v \lrcorner \theta s) \\ &= -(u_x \lrcorner d_x v) \lrcorner \theta s - v_x \lrcorner (u_x \lrcorner d_x \lrcorner d_x \theta) s + (v_x \lrcorner \theta) \cdot (u_x \lrcorner \theta) \cdot s \\ &= -u_x \lrcorner d_x(-v \lrcorner \theta) \cdot s + (v \lrcorner \theta \cdot u \lrcorner \theta s).\end{aligned}$$

Recall that

$$d\theta(u,v) \cdot s = u \lrcorner d(v \lrcorner \theta) - v \lrcorner d(u \lrcorner \theta) - [u,v] \lrcorner \theta.$$

This implies that

$$W = -[u,v] \lrcorner \theta s - d\theta(u,v) \cdot s - \theta \wedge \theta(u,v) \cdot s.$$

Overall, we see that

$$[A_D u, A_D v] = ([u,v], -[u,v] \lrcorner \theta s - (d\theta + \theta \wedge \theta)(u,v) \cdot s),$$

and hence by definition,

$$\begin{aligned}-F_D(A_D u, A_D v) &= \theta_D[A_D u, A_D v] \\ &= (0_x, -[u,v] \lrcorner \theta s - (d\theta + \theta\wedge\theta)(u,v)s + [u,v] \lrcorner \theta s) \\ &= (0_x, -F(u,v) \cdot s) = -\Phi(s, F(u,v)s).\end{aligned}$$

□

From now on, we write $F_D = F \in \Omega^2(\mathrm{End}E)$, using the identification from the claim. We turn now to the Bianchi identity, which gives a relation between the covariant derivative and the curvature.

We first discuss associated connections: Given two bundles (E, D^E), (F, D^F), we can do the following:

a) Endow E^* with the connection $D^{E^*}(\alpha(s)) := d(\alpha(s)) - \alpha(D^E s)$ for $\alpha \in \Omega^0(E^*), s \in \Omega^0(E)$. One can check that this satisfies the Leibniz rule.

b) Endow $E \oplus F$ with the connection $D^{E\oplus F}(s_1 \oplus s_2) = D^E s_1 \oplus D^F s_2$, for $s_1 \in \Omega^0(E), s_2 \in \Omega^0(F)$.

c) Endow $E \otimes F$ with the connection $D^{E\otimes F}(s_1 \otimes s_2) = (D^E s_1) \otimes s_2 + s_1 \otimes D^F s_2$.

In particular, we have a connection on $\mathrm{End}(E)$ given by

$$D^{\mathrm{End}E}(B)s = D^E(Bs) - B(D^E s)$$

for $B \in \Omega^0(\mathrm{End}E), s \in \Omega^0(E)$. Locally, write $D^E = D = d + \theta$. Then

$$D^{\mathrm{End}E}(B) = dB + \theta B - B\theta = (d + [\theta, \cdot])B.$$

Given (E, D), we will now extend D to $D : \Omega^p(E) \to \Omega^{p+1}(E)$. Locally, for $\xi \in \Omega^p(E|_U)$ where $E|_U$ is trivialized via a local frame f on $U \subseteq M$ open, we let

$$(D\xi)(f) = d(\xi(f)) + \theta(f) \wedge \xi(f) \in \Omega^{p+1}(E|_U).$$

Exercise 5.3.3. Check that this defines $D\xi$ globally by using that $\xi(fg) = g^{-1}\xi(f)$ for $g \in C^\infty(U, \mathrm{GL}(\mathbb{C}^r))$ together with the formula for $\theta(fg)$.

Note that $D^2\xi = (D \circ D)\xi = F_D \wedge \xi$: In the frame f,

$$\begin{aligned} D^2\xi &= (d + \theta)(d + \theta)\xi = (d + \theta)(d\xi + \theta \wedge \xi) \\ &= d^2\xi + d(\theta \wedge \xi) + \theta \wedge d\xi + \theta \wedge \theta \wedge \xi \\ &= (d\theta) \wedge \xi - \theta \wedge d\xi + \theta \wedge d\xi + \theta \wedge \theta \wedge \xi \\ &= (d\theta + \theta \wedge \theta)\xi = F_D \wedge \xi. \end{aligned}$$

From this, we obtain an abstract version of the "non-integrability" of H_D. Namely, we obtain a sequence

$$\Omega^0(E) \to \Omega^1(E) \to \Omega^2(E) \to \cdots \to \Omega^p(E) \to \cdots.$$

We conclude that the curvature is the obstruction for this sequence to be a complex, since $D^2 = 0$ if and only if $F_D = 0$.

Proposition 5.3.4 (Bianchi identity)**.** *Consider the sequence*

$$\Omega^0(\mathrm{End}E) \overset{D^{\mathrm{End}E}}{\to} \cdots \overset{D^{\mathrm{End}E}}{\to} \Omega^p(\mathrm{End}E) \overset{D^{\mathrm{End}E}}{\to} \cdots.$$

Then $D^{\mathrm{End}E}(F_D) = 0$.

Proof. Locally, $D^{\mathrm{End}E} : \Omega^0(\mathrm{End}E) \to \Omega^1(\mathrm{End}E)$ can be written $D^{\mathrm{End}E} = d + [\theta, \cdot]$. We will extend the bracket so that this formula is true for $D^{\mathrm{End}E} : \Omega^p(\mathrm{End}E) \to \Omega^{p+1}(\mathrm{End}E)$ in general.

Let $\chi \in \Omega^p(\mathrm{End}E)$, $\psi \in \Omega^q(\mathrm{End}E)$ and define

$$[\psi, \chi](f) = \chi(f) \wedge \psi(f) - (-1)^{pq}\psi(f) \wedge \chi(f).$$

Using that $\chi(fg) = g^{-1}\chi(f)g$, one checks that this bracket is globally defined. We find that

$$D^{\mathrm{End}E}\chi = d\chi + [\theta, \chi] = (d + [\theta, \cdot])\chi.$$

From this, we obtain

$$\begin{aligned} D^{\mathrm{End}E}(F_D) &= (d + [\theta, \cdot])(d\theta + \theta \wedge \theta) \\ &= d^2\theta + d(\theta \wedge \theta) + [\theta, d\theta] + [\theta, \theta \wedge \theta] \\ &= d\theta \wedge \theta + \theta \wedge d\theta + \theta \wedge \theta \wedge \theta - (-1)^2(d\theta \wedge \theta + \theta \wedge \theta \wedge \theta) \\ &= (d\theta) \wedge \theta - \theta \wedge d\theta + \theta \wedge d\theta + \theta \wedge \theta - d\theta \wedge \theta - \theta \wedge \theta \wedge \theta \\ &= 0. \end{aligned}$$

□

Remark 5.3.5. This identity is Prop. 1.10 of [Wel08] which states that $-[\theta, F] = [F, \theta]$ and $dF = [F, \theta]$ (with the notation $F = \Theta$).

5.4 Unitary connections

Let (E, h) be a complex hermitian smooth bundle over M. Using h we construct the map

$$\begin{aligned} h : \Omega^p(E) \times \Omega^q(E) &\to \Omega^{p+q}(M) \otimes \mathbb{C} \\ (\omega_1 \otimes s_1, \omega_2 \otimes s_2) &\mapsto \omega_1 \wedge \overline{\omega_2} \cdot h(s_1, s_2) \end{aligned}$$

for $\omega_1 \in \Omega^p(M)$, $\omega_2 \in \Omega^q(M)$, $s_1, s_2 \in C^\infty(E)$. Here, we use the identification

$$\wedge^k T^*M \otimes_{\mathbb{R}} E \cong (\wedge^k T^*M \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} E.$$

Definition 5.4.1. A connection D is called *h-unitary* (or *h-compatible*) if

$$d(h(s_1, s_2)) = h(Ds_1, s_2) + h(s_1, Ds_2)$$

Locally, on $U \subseteq M$, such that $E|_U \cong U \times \mathbb{C}^r$, we have

$$h(s_1, s_2) = \overline{s_2}^T H s_1,$$

where $H \in C^\infty(U, M_{r\times r}(\mathbb{C}))$ satisfies $\overline{H}^T = H$ in every point. Then, since

$$d(h(s_1, s_2)) = (d\overline{s_2}^T)Hs_1 + \overline{s_2}^T(dH)s_1 + \overline{s_2}^T H ds_1,$$

and since

$$h(Ds_1, s_2) + h(s_1, Ds_2) = \overline{s_2}^T H(ds_1 + \theta s_1) + \overline{ds_2 + \theta s_2}^T H s_1,$$

the compatibility condition becomes

$$dH = H\theta + \overline{theta}^T H.$$

A compatible connection always exists: Choose a trivializing cover $\{U_\alpha\}_{\alpha \in I}$ and a partition of unity φ_α subordinate to this cover. Then $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^r$ give rise to frames f_α which we chan orthonormalize using the Gram–Schmidt process. Thus we can assume $H_\alpha = \mathrm{Id}$ for all $\alpha \in I$. Then this must solve

$$\theta_\alpha + \overline{\theta_\alpha}^T = 0$$

and in particular, we can take $\theta_\alpha = 0$ for all α, and we can take

$$Ds = \sum_\alpha \varphi_\alpha d(s(f_\alpha)).$$

19th lecture, November 8th 2011

5.5 Chern connection

Let $E \to X$ be a holomorphic vector bundle over a complex manifold, and let h be a hermitian metric on E. Then we have

$$\Omega^k(E) = \sum_{p+q=k} \Omega^{p,q}(E).$$

Note that $\wedge^k T^* X_0 \otimes_{\mathbb{R}} E \cong ((\wedge^k T^* X_0) \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} E$. If $D : \Omega^0(E) \to \Omega^1(E)$ is a connection, we split

$$\partial_D = D' : \Omega^0(E) \to \Omega^{1,0}(E), \quad \overline{\partial}_D = D'' : \Omega^0(E) \to \Omega^{0,1}(E).$$

Theorem 5.5.1 (Chern)**.** *Let (E, h) be a holomorphic hermitian bundle. There exists a unique connection $D = D_h$, called the* Chern connection *in E such that*

1. *D is h-compatible, and such that*
2. *$D''s = 0$ for all holomorphic sections $s \in \mathcal{O}_E(U)$ of $E|_U$ for all $U \subseteq X$ open.*

Proof. Take any h-compatible connection D in E (which we have seen is possible). Then locally, $D = d + \theta$ and decompose $\theta = \theta^{1,0} + \theta^{0,1}$. If $s \in \mathcal{O}_E(U)$, we have $\overline{\partial}_D s = \overline{\partial} s + \theta s$. If D satisfies the properties of the Theorem, then $0 = \overline{\partial}_D s$. If the local frame chosen is holomorphic, then $\overline{\partial} s = 0$ and so $\theta^{0,1} = 0$.

The first property implies that $dH = \theta^* H + H\theta$, where H is the matrix for h. Comparing types of this equation we find $\overline{\partial} H = \theta^* H$ and $\partial H = H\theta$. Thus the condition becomes $\theta = H^{-1}\partial H$. This implies that the connection, if it exists, is unique. To see that it exists, it suffices to see that the expression for θ behaves as it should under a holomorphic change of frame $g : U \to \mathrm{GL}(\mathbb{C}^r)$ (i.e. $\overline{\partial} g = 0$) which is left as an exercise. □

Remark 5.5.2. We can recover completely the holomorphic structure on E from $D_h' =: D'$. This follows from the fact that from the complex structure J on X and D_h (the Chern connection of some h), one can construct the canonical almost structure on E.

The map $D' : \Omega^0(E) \to \Omega^{0,1}(E)$ is usually called the *Dolbeault operator* (also denoted $\overline{\partial}_E$).

Proposition 5.5.3. *The Chern connection satisfies the following (in a holomorphic frame):*

a) *θ is of type $(1, 0)$ and $\partial\theta = -\theta \wedge \theta$.*

b) *$F_h = D_{D_h} = \overline{\partial}\theta$ and so $F_h \in \Omega^{1,1}(\mathrm{End}E)$.*

c) *$\overline{\partial} F = 0$ and $\partial F + [\theta, F] = 0$.*

Proof. a) Since $\theta = H^{-1}\partial H$, we have

$$\begin{aligned}\partial\theta &= \partial(H^{-1}) \wedge \partial H + H^{-1}\partial^2 H = -H^{-1}(\partial H) \wedge H^{-1}\partial H \\ &= -\theta \wedge \theta.\end{aligned}$$

Here we have used that, in general, if $t \mapsto A(t) \in M_{r\times r}(\mathbb{C})$ is a matrix-valued function, then $\frac{d}{dt}A(t)^{-1} = -A(t)^{-1}\frac{dA(t)}{dt}A(t)^{-1}$.

b) We have

$$\begin{aligned}F &= d\theta + \theta \wedge \theta = \partial\theta + \overline{\partial}\theta + \theta \wedge \theta \\ &= \overline{\partial}\theta.\end{aligned}$$

c) This follows from the Bianchi identity: The identity tells us that $dF = [\theta, F] = 0$, which implies that

$$\overline{\partial} F = \overline{\partial}^2 \theta = 0.$$

□

Note that $\overline{F_h}^T H + HF = 0$. Globally, this means that

$$h(Fs_1, s_2) + h(s_1, Fs_2) = 0$$

for all $s_1, s_2 \in C^\infty(E)$. This happens for all F_D when D is h-compatible.

We discuss now the notion of *adapted coordinates*: Note that locally, $F = \overline{\partial}\theta = \overline{\partial}(H^{-1}\partial H)$. Inverting a matrix generally takes some computation, so we will find coordinates around a point p such that $F|_p = \overline{\partial}\partial H|_p$.

Pick $x \in X$ and $U \subseteq X$ an open neighbourhood with holomorphic trivialization $E|_U \cong U \times \mathbb{C}^r$. Identify $U \cong \mathbb{C}^n$, $n = \dim_{\mathbb{C}} X$, with $x \equiv 0 \in \mathbb{C}^n$.

Proposition 5.5.4. *There is a holomorphic frame in U such that*

a) $H = \operatorname{Id} + O(|x|^2)$, *and*

b) $F_U|_0 = \overline{\partial}\partial H|_0$.

Proof. Assume that a) holds. This means that $|H - \operatorname{Id}| \leq C \cdot |x|^2$ in some neighbourhood of $0 \in \mathbb{C}^n$ for some C. Then

$$\begin{aligned} H &= \operatorname{Id} + P, \\ H^{-1} &= \operatorname{Id} - P + P^2 - P^3 + \cdots . \end{aligned}$$

At a point x, we have

$$F_h|_x = \overline{\partial}(H^{-1}\partial H)|_x = \overline{\partial}\partial H|_x + O(|x|),$$

which implies that $F_h|_0 = \overline{\partial}\partial H|_0$.

If we expand H in Taylor series, then a) means that the first order piece vanishes. To see this there are two steps. The first is to see that $H|_0 = \operatorname{Id}$ and the second one to show that $dH|_0 = 0$.

To see the first step, we show that $\overline{H}(0)^T = H(0)$. We have $H(0) \in \operatorname{GL}(\mathbb{C}^r)$. Polar decompostion tells us that for $g' \in \operatorname{GL}(\mathbb{C}^r)$, we can write $g' = Ke^{i\xi}$ for $\xi \in \operatorname{Lie}\operatorname{U}(r)$, i.e. $\overline{\xi}^T + \xi = 0$. In our case, we write $H(0) = e^{i\xi}$ for $\xi \in \operatorname{Lie}\operatorname{U}(r)$. Denote $g = e^{-i\xi/2}$. Write $h(f) \equiv H$. Then $H(0) = (gg^*)^{-1}$ and $h(fg) = g^* h(f) g = g^* H g = \operatorname{Id} + O(|x|)$.

Assume by this change of frame that $H = \operatorname{Id} + O(|x|)$. For the second step, choose $g : U \to \operatorname{GL}(\mathbb{C}^r)$ to be $g = \operatorname{Id} + A$, where $A(x) = \sum_j A^j x_j$ for constant matrices $A^j = -\frac{\partial H}{\partial x_j}(0)$. Note that in the Taylor expansion of H, there are also terms of the form $\frac{\partial}{\partial \overline{x}_j} H(0)$.

Computing, we find

$$dA(0) = \partial A(0) = -\partial H|_0.$$

In the new basis,

$$\begin{aligned} d(h(fg))(0) &= d((\operatorname{Id} + A)^* H(\operatorname{Id} + A)(0)) = d(H + A^*H + HA + \cdots)(0) \\ &= dH(0) + dA^*(H(0)) + H(0)dA(0) = dH(0) + dA^*(0) + dA(0) \\ &= \partial H(0) + \overline{\partial} H(0) + (-\partial H)^*(0) - \partial H(0) = \overline{\partial} H(0) - \overline{\partial} H(0) = 0. \end{aligned}$$

□

Example 5.5.5. Consider the Fubini–Study metric h_{FS} on $\mathcal{O}_{\mathbb{P}(\mathbb{C}^n)}(1) = Q_1 \to G_1(\mathbb{C}^n) = \mathbb{P}(\mathbb{C}^n) = \{[z_1 : \cdots : z_n]\}$ and write $\omega_{\mathrm{FS}} = iF_{h_{\mathrm{FS}}}$. Note that $F_{h_{\mathrm{FS}}} \in \Omega^2(\operatorname{End} Q_{1_{h_{\mathrm{FS}}}}) \cong \Omega^2(\mathbb{P}(\mathbb{C}^n)) \otimes \mathbb{C}(?)$, $\operatorname{End} Q_1 \cong Q_1 \otimes Q_1^* \cong \mathbb{P}(\mathbb{C}^n) \times \mathbb{C}$. Since $h_{\mathrm{FS}}(F_{h_{\mathrm{FS}}} \cdot, \cdot) + h_{\mathrm{FS}}(\cdot, F_{h_{\mathrm{FS}}} \cdot) = 0$, we have $\overline{F_{h_{\mathrm{FS}}}} = -F_{h_{\mathrm{FS}}}$ so $F_{h_{\mathrm{FS}}} \in i\Omega^2(\mathbb{CP}^n)$ and $\omega_{\mathrm{FS}} \in \Omega^2(\mathbb{C}(\mathbb{P}^n))$. By the Bianchi identity $d\omega_{\mathrm{FS}} = 0$: Locally,

$$[\theta, \omega_{\mathrm{FS}}] = \theta \wedge \omega_{\mathrm{FS}} - (-1)^2 \omega_{\mathrm{FS}} \wedge \theta = 0.$$

We claim that $\omega_F S$ is non-denerate, i.e. that $v \lrcorner \omega_{\mathrm{FS}} = 0$ implies $v = 0$ for $v \in C^\infty(T\mathbb{P}(\mathbb{C}^n)_0)$.

The action of $\mathrm{U}(n)$ on $\mathbb{P}(\mathbb{C}^n)$ induced by the inclusion $\mathrm{U}(n) \to \mathrm{GL}(\mathbb{C}^r)$ is transitive in the sense that for all $z_1, z_2 \in \mathbb{P}(\mathbb{C}^n)$ there exists $g \in \mathrm{U}(n)$ such that $gz_1 = z_2$. Moreover, ω_{FS} is preserved by this action.

Recall our formula for h_{FS}: For $[v_1]|_{[z_1:\cdots:z_n]} \in Q_1|_{[z_1:\cdots:z_n]}$, $[v_1] \in \mathbb{C}^n/\ker[z_1 : \cdots : z_n]$, where $[z_1 : \cdots : z_n] : \mathbb{C}^n \to \mathbb{C}$ is the map $v \mapsto z \circ v = \sum z_i v_i$, we have

$$h_{\mathrm{FS}}([v_1],[v_2])|_z = \frac{\overline{v_2}^T \overline{z}^T z v_1}{|z|^2}.$$

On coordinates $z_j \neq 0$, $Q_1|_{U_j} \to \mathbb{C}^{n-1} \times \mathbb{C}$, $[v]|_z \mapsto (w, \frac{zw}{z_j})$. Write $w_i = z_i/z_j$, $i = 1, \dots, \hat{j}, \dots, n$. Then

$$h_{\mathrm{FS}}((w,\lambda_1),(w,\lambda_2)) = \frac{\lambda_1 \overline{\lambda_2}}{1+|w|^2},$$

and the curvature becomes

$$\begin{aligned} F_{h_{\mathrm{FS}}} &= \overline{\partial}((1+|w|^2)\partial((1+|w|^2)^{-1})) = -\overline{\partial}\partial \log(1+|w|^2) \\ &= \partial\left(\frac{\sum_l w_l d\overline{w}_l}{1+|w|^2}\right) \\ &= \frac{\sum_l dw_l \wedge d\overline{w}_l}{1+|w|^2} - \frac{\sum_{l,k} w_l \overline{w}_k dw_l \wedge d\overline{w}_k}{(1+|w|^2)^2}, \end{aligned}$$

and so

$$\omega_{\mathrm{FS}}|_{w=0} = i \sum_{l=1}^{n-1} dw_l \wedge d\overline{w}_l.$$

As we will discuss later, this is the expression for the standard "symplectic form" on $\mathbb{C}^n$.

We conclude that ω_{FS} is closed and non-degenerate. Moreover, if J is the canonical almost complex structure on $\mathbb{P}(\mathbb{C}^n)$, then $\omega_{\mathrm{FS}}(\cdot, J\cdot)$ is positive in the sense that $\omega_{FS}(v, Jv) \geq 0$ for all $v \in C^\infty(T\mathbb{P}(\mathbb{C}^n)_0)$. We claim (without having defined it) that $g_{\mathrm{FS}} = \omega_{\mathrm{FS}}(\cdot, J\cdot)$ is a Kähler metric on $\mathbb{P}(\mathbb{C}^n)$.

Note now that if $L \to X$ is an ample line bundle over a complex manifold X, then by definition of ampleness, there is a $k \gg 0$ and an embedding $X \hookrightarrow \mathbb{P}(\mathbb{C}^{N_k})$ given by holomorphic section, where $N_k = \dim H^0(X, L^k)$. Then we can pull-back all the structure of $\mathbb{P}(\mathbb{C}^{N_k})$, i.e. h_{FS}, ω_{FS}, g_{FS}, to (X, L). Thus, every *polarized manifold* (X, L) like this has a Kähler structure. What we will see is the converse to this statement.

20th lecture, November 14th 2011

6 Kähler geometry

6.1 Preliminaries on linear algebra

Let V be a d-dimensional real vector space with complex structure $J : V \to V$. Fix g an inner product on V (so g is symmetric and $g(v,v) > 0$ for all $v \in V \setminus \{0\}$).

Definition 6.1.1. The inner product g is called hermitian if $g(J\cdot, J\cdot) = g$.

Given a hermitian inner product g, we can consider $\omega = g(J\cdot, \cdot)$, which, since g is symmetric and J is an isometry, defines an element $\omega \in \wedge^2 V^*$, i.e. ω is skew-symmetric. Furthermore, since $J^2 = -\mathrm{Id}$, we get that $\omega(J\cdot, J\cdot)\omega$.

Let $V_c = V \otimes_{\mathbb{R}} \mathbb{C}$. Then the hermitian inner product g extends uniquely to g_c a $\mathbb{C}$-bilinear symmetric form on V_c, such that

1. $g_c(\overline{v}, \overline{w}) = \overline{g_c(v, w)}$, for all $v, w \in V_c$,
2. $g_c(v, \overline{v}) > 0$, for all $v \in V_c \setminus \{0\}$,
3. $g_c(v, \overline{w}) = 0$, if $v \in V^{1,0}$, $w \in V^{0,1}$.

Consider $\omega_c = g_c(J\cdot, \cdot)$, where $J : V_c \to V_c$ is $\mathbb{C}$-linear extension of J. Consider $\wedge V^* = \bigoplus_{r\geq 0} \bigoplus_{p+q=r} (V^*)^{p,q}$.

Proposition 6.1.2. *We have $\omega_c \in (V^*)^{1,1}$.*

Proof. If $v, w \in V^{1,0}$, then $\omega_c(v, w) = ig_c(v, w) = 0$ by the third property above. We get what we want from the identification

$$(V^*)^{1,0} = \{\alpha \in V_c^* : v \lrcorner \alpha = 0 \ \forall v \in V^{0,1}\}.$$

□

Consider a basis $\{\frac{\partial}{\partial z_j}\}_{j=1}^n$, $2n = d$, of $V^{1,0}$ with a dual basis $\{dz_j\}_{j=1}^n$. Then we can write

$$g_c = 2 \sum_{i,j} h_{ij} dz_i \odot d\overline{z}_j,$$

where we write

$$dz_i \odot d\overline{z}_j = \frac{1}{2}(dz_i \otimes d\overline{z}_j + d\overline{z}_j \otimes dz_i), \quad h_{ij} = g_c \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_j} \right), \quad \frac{\partial}{\partial \overline{z}_j} = \overline{\frac{\partial}{\partial z_j}}.$$

We also have

$$\omega_c = g_c(J\cdot, \cdot) = i \sum h_{i\overline{j}} dz_i \wedge d\overline{z}_j,$$

where here $dz_i \wedge d\overline{z}_j = dz_i \otimes d\overline{z}_j - d\overline{z}_j \otimes dz_i$.

Note that $\overline{h_{i\overline{j}}} = h_{j\overline{i}}$, so h defines a hermitian metric on $V^{1,0}$.

Let V be a complex vector space of dimension $\dim_{\mathbb{C}} V = n$, and let V_0 be the underlying real $2n$-dimensional vector space. As always, we can endow V_0 with a complex structure J given by multiplication by i. Fix a hermitian metric on V, $h : V \otimes_{\mathbb{C}} V \to \mathbb{C}$ (not $\mathbb{C}$-linear), i.e. h satisfies $h(v, w) = \overline{h(w, v)}$ and $h(\lambda v, w) = \lambda h(v, w) = h(v, \overline{\lambda} w)$ for $\lambda \in \mathbb{C}$. Define

$$g(u, v) = 2 \operatorname{Re} h(u, v), \quad \omega = -2 \operatorname{Im} h,$$

for $u, v \in V_0$. Then g is a hermitian inner product on (V_0, J) since

$$g(J\cdot, J\cdot) = 2 \operatorname{Re} h(J\cdot, J\cdot) = 2 \operatorname{Re} i(-i)h = 2 \operatorname{Re} h,$$

and we have $\omega = g(J\cdot, \cdot)$.

Choose a complex basis $\{w_j\}_{j=1}^n$ on V and consider $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\}$ the real basis of V_0, so $\frac{\partial}{\partial x_j} = w_j$, $\frac{\partial}{\partial y_j} = iw_j$. Let $\{w_j^*\}$ be the dual basis of $\{w_j\}$. Define the $\mathbb{R}$-linear map $\overline{w_j^*} : V_0 \to \mathbb{C}$ by $v \mapsto \overline{w_j^*(v)}$. Consider the $\mathbb{C}$-linaer extension $(\overline{w_j^*})_c$ of $\overline{w_j^*}$ to $(V_0)_c$. Then

$$(\overline{w_j^*})_c = dx_j - idy_j =: d\overline{z}_j$$
$$(w_j^*)_c = dx_j + idy_j =: dz_j,$$

so dz_j is the dual of $\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$. Setting $h_{i\overline{j}} = h(w_i, w_j)$, we have

$$g_c = 2 \sum_{i,j} h_{j\overline{k}} dz_j \odot d\overline{z}_k,$$
$$\omega_c = i \sum_{i,j} h_{j\overline{k}} dz_j \wedge d\overline{z}_k.$$

In other words, the hermitian metric induced by $g_c = (2 \operatorname{Re} h)_c$ in the $(1, 0)$-part of V_0 is given by the isomorphism $V \cong (V_0, J) \cong V_0^{1,0}$ given as always by $w \mapsto \frac{1}{2}(w - iJw)$.

So, Kähler geometry at a point is simply hermitian linear algebra. Things first begin to happen when we impose global integrability conditions.

6.2 Kähler manifolds

Let (M, J) be an almost complex manifold.

Definition 6.2.1. A Riemannian metric on M is *hermitian* if $g(J\cdot, J\cdot) = g$. The triple (M, J, g) is called an *almost hermitian manifold*. It is called a *hermitian manifold* if $N_J = 0$.

By the pointwise nonsense of the previous section, if (M, J, g) is hermitian, then the complex manifold $X = (M, J)$ has a canonical hermitian metric h on $TX \cong (TM)^{1,0}$.

Given a hermitian manifold (M, J, g), we obtain two connections. One is the Levi-Civita connection (∇, TM) of g, and the other is the Chern connection (D, TX) of h. We want to compare ∇ and D. To do so, we extend ∇ $\mathbb{C}$-linearly to the smooth complex bundle $TM_c \to M$. Note that D, via the isomorphism $TX \cong TM^{1,0}$, defined only $TM^{1,0}$, and we want to extend it. Recall that $C^\infty(TM^{1,0})$ is generated by $v - iJv$ for $v \in C^\infty(TM)$. We claim that ∇_c preserves $TM^{1,0}$ if and only if $\nabla J = 0$, where $(\nabla J)v = \nabla(Jv) = J\nabla v$.

It is clear that if $\nabla J = 0$, then ∇_c preserves $TM^{1,0}$ since $\nabla_c(v - iJv) = (\mathrm{Id} - iJ)\nabla_c v$, and $\nabla_c v = \nabla v$ for $v \in C^\infty(TM)$.

For the other direction, note that if ∇_c preserves $TM^{1,0}$, we have

$$J\nabla_c v^{1,0} = i\nabla_c v^{1,0} = \nabla_c(Jv^{1,0}),$$

which implies that $\nabla J = 0$.

The condition $\nabla J = 0$ is very strong.

Proposition 6.2.2. *Let (M, J, g) be almost hermitian. Then the following are equivalent:*

1) $\nabla J = 0$.

2) $N_J = 0$ *and* $d\omega = 0$, *where* $\omega = g(J\cdot, \cdot)$.

Moreover, if any one of these happens, then $D = \nabla_c|_{TM^{1,0}}$.

Definition 6.2.3. A *Kähler manifold* is an almost hermitian manifold such that 1) or 2) holds.

One of the nicest things about this story is that this links with the geometry for pairs (M, ω), where $\omega \in \Omega^2(M)$ satisfies the conditions that $d\omega = 0$ and that ω^n never vanishes. Such pairs are called *symplectic manifolds*, and ω is called the *symplectic form*.

21th lecture, November 15th 2011

Recall that (M, J, g) is called almost hermitian if $g(J\cdot, J\cdot) = g$.

Proposition 6.2.4. *Let (M, J, g) be almost hermitian. Then the following are equivalent:*

1) $\nabla J = 0$.

2) $N_J = 0$ *and* $d\omega = 0$, *where* $\omega = g(J\cdot, \cdot)$.

Moreover, if any one of these happens, then $D = \nabla$.

Remark 6.2.5. In the previous lecture, we compared D with $\nabla_c|_{TM^{1,0}}$ where the latter made sense if and only if $\nabla J = 0$. We can do better and compare ∇ and D though, since as smooth complex bundles (TM, J) and $TM^{1,0}$ are isomorphic through the isomorphism $v \mapsto \frac{1}{2}(v - iJv)$. Take the connection on (TM, J) induced by D and denote this by D as well. Note that D being a connection on (TM, J) is equivalent to D being a connection on TM with $DJ = 0$. Then it is clear that the eqution $D = \nabla$ makes sense.

Remark 6.2.6. If (M, J, g) is hermitian (i.e. $N_J = 0$) and $\nabla = D$, then (M, J, g) is Kähler. This follows since $\nabla J = DJ = 0$.

Note that this makes sense for almost hermitian manifolds (for a suitable notion of Chern connection D on almost hermitian manifolds), and still in this generalized setting, we have that $\nabla = D$ implies that (M, J, g) is Kähler.

Proof of Proposition 6.2.4. Assume that 1) and 2) are equivalent and let us prove that $\nabla J = 0$ implies that $D = \nabla$. We will use that $N_J = 0$ implies that we have holomorphic coordinates.

By definition, ∇ is the unique g-compatible connection such that the *torsion* of ∇ vanishes, i.e.

$$T_\nabla(v, w) = \nabla_v w - \nabla_w v - [v, w] = 0$$

for $v, w \in C^\infty(TM)$. The strategy of the proof is the following: We know that D preserves J (i.e. that $DJ = 0$), and D preserves the hermitian metric on $TM^{1,0}$. This implies that $Dg = 0$. Thus all we need to check is that the torsion $T_D = 0$. For this, we compute T_{D_c}, the $\mathbb{C}$-linear extension of T_D to TM_c. Since $N_J = 0$, we can restrict

$$T_D(v, w) = D_v w - D_w v - [v, w]$$

to $TM^{1,0}$ and since T_{D_c} comes from T_D, it is enough to check that $T_{D_c}|_{TM^{1,0}} = 0$.

We therefore want to check $T_{D_c}(v^{1,0}, w^{1,0}) = 0$ for all $v^{1,0}, w^{1,0} \in C^\infty(TM^{1,0})$. Take $x \in X \equiv (M, J)$. Consider adapted coordinates around x. Then consider $\theta = \theta(D)$, the local connection matrix $\theta = H^{-1}\partial H$. Then H is the matrix determined by $h = h_{g,J}$ on $TM^{1,0} \equiv TX$ satisfies $H = \mathrm{Id} + O(|x|^2)$, where we identify $x \equiv 0 \in \mathbb{C}^n$, and $\theta|_x = 0$. Then the torsion is

$$\begin{aligned} T_{D_c}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)|_x &= D_{\frac{\partial}{\partial z_i}}\left(\frac{\partial}{\partial z_j}\right)|_x - D_{\frac{\partial}{\partial z_j}}\left(\frac{\partial}{\partial z_i}\right)|_x \\ &= \frac{\partial}{\partial z_i} \lrcorner \theta \frac{\partial}{\partial z_j}|_x - \frac{\partial}{\partial z_j} \lrcorner \theta \frac{\partial}{\partial z_i}|_x = 0. \end{aligned}$$

In conclusion $T_{D_c|_{TM^{1,0}}} = 0$ implies that $T_{D_c} = 0$ and thus that $T_D = 0$ and $\nabla = D$.

We now prove that $\nabla J = 0$ if and only if $N_J = 0$ and $d\omega = 0$ where $\omega = g(J\cdot, \cdot)$.

We claim that

$$2g((\nabla_{v_2} J)v_0, Jv_1) = d\omega(Jv_0, v_1, v_2) + d\omega(v_0, Jv_1, v_2) - g(v_2, N_J(v_0, v_1)) \tag{2}$$

for $v_0, v_1, v_2 \in C^\infty(TM)$. If this holds, then clearly 2) implies 1). We want to see that it also tells us that 1) implies 2). To see this, note that

$$\begin{aligned} d\omega(v_0, v_1, v_2) &= v_0(\omega(v_1, v_2)) - v_1(\omega(v_0, v_2)) + v_2(\omega(v_0, v_1)) \\ &= -\omega([v_0, v_1], v_2) + \omega[v_0, v_2], v_1) - \omega([v_1, v_2], v_0). \end{aligned}$$

We will use that the g-compatibility

$$v_0(g(v_1, v_2)) = g(\nabla_{v_0} v_1, v_2) + g(v_1, \nabla_{v_0} v_2),$$

and that ∇ is torsion free,

$$\nabla_{v_0} v_1 - \nabla_{v_1} v_0 = [v_0, v_1].$$

From this we will show that

$$\Delta := g((\nabla_{v_0} J)v_1, v_2) + g((\nabla_{v_1} J)v_2, v_0) + g((\nabla_{v_2} J)v_0, v_1) = d\omega(v_0, v_1, v_2), \tag{3}$$

which together with (2) shows that 1) implies 2). We find

$$\begin{aligned}\Delta &= g(\nabla_{v_0}(Jv_1) - J\nabla_{v_0}v_1, v_2) + g(\nabla_{v_1}(Jv_2) - J\nabla_{v_1}v_2, v_0) + g(\nabla_{v_2}(Jv_0) - J\nabla_{v_2}v_0, v_1)\\ &= v_0(g(Jv_1, v_2)) - g(Jv_1, \nabla_{v_0}v_2) - g(J\nabla_{v_0}v_1, v_2)\\ &\quad + v_1(g(Jv_2, v_0)) - g(Jv_2, \nabla_{v_1}v_0) - g(J\nabla_{v_1}v_2, v_0)\\ &\quad + v_2(g(Jv_0, v_1)) - g(Jv_0, \nabla_{v_2}v_1) - g(J\nabla_{v_2}v_0, v_1)\\ &= v_0(\omega(v_1, v_2)) + v_1(\omega(v_2, v_0)) + v_2(\omega(v_0, v_1))\\ &\quad - \omega(\nabla_{v_0}v_1 - \nabla_{v_1}v_0, v_2) + \omega(\nabla_{v_0}v_2 - \nabla_{v_2}v_0) - \omega(\nabla_{v_1}v_2 - \nabla_{v_2}v_1, v_0)\\ &= d\omega(v_0, v_1, v_2)\end{aligned}$$

which proves 3. It now remains to prove (2). Note that $J(\nabla J) = -(\nabla J)J$ and

$$g([\nabla_{Jv}J - J\nabla_v J]\cdot, \cdot) = -g(\cdot, [\nabla_{Jv}J - J\nabla_v J]\cdot).$$

This together with (3) implies that

$$\begin{aligned}&d\omega(Jv_0, v_1, v_2) + d\omega(v_0, Jv_1, v_2)\\ &\quad = g((\nabla_{JV_0}J)v_1, v_2) + g((\nabla_{v_1}J)v_2, Jv_0) + g((\nabla_{v_2}J)Jv_0, v_1)\\ &\qquad + g((\nabla_{v_0}J)Jv_1, v_2) + g((\nabla_{Jv_1}J)v_2, v_0) + g((\nabla_{v_2}J)v_0, Jv_1)\\ &\quad = 2g((\nabla_{v_2}J)v_0, v_1) + g(v_2, \nabla_{Jv_0}v_1 - J(\nabla_{v_0}J)v_1) + g(v_0, (\nabla_{Jv_1}J)v_2 - J(\nabla_{v_1}J)v_2)\\ &\quad = 2g((\nabla_{v_2}J)v_0, v_1) + g(v_2, [\nabla_{Jv_0} - J\nabla_{v_0}J]v_1 - [\nabla_{Jv_1}J - J\nabla_{v_1}J]v_0)\end{aligned}$$

Finally, using that ∇ is torsion-free, one finds that

$$[\nabla_{Jv_0} - J\nabla_{v_0}J]v_1 - [\nabla_{Jv_1}J - J\nabla_{v_1}J]v_0) = -N_J(v_0, v_1),$$

which proves (2). ☐

Note that from the last formula of the proof, we see that $\nabla J = 0$ implies $N_J = 0$. One can check that $\nabla J = 0$ also implies that $\nabla_{Jv}J - J\nabla_v J = 0$. We know a priori that $N_J = 0$ if and only if $L_{Jv}J - JL_vJ = 0$, and so we have obtained a metric version of this formula.

The Proposition allows us to prove the following, using $\nabla = D$.

Proposition 6.2.7. *The curvature F_∇ and the* Ricci tensor *of g satisfy*

1) $F_\nabla \cdot J = JF_\nabla$, $F_\nabla(J\cdot, J\cdot) = F_\nabla$,

2) $\mathrm{Ric}(J\cdot, J\cdot) = \mathrm{Ric}$, *and* $\mathrm{Ric} = J\operatorname{tr} F_\nabla(\cdot, J\cdot)$.

Proof. The proof is an exercise. The idea is to work in TM_c using $D = \nabla$. Then the first formula of 1) just corresponds to the fact that $F_D \in \Omega^2(\mathrm{End}(TM, J))$ which holds since D is the Chern connection. The second one corresponds to $F_D \in \Omega^{1,1}(\mathrm{End}(TM, J))$. ☐

Remark 6.2.8. In terms of D on TX, $X \equiv (M, J)$, the second formula of the Proposition tells us that

$$\mathrm{Ric} = i\operatorname{tr} F_D(\cdot, J\cdot).$$

Now Ric is symmetric, and we can define the *Ricci form* ρ of the Kähler structure (g, J, ω) by

$$\rho = i\operatorname{tr} F_D \in \Omega^{1,1}(X).$$

Usually, if J is a fixed complex structure, in Kähler geometry one works with ω instead of g. It is therefore common to call ω the *Kähler metric*, even if it is not a metric.

Fix a complex structure. Locally, in holomorphic coordinates, we can write

$$\rho = \rho_\omega = i\operatorname{tr}\overline{\partial}(H^{-1}\partial H) = i\overline{\partial}\partial\log(\det H),$$

where for a hermitian metric h on the tangent bundle, we define $\det H \equiv \det h$ the natural hermitian metric on $K_X^* := \wedge^n TX$, $n = \dim_{\mathbb{C}} X$. Here, K_X is called the *canonical bundle* and K^*X the *anti-canonical bundle*. Thus ρ_ω is i times the curvature of the natural metric on the anti-canonical bundle.

$$\det$$

6.3 Examples

Example 6.3.1 (The Kähler–Einstein equation)**.** In Riemannian geometry, a Riemannian metric g is called *Einstein* if $\mathrm{Ric}_g = \lambda \cdot g$ for $\lambda \in \mathbb{R}$. This is what happens if one copies the Einstein equation from Lorentz geometry and writes what it means in Riemannian geometry.

In Kähler geometry, we say that a Kähler structure (g, J, ω) is *Kähler–Einstein* if $\mathrm{Ric}_g = \lambda g$. Contracting with J this tells us that

$$\rho_\omega = \lambda \omega,$$

which is the *Kähler–Einstein equation.* A big amount of work being done in Kähler geometry amounts to studying the Kähler–Einstein equations, and it is an open question which manifolds admit Kähler–Einstein metrics.

Example 6.3.2. If (X, g, ω) is Kähler, if $i : Z \hookrightarrow X$ is a complex submanifold, then Z is Kähler, that is, $(Z, i^*g, i^*\omega)$ defines a Kähler structure on Z with the complex structure from X. This follows from the facts that $N_J^Z = 0$, i^*g is hermitian, and $di^*\omega = i^*d\omega = 0$.

Example 6.3.3. Another example is $(\mathbb{C}^n, \omega = i \sum_j dz_j \wedge d\bar{z}_j)$ which is a non-compact Kähler manifold which is flat in the sense that the curvature vanishes.

Example 6.3.4. Projective space $(\mathbb{P}(\mathbb{C}^n), \omega_{\mathrm{FS}})$ is a Kähler manifold. Here,

$$\omega_{\mathrm{FS}} = iF_{h_{\mathrm{FS}}},$$

where h_{FS} is the Fubini–Study hermitian metric on $\mathcal{O}_{\mathbb{P}(\mathbb{C}^n)}(1)$.

Remark 6.3.5. On a Kähler manifold we have a closed form ω, i.e. $d\omega = 0$, and we have a chomology class $[\omega] \in H^2(X, \mathbb{R})$. When $\omega = iF_h$ for h a hermitian metric on a line bundle $L \to X$ as in the previous example, then $[\omega]$ has special properties (e.g. it is in $H^2(X, \mathbb{Z})$).

Example 6.3.6. Combining Examples 6.3.2 and 6.3.4 we see that any projective manifold X is Kähler with Kähler form $\omega = iF_h$ for h a hermitian metric on $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}(\mathbb{C}^n)}(1)|_X$. Our original motivation was to determine which complex manifolds were projective, so we are led to consider compact Kähler manifolds.

If X is a compact complex manifold endowed with an ample line bundle $L \to X$, then $(X, \varphi^*\omega_{\mathrm{FS}})$ is Kähler, where $\varphi : X \hookrightarrow \mathbb{P}(\mathbb{C}^N)$, where $N = \dim_{\mathbb{C}} H^0(X, L^k)$ for some $k \gg 0$, and $\varphi^*\omega_{\mathrm{FS}} = iF_{\varphi^*h_{\mathrm{FS}}}$.

7 Kodaira's Embedding Theorem

7.1 Statement of the theorem

Definition 7.1.1. A line bundle $L \to X$ is called *positive* if there exists a hermitian metric h on L such that $\omega = iF_h$ is a Kähler form.

Theorem 7.1.2 (Kodaira, version 1)**.** *A line bundle $L \to X$ over a compact complex line bundle is ample if and only if it is positive.*

As we saw in Example 6.3.6, if L is ample, it is positive. To prove the converse, we use the concept of Bergman kernels.

7.2 Bergman kernels

Let $E \to X$ be a holomorphic vector bundle over a compact complex manifold, and let $L \to X$ be a positive line bundle, so there exists h such that $\omega = iF_h$ is a Kähler form.

Assume that $H^0(X, E \otimes L^k)$ is finite dimensional. Fix H a hermitian metric on E and consider the metric $H_k = H \otimes h^k$ on $E \otimes L^k$.

Consider the hermitian metric on $H^0(X, E \otimes L^k)$ given by

$$\langle s_1, s_2 \rangle_{L^2} = \int_X (s_1, s_2)|_{H_k} \frac{\omega^n}{n!},$$

where $n = \dim_{\mathbb{C}} X$.

Remark 7.2.1. If ω is Kähler, then $\frac{\omega^n}{n!}$ is a volume form on X, as one finds that in coordinates,

$$\frac{\omega^n}{n!} = \sqrt{|\det g|} dx_1 \wedge \cdots \wedge dx_{2n}.$$

The holomorphic structure determines an orientation, i.e. for a holomorphic atlas, the differential of changes of coordinates have the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

with positive determinant.

In general any almost complex structure induces an orientation; see [KN96, Prop. 2.1].

Definition 7.2.2. The *Bergman kernel* K is the kernel of integration for the orthogonal projection

$$\Pi : C^\infty(E \otimes L^k) \to H^0(E \otimes L^k).$$

Thus, K is a section of the bundle $\overline{E \otimes L^k} \boxtimes E \otimes L^k \to X \times X$.

Here, for projections $p_1 : Z \to Y_1$, $p_2 : Z \to Y_2$ and bundles $E_1 \to Y_1$, $E_2 \to E_2$, we can form the bundle $E_1 \boxtimes E_2 \to Z$ given by $E_1 \boxtimes E_2 = p_1^* E_1 \otimes p_2 E_2$. In our case, $Z = X \times X$, and p_i, $i = 1, 2$, is just projection onto the i'th factor.

By definition of kernel of integration, we have

$$\Pi s(x) = \int_X (s, K(\cdot, x))|_{H_k} \frac{\omega^n}{n!}.$$

From now on, we simply write $EL^k = E \otimes L^k$. As $H^0(EL^k)$ is finite dimensional, consider $\{s_j\}_{j=1}^{N_k}$ be an orthonormal basis with respect to $\langle \, , \, \rangle_{L^2}$. Consider

$$B^k = B^k(x, y) = \sum_j s_j(x)(\cdot, s_j(y))_{H^k} \in C^\infty(X \times X, (EL^k)^* \boxtimes EL^k).$$

Then we can rewrite

$$(\cdot, K(x, y))_{H_k} = B^k(x, y). \tag{4}$$

We will typically write everything in terms of B^k, which is typically known as the *density of states*. Using (4), we can write

$$\Pi s(x) = \int_X B^k(\cdot, x) s \frac{\omega^n}{n!}.$$

We now sketch how to prove Kodaira's theorem using B^k. The idea is the following: Consider B^k restricted to the diagonal $X \subseteq X \times X$. Then

$$B^k(x) = B^k(x, x) \in C^\infty(X, (EL^k)^* \otimes (EL^k)) = C^\infty(X, \mathrm{End}(EL^k)) \cong C^\infty(X, \mathrm{End}(E))$$

can be written

$$B^k(x) = \sum_j s_j(\cdot, s_j)_{H^k}(x).$$

We will see that B^k admits an asymptotic expansion in k,

$$(2\pi)^n B^k = k^n B_0 + k^{n-1} B_1 + \cdots ,$$

where $B_0 = \mathrm{Id}$ and $B_j \in C^\infty(\mathrm{End}(E))$. So far, this expansion is only formal. For $k \gg 0$ and $x \in X$, choose a frame $\{e_1, \dots, e_r\}$ around that point. Then we can write $(2\pi)^n B^k_x e_k(x) - e_k(x) k^n$ as k^{n-1} times something. This is equivalent to

$$\frac{(2\pi)^n}{k^n} \sum_j s_j(x)(e_k(x), s_j(x))_{H_k} - e_k(x)$$

being k^{-1} times something. Morally, this k^{-1} times something is small because $k \gg 0$. We are thus comparing the $r \times r$-matrices

$$A = \frac{(2\pi)^n}{k^n} \begin{pmatrix} \sum_j s_j(x)(e_1(x), s_j(x))_{H_k} \\ \vdots \\ \sum_j s_j(x)(e_r(x), s_j(x))_{H_k} \end{pmatrix}, \quad B = \begin{pmatrix} e_1(x) \\ \vdots \\ e_r(x) \end{pmatrix}.$$

Then since we know that $\det(B) \neq 0$, we must also have $\det(A) \neq 0$. This implies that $\{s_1(x), \dots, s_N(x)\}$ generate $E \otimes L^k|_x$.

Note that if $E = X \times \mathbb{C}$ there exists a j such that $s_j(x) \neq 0$. This means that the map $\varphi_s : X \to G_r(\mathbb{C}^N)$, where $N = \dim H^0(EL^k)$ is well-defined at $x \in X$.

We will prove that the asymptotic expansion above is uniformly well-behaved over X, since then $E \otimes L^k$ is generated by global sections which will then define φ_s as a holomorphic map. We will in fact prove more and see that the expansion carries metric information which will allow us to compare the initial Kähler metric on X with the pullback of the metric on $G_r(\mathbb{C}^N)$. We will see that, asymptotically, the embedding tends to an isometry.

22nd lecture, November 22th 2011

7.2.1 Symplectic geometry

As further motivation for Bergman kernels, we discuss symplectic geometry. The story begins with classical mechanics. Consider a physical system whose configurations are described by points on a manifold M. The evolution of an element in the system (at a configuration $x(0)$ at time $t = 0$) is governed by the "Principle of Least Action". The curve $x(t) \subseteq M$ describing this evolution is a solution of a variational principle. That is, x is a minimum (critical point) of a functional I defined on a space of paths on M. To be more precise, we consider a *Lagrangian* $L : TM \times \mathbb{R} \to \mathbb{R}$, and the functional I, known as the *action functional*, is of the form

$$I(x(t)) = \int_{t_0}^{t_1} L(\dot{x}(t), t)\, dt.$$

Here, we consider I as defined on the space $P = C^1([t_0, t_1], M)_{x_0, x_1}$ of paths with $x(t_0) = x_0$, $x(t_1) = x_1$. Critical points of I satisfy the *Euler–Lagrange equation* for $x \in P$ is

$$dI(\dot{x}) = 0,$$

for $\dot{x} \in T_x P$. This equation is equivalent to the following local expression in coordinates (x, p) for TM:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial p}(\dot{x}(t), t)\right) = \frac{\partial L}{\partial x}(\dot{x}(t), t).$$

This is a system of N second order differential equations, where $n = \dim M$.

Example 7.2.3. Take $M = \mathbb{R}^n$, and let $V \in C^\infty \mathbb{R}^n$ (or possibly a space allowing for singularities) be a *potential*. Take the Lagrangian to be

$$L = \frac{1}{2}|p|^2 - V(x).$$

Then the Euler–Lagrange equation is

$$\frac{\partial^2 x}{\partial t^2} = -\frac{\partial V}{\partial x}(x(t)).$$

For example, one could consider the potential $V = -1/|x|$ describes a system udner foces like gravity or electromagnetism. In such a system, dynamics are described by a force towards the origin which varies with inverse the square of distance.

The idea that leads to symplectic geometry is to rewrite the Euler–Lagrange equation as a system of $2n$ first order equations. To do so, we work in T^*M rather than TM. For this we need the Legendre transform.

Let $p : TM \to M$ be the projection. Consider $\Phi : p^*TM \to VTM$ given by $(v, w) \mapsto \frac{d}{dt}|_{t=0}(v + tw)$.

Definition 7.2.4. The *Legendre transform* of L is the map $\tilde{L} : TM \to T^*M$ which maps

$$v \mapsto dL|_v \Phi(v, \cdot)$$

Example 7.2.5. Let g be a metric on M and let $L(v) = \|v\|_g^2$. Then $\tilde{L}(v) = g(v, \cdot)$. This defines an isomorphism $TM \cong T^*M$.

In coordinates (x, p) on TM, we have

$$\tilde{L}(x, p) = \left(x, \frac{\partial L}{\partial p}\right) =: (x, q).$$

We assume that $\tilde{L}$ is an isomorphism. Locally, the condition for this is

$$\det\left(\frac{\partial^2 L}{\partial p^2}\right) \neq 0,$$

which is known as the *Legendre condition*. We write (x, q) for coordinates on T^*M. Define the *Hamiltonian* $H : T^*M \to \mathbb{R}$ by

$$\sigma \mapsto \langle \tilde{L}^{-1}(\sigma), \sigma\rangle - L \circ \tilde{L}^{-1}(\sigma).$$

In coordinates, this can be written as

$$\begin{aligned} H(x, q) &= \sum_j q_j p_j - L(x, p), \\ (x, p) &= \tilde{L}^{-1}(x, q) = (x, G(t, x, q)). \end{aligned}$$

We thus obtain the equations

$$\begin{aligned} \frac{\partial H}{\partial x} &= -\frac{\partial L}{\partial X}, \\ \frac{\partial H}{\partial Q} &= G + q \cdot \frac{\partial G}{\partial q} - \frac{\partial L}{\partial p}\frac{\partial G}{\partial q} = G, \end{aligned}$$

and the Euler–Lagrange equations transform into

$$\frac{\partial q}{\partial t} = -\frac{\partial H}{\partial x}, \quad \frac{\partial x}{\partial t} = \frac{\partial H}{\partial p}$$

for a curve $((x(t), q(t))$ in T^*M. These are *Hamilton's differential equations.* An observation made in the 19th century is the following: If a set of particles in our system have positions x and momenta q in the region s_1 (i.e. a 2-dimensional manifold) at time t_1, then at any later time t_2, their positions and momenta another region s_2 with the same area. Here, the regions are oriented, so "area" makes sense.

To express this observation formula, we need the notion of a *symplectic structure* ω. Such an ω will satisfy two conditions. It is an element of $\Omega^2(T^*M)$, since we want to measure the volume of surfaces on T^*M. It will also be non-degenerate, $d\omega = 0$, since the area of a surface involves integrals like $\int_S \omega$, and one uses Stokes' theorem to describe how it evolves.

Define the *Liouville 1-form* $\lambda \in \Omega^1(T^*M)$ by

$$\lambda(w_\sigma) = \langle \sigma, d\pi(w_\sigma) \rangle,$$

for $\pi : T^*M \to M$ projection, and $w_\sigma \in T_\sigma T^*M$. Then $\omega = -d\lambda$ satisfies the properties above and is thus a symplectic structure on T^*M. In coordinates, if $\lambda = \sum_j q_j dx_j$, then

$$\omega = \sum_j dx_j \wedge dq_j.$$

A curve $\gamma(t)$ on T^*M satisfies Hamilton's equations if and only if

$$dH|_{(t,\gamma(t))} = \dot{\gamma}(t) \lrcorner \omega.$$

As ω is non-degenerate, $dH = Y \lrcorner \omega$ defines a vector field on T^*M and solutions of Hamilton's equations are given by the flow of Y.

Definition 7.2.6. Let M be a manifold. A 2-form $\omega \in \Omega^2(M)$ is a *symplectic structure* on M if $d\omega = 0$, and ω^n is a volume form. Here, $2n = \dim_{\mathbb{R}} M$. The pair (M, ω) is called a *symplectic manifold.*

23rd lecture, November 28th 2011

Remark that if (M, ω) is a symplectic manifold then $\dim_{\mathbb{R}} M$ is even. Remark also that ω^n is a volume form if and only if it satisfies the non-degeneracy condition: If $Y \lrcorner \omega = 0$, then $Y ==$ for all $Y \in C^\infty(TM)$.

Given $f \in C^\infty(M)$, we can write $df = Y_f \lrcorner \omega$, where $Y_f \in C^\infty(TM)$ is called the *hamiltonian vector field* of f.

The content of the next theorem is that, locally, all symplectic structures are "the same" (note that e.g. Riemannian metrics do not have this properties, so symplectic structures are something more subtle).

Theorem 7.2.7 (Darboux Theorem)**.** *Any symplectic manifold is locally symplectomorphic to* $(T^*\mathbb{R}^n, \omega)$*, where* ω *is the standard symplectic structure defined previously.*

Here, given two (M_1, ω_1) and (M_2, ω_2) symplectic manifolds we call a diffemorphism $f : M_1 \to M_2$ a *symplectomorphism* if $f^*\omega_2 = \omega_1$. The two symplectic manifolds are called *symplectomorphic.*

The question now is how to identify a symplectic structure locally; i.e., which information does ω on $T^*\mathbb{R}^n$ carry? The idea is that a Riemannian metric rigidifies the manifold, as we can not deform a Riemannian manifold in an arbitrary way (i.e. in general, diffeomorphisms are not necessarily isometries). How does a symplectic structure rigidify $T^*\mathbb{R}^n \cong \mathbb{R}^n$? This leads to the idea of the *symplectic camel*: "It would be easier for a camel to go through the hole of a needle than a rich man to get into heaven". Write $\omega = \sum_{j=1}^n dx_j \wedge dq_j$ on $\mathbb{R}^{2n} \equiv (x, q)$. Consider the unit ball B and a wall W,

$$B = \{(x, q) \in \mathbb{R}^{2n} \mid \sum_j |x_j|^2 + |q_j|^2 = 1\},$$

$$W = \{(x, q) \in \mathbb{R}^{2n} \mid q_1 = 0\}$$

and a hole H_ε in the wall,

$$H_\varepsilon = \{(x,q) \in W \mid \sum |x_j|^2 + |q_j|^2 < \varepsilon\}$$

for $\varepsilon \in \mathbb{R}$. The question is: Can we find a continuous family $\varphi : B \times [0,1] \to \mathbb{R}^n$ of symplectic embeddings (i.e. $\varphi_t^*\omega = \omega$) which go through the hole? That is, satisfying

$$\varphi_t(B) \subseteq \mathbb{R}^{2n} \setminus (W \setminus H_\varepsilon),\ \varphi_0(B) \subseteq \{q_1 > 0\},\ \varphi_1(B) \subseteq \{q_1 < 0\}.$$

The answer is given by the following theorem:

Theorem 7.2.8 (Gromov)**.** *No, if* $n > 1$ *and* $\varepsilon < 1$*.*

A similar result (with a similar proof) is the *Nonsqueezing Theorem* of Gromov, which is a fundamental theorem in symplectic geometry. Consider a ball B^R and a cylinder Z^r,

$$B^R = \{(x,q) \in \mathbb{R}^{2n} \mid |(x,q)|^2 \leq R^2\}, Z^r = \{(z,q) \in \mathbb{R}^{2n} \mid |x_1|^2 + |q_1|^2 \leq r^2\}.$$

The question then is, whether or not it is possible to embed B^R into Z^r. Clearly, this is possible by a volume preserving map by “squeezing” B^R, but it turns out not to be possible to do symplectically.

Theorem 7.2.9 (Nonsqueezing theorem (Gromov))**.** *If* $r < R$ *and* $n > 1$*, there is no symplectic embedding* $\varphi : B^r \to Z^R$*.*

The interpretation in terms of classical mechanics is the following: If a collection of particles initially spread out all over B^R, then they do not evolve in such a way that the position and momentum (x_1, q_1) spread out less than initially; one can think of this as a classical analogue of the Heisenberg uncertainty principle.

7.2.2 Geometric quantization

The reference for the following is [AE05] and [MS98].

Feynman said “I think I can safely say that noone understands quantum mechanics”. We will discuss the trials of mathematics to make sense of what quantum mechanics does. Quantization in general is the transition from classical mechanics to quantum mechanics. Rigorously, this is treated by geometric quantization among other theories.

For us, geometric quantization is a theory whose goal is to find a recipe to make the following assignments: To a symplectic manifold (M,ω), we associate a Hilbert space H. The projectivization $\mathbb{P}(H)$ is called the *space of states*. To the Lie algebra $(C^\infty(M), \{\cdot,\cdot\})$, where $\{\cdot,\cdot\}$ is the Poisson bracket which we will define below, we associate an algebra $\mathcal{A}$ of self-adjoint operators on H called *observables*.

Given a symplectic manifold (M,ω) and $f, g \in C^\infty(M)$, we define the *Poisson bracket*

$$\{f,g\} = \omega(Y_f, Y_g),$$

where as before, $df = Y_f \lrcorner \omega$ and $dg = Y_g \lrcorner \omega$. This defines a Lie bracket on $C^\infty(M)$.

The assignment of quantization must satisfy some strong properties. Usually one restricts attention to a subalgebra $\mathrm{Obs} \subseteq (C^\infty(M), \{\cdot,\cdot\})$. Write Q_f for the operator associated to $f \in C^\infty(M)$. We require the following:

- Q1) $Q_1 = \mathrm{Id}$.
- Q2) The map $f \mapsto Q_f$ is $\mathbb{R}$-linear.
- Q3) $[Q_f, Q_g] = \frac{i\hbar}{2\pi} Q_{\{f,g\}}$, where $\hbar \in \mathbb{R}_{>0}$.
- Q4) Functoriality: For a smooth map $(M_1,\omega_1) \xrightarrow{\varphi} (M_2,\omega_2)$ such that $\varphi^*\omega_2 = \omega_1$, the map $C^\infty(M_2) \to C^\infty(M_1)$ given by $f \mapsto f \circ \varphi$ restricts to a map $\mathrm{Obs}_2 \to \mathrm{Obs}_1$, and moreover, there exists a unitary operator $U_\varphi : H_1 \to H_2$, and $Q^1_{f\circ\varphi} = U^*_\varphi Q^2_f U_\varphi$.

Q5) Canonical quantization: If $(M,\omega) = (T^*\mathbb{R}^n, \omega)$, then $H \cong L^2(\mathbb{R}^N, \mathbb{C})$ in the variable q, and

$$Q_{q_j}\psi = q_j\psi,\ Q_{x_j}\psi = \frac{-i\hbar}{2\pi}\frac{\partial\psi}{\partial q_j}$$

for all $\psi \in H$.

Kostant and Sourier proposed a solution in two steps: 1) Prequantization and 2) Polarization. Step 1) amounts to finding a hermitian line bundle $(L,h) \to (M,\omega)$ with a unitary connection D such that $\frac{i}{2\pi}F_D = \omega$. Then, consider a family of pre-Hilbert spaces

$$H_k' = C^\infty(M, L^k)_{L^2} := \{s \in C^\infty(M, L^k) \mid \|s\|_{L^2}^2 = \int_M (s,s)_{h^k}\frac{\omega^k}{n!} < \infty\}.$$

Consider self-adjoint operators

$$C^\infty(M) \to \mathcal{A}(H) := \{\text{self-adjoint operators on } H\},$$
$$f \mapsto Q_f = \frac{-i}{2\pi k}D_{Y_f} + f.$$

Note that H_k' is only pre-Hilbert, so make a completion. Then Q1), Q2), Q3), and Q4) above are satisfied (with $\hbar = k^{-1}$). However, Q5) fails: If $(M,\omega) = T^*\mathbb{R}^n, \omega)$, we can identify $\overline{H_k'} = L^2(\mathbb{R}^{2n}, \mathbb{C})$, so we have twice as many variables as we should.

The polarization step roughly amounts to choosing a subbundle $P \subseteq TM^c$ of half the dimension, such that P is closed under Lie bracket. Then consider $H_k \subseteq H_k'$ given by sections that are covariantly constant along the directions in P.

In the particular case we are interested in, that is when (M,ω) admits a Kähler structure J, we can define

$$P := T^{0,1}M \subseteq TM^c, \quad H_k = H^0(X, L^k),$$

where $X = (M,J)$. When X is compact, the spaces H_k are finite dimensional and therefore Hilbert. To see the last equality, recall that $\omega \in \Omega^{1,1}(X)$, and so, since $F_D = \frac{i}{2\pi}\omega$, the connection D induces a holomorphic structure on $L \to X$.

Remark 7.2.10. The algebra $\mathrm{Obs} \subseteq (C^\infty(M), \{\cdot,\cdot\})$ of observables, mapping $\mathrm{Obs} \to \mathcal{A}(H_k)$, is defined to be those $f \in C^\infty(M)$ with $[Y_f, P] \subseteq P$.

In the Kähler case, we claim that the condition $[Y_f, T^{0,1}M] \subseteq T^{0,1}M$ implies that Y_f is holomorphic. To see this, write $Y_f = Y_f^{1,0} + \overline{Y_f^{1,0}}$, and since $N_J = 0$, $P = T^{0,1}M$ is integrable, so $[Y_f^{0,1}, P] \subseteq P$. Hence the only thing to check is that $[Y_f^{1,0}, TM^{0,1}] \subseteq TM^{0,1}$. Locally, $Y_f^{1,0} = \sum_j v^j\frac{\partial}{\partial z_j}$, and we find

$$\left[Y_f^{1,0}, \frac{\partial}{\partial \overline{z}_k}\right] = \sum_j \frac{\partial v^j}{\partial \overline{z}_k}\frac{\partial}{\partial z_j},$$

so the condition implies that $\frac{\partial v^j}{\partial \overline{z}_k} = 0$.

Now since

$$0 = L_{Y_f}\omega = d(Y_f \lrcorner \omega) + Y_f \lrcorner d\omega = ddf = 0,$$

we find that for $f \in \mathrm{Obs}$ we have $L_{Y_f}\omega = 0$, and $L_{Y_f}J = 0$, which implies that $L_{Y_f}g = 0$, so Y_f is an infinitesimal isometry (a Killing vector field) with respect to $g = \omega(\cdot, J\cdot)$.

Theorem 7.2.11 (Myers–Steenrod)**.** *The group of isometries of a Riemannian manifold is a finite dimensional Lie group.*

This means that Obs is finite-dimensional as a vector space. That is, we have drastically restrained the classical algebra of observables, which is unsatisfactory (and often, the algebra consists only of constant functions).

Note also that we picked a complex structure on M, and in general the dependence on this complex structure is a hard problem.

7.2.3 Berezin quantization

The reference for this section is [RCG90]

To increase Obs, we consider Berezin quantization. We consider the same setting as before: $L \to (X, \omega)$ is a line bundle over a compact Kähler manifold, $\omega = \frac{i}{2\pi} F_h$, and $H_k = (H^0(X, L^k), \langle\, , \,\rangle_{L^2})$.

The idea is that we want to reverse engineer the association $C^\infty(X) \to \mathcal{A}(H_k)$ to obtain the Berezin symbol $\sigma^k : \mathcal{A}(H_k) \to C^\infty(X) \otimes \mathbb{C}$.

The *Berezin construction* is the following: Given $q \in L^k \setminus \{0\}$, we define a continuous linear functional l_q on H_k by

$$l_q(s) = s(x)/q,$$

where $x = p(q)$. By the Riesz–Fischer representation theorem (which is really not necessary here as everything is finite dimensional, but the construction also works out in the infinite dimensional case), there is an element $e_q \in H_k$ such that $l_q(s) = \langle s, e_q\rangle_{L^2}$ for all $s \in H_k$. The element e_q is usually called the *coherent vector* of q. Note that $l_{cq}(s) = c^{-1} l_q(s)$, which implies that $e_{cq} = \overline{c}^{-1} e_q$, so for $x \in X$, $q \in p^{-1}(x) \setminus \{0\}$, the *coherent state* of x, $e_x := [e_q] \in \mathbb{P}(H_k)$, is well-defined.

Consider A a bounded linear operator on H_k and define the *Berezin covariant symbol* by

$$\sigma^k(A)(x) = \frac{\langle A e_q, e_q\rangle_{L^2}}{\langle e_q, e_q\rangle_{L^2}} \in \mathbb{C}$$

for $x \in X$, $q \in L^k \setminus \{0\}$, $p(q) = x$. Now, we can recover A from $\sigma^k(A)$: By definition, $\sigma^k(A)$ is real analytic, and $\sigma^k(A^*) = \overline{\sigma^k(A)}$, which allows us to extend $\sigma^k(A)$ to an open dense subset of $X \times X$ containing the diagonal,

$$\sigma^k(A)(x, y) = \frac{\langle A e_q, e_{q'}\rangle_{L^2}}{\langle e_q, e_{q'}\rangle_{L^2}},$$

where $p(q) = x, p(q') = y$, $q, q' \in L^k \setminus \{0\}$, so $\sigma^k(A)$ is defined on

$$\{(x, y) \in X \times X \mid \langle e_q, e_{q'}\rangle \neq 0\}.$$

Note that $\sigma^k(A)(x, y)$ is just the analytic continuation of $\sigma^k(A)$.

24th lecture, November 29th 2011

We begin with a comment on geometric quantization: For the axiom Q4) to hold in this setup, the map φ must take $((L_1, h_1, D_1), P_1)$ to $((L_2, h_2, D_2), P_2)$, i.e. $\varphi^*(L_2, h_2, D_2) = (L_1, h_1, D_1)$ (which is not a very restrictive requirement), and $\varphi_* P_1 \subseteq P_2$, which is an unsatisfactory requirement: We would like to have the quantization be independent of P. If we have this, we get a representation on H of the group $\mathrm{Sp}(M, \omega)$ of self-symplectomorphisms of (M, ω). Independence of P is hard to obtain though. Although geometric quantization is unsatisfactory for quantization, since it has very few observables, it is a very helpful mathematical tool.

We turn back to Berezin quantization. Recall that we have $L \to X$ a positive line bundle over a compact Kähler manifold, so we have a metric h on L such that $\frac{i}{2\pi} F_h = \omega$ is the Kähler form on X. Given k, we have the Hilbert space $H_k = (H^0(X, L^k), \langle\, , \,\rangle_{L^2})$ with the inner product determined by h and $\omega^n/n!$. Recall that to a bounded linear operator $A \in \mathrm{Op}(H_k)$, we associate its Berezin symbol $\sigma^k(A) : X \to \mathbb{C}$, defined by

$$\sigma^k(A)(x) = \frac{\langle A e_q, e_q\rangle_{L^2}}{\langle e_q, e_q\rangle_{L^2}},$$

where $\langle s, e_q\rangle_{L^2} = s(x)/q$ for $q \in L^k \setminus \{0\}$ with $p(q) = x$, and $e_q, s \in H^0(X, L^k)$. Thus we have a map $\mathrm{Op}(H_k) \to C^\infty(X, \mathbb{C})$ which restricts to a map on self-adjoint operators, $\mathcal{A}(H_k) \to C^\infty(X, \mathbb{R})$

(which gives real-valued functions since $\sigma^k(A^*) = \overline{\sigma^k(A)}$). The idea of Berezin is to invert this last map from $\sigma^k(\mathcal{A}(H_k)) = \mathrm{Obs}_k$. So our goal is to recover A from $\sigma^k(A)$.

For this, we claim that $\sigma^k(A)$ is real analytic. To prove this, we take an orthonormal basis $\{s_j\}$ of H_k. If $s_j(x) = \lambda_j q$, and $p(q) = x$, $q \neq 0$, then we can write

$$e_q = \sum_j \langle e_q, s_j\rangle_{L^2} s_j = \sum_j \overline{\lambda_j} s_j.$$

Write $A = (A_{ij})$, so that

$$\sigma^k(A)(x) = \frac{\sum_{j,l} \lambda_j \overline{\lambda_l} A_{jl}}{\sum_j |\lambda_j|^2} = \frac{\sum_{j,l} s_j(x)\overline{s_l}(x) A_{jl}}{\sum_j s_j(x)\overline{s_j(x)}},$$

which is a well-defined analytic function. Consider the analytic extension $\sigma^k(A)(x,y)$. This satisfies $\sigma^k(A)(x,\overline{x}) = \sigma^k(A)(x)$ and $\sigma^k(A)(x,y)$ is holomorphic in x and anti-holomorphic in y. Recall that the extension is defined on

$$V = \{(x,y) \mid \langle e_{q'}, e_q\rangle \neq 0, p(q') = y, p(q) = x\}.$$

Now, we can recover A from its symbol since for $q \in L^k \setminus \{0\}$, $p(q) = x$, we have

$$\begin{aligned} As(x) = \langle As, e_q\rangle_{L^2} q = \langle s, A^* e_q\rangle_{L^2} q &= \left(\int_X (s(y), (A^* e_q)(y))_{h^k} \frac{\omega^n}{n!}\right) q \\ &= \left(\int_X (s(y), e_q(y))_{h^k} \sigma^k(A) \frac{\omega^n}{n!}(y)\right) q. \end{aligned}$$

The last equality follows from

$$A^* e_q(y') = \langle e_q, A e_{q'}\rangle_{L^2} q' = \frac{\langle e_q, A e_{q'}\rangle_{L^2}}{\langle e_q, e_{q'}\rangle} e_q(y),$$

which, on the other hand, holds since $q' = e_q(y')/\langle e_q, e_{q'}\rangle_{L^2}$.

Then we can define from this the map $\mathrm{Obs}_k = \sigma^k(\mathcal{A}(H_k)) \to \mathcal{A}(H_k)$. The question now is whether this is a quantization. It turns out that it is, under very strong assumptions that have to do with the Bergman kernel.

Remark 7.2.12. Since H_k is finite dimensional, any $A \in \mathrm{Op}(H_k)$ is generated by a rank 1-operators.

Given $u, v \in H_k$, we can define $As = \langle s, u\rangle_{L^2} v$, which has symbol

$$\sigma^k(A)(x) = \frac{\langle\langle e_q, u\rangle v, e_q\rangle_{L^2}}{\langle e_q, e_q\rangle_{L^2}} = \frac{\langle e_q, u\rangle\overline{\langle e_q, v\rangle}}{\langle e_q, e_q\rangle} = \frac{h^k(u,v)}{|q|^2\langle e_q, e_q\rangle}.$$

Define the *Rawnsley function* $\rho^k(x) := |q|^2\langle e_q, e_q\rangle$. Take an orthonormal basis $\{s_j\}$ and write $e_q = \sum_j \overline{\lambda_j} s_j$. Then

$$|q|^2\langle e_q, e_q\rangle = |q|^2 \sum_j |x_j|^2 = \sum_j |s_j|^2_{h^k}(x).$$

The right hand side is the density of states of the Bergman kernel on L^k.

The following theorem gives conditions on ρ^k that the quantization procedure make sense.

Theorem 7.2.13. *If for all k, ρ^k is a constant depending on k, then $\bigcup_{k=1}^\infty \sigma^k(\mathcal{A}(H_k)) \subseteq C^\infty(M)$ is a dense subalgebra of $C^\infty(M)$.*

Remark 7.2.14. The very strong condition on the ρ^k being constants is related to the Kähler metric being *balanced*.

Note that we need L^k to be generated by global sections to make sense of the condition.

7.2.4 Berezin–Toeplitz quantization

To avoid the restriction on the ρ_k above and still make the set of quantizable observables large, we turn to Berezin–Toeplitz quantization. The idea is to relax the axiom Q3) to some asymptotic expression, formally

$$[Q_f, Q_g] = \frac{-i\hbar}{2\pi} Q_{\{f,g\}} + O(\hbar^2).$$

Consider the L^2-projection $\Pi^k : C^\infty(X, L^k) \to H_k = H^0(X, L^k)$. Recall that the Bergman kernel $K(x,y) \in C^\infty(X \times X, \overline{L^k} \boxtimes L^k)$ is the kernel of integration for Π^k, i.e.

$$\Pi^k(s)(x) = \int_X (s(y), K(y,x))_{h^k} \frac{\omega^n}{n!}(y).$$

Given $f \in C^\infty(M)$, define for $s \in H_k$ the *Toeplitz operator* on H_k by

$$T_f^k s = \Pi^k(fs) = \int_X f \cdot (s, K^k(\cdot, x)) \frac{\omega^n}{n!} = \int_X f \cdot B^k(\cdot, x) s \frac{\omega^n}{n!},$$

where $B^k = (\cdot, K)_{h^k} \in C^\infty((L^k)^* \boxtimes L^k)$. Remark that if $\{s_i\}$ is an orthonormal basis, we have

$$B^k(x,x) = \sum_j |s_j|^2_{h^k}(x).$$

The assignment $f \mapsto T_f^k$ defines a quantization in the following sense:

Theorem 7.2.15 (Bordemann–Meinrenken–Schlichenmaier, [BMS94]). *We have*

$$\lim_{k\to\infty} \|T_f^k\| = \|f\|_\infty,$$
$$\lim_{k\to\infty} \|ki[T_f, T_g] - T^k_{\{f,g\}}\| = O(k^{-1})$$
$$\lim_{k\to\infty} \|T_f^k T_g^k - T^k_{fg}\| = O(k^{-1})$$

We saw that the second of these had to do with relaxing property Q3) of quantization. The last equality has to do with deformation quantization, which we will not discuss. A new proof of this theorem, using Bergman kernel asymptotics, is given in [MM07, Theorem 4.1.1]. The idea is to use an off-diagonal expansion of the density B^k and compute explicitly the first order terms of this expansions.

7.3 Proof of the Kodaira embedding theorem

7.3.1 Bergman kernel asymptotics

Let $(E, H) \to X$ be a holomorphic hermitian vector bundle of rank r over X compact, and assume that X has a positive line bundle $L \to X$, i.e. there is a hermitian metric h on L such that $iF_h = \omega$ is a Kähler form on X. Write $g = \omega(\cdot, J\cdot)$.

Note that H and g induce metrics on the smooth bundles $E_l = (T^*X_0)^{\otimes l} \otimes E$ for all l, and the Chern connection D_H of H and the Levi-Civita connection ∇ of g induce connections D_l on E_l. Define a norm on $C^\infty(X, E_l)$ by

$$\|B\|_{C^l} = \sum_{j=1}^{l} \sup_{x\in X} |D_{j-1}(D_{j-2}(\cdots(D_0 B)\cdots))|,$$

where one the right hand side we use the point-wise norm on E_l induced by H and h.

Remark 7.3.1. Starting with a connection D on some bundle F, we can extend it to $D : \Omega^k(F) \to \Omega^{k+1}(F)$. That is *not* what we do in the above definition of $\|\cdot\|_{C^l}$.

Recall that in a given orthonormal basis $\{s_i\}$ of $H^0(X, E \otimes L^k)$, the density of states of the Bergman kernel $B^k(x) = \sum_{j=1} s_j(x)(\cdot, s_j(x))_{H^k}$, where $H_k = H \otimes h^k$. Recall that $B^k \in C^\infty(\mathrm{End}(E \otimes L^k)) \cong C^\infty(\mathrm{End}(E))$.

Theorem 7.3.2 (Catlin–Zelditch et. al., Theorem 4.1.1 of [MM07])**.** *Let $n = \dim_{\mathbb{C}} X$. There exists a C^∞-expansion of B^k as $k \to \infty$,*

$$(2\pi)^n B^k = B_0 k^n + B_1 k^{n-1} + \cdots ,$$

where $B_j \in C^\infty(\mathrm{End}(E))$, depending on the metrics h and H, and $B_0 = \mathrm{Id}$. More precisely, for any $l, N \geq 0$, there exists a constant $C = C(l, n, h, H)$ such that

$$\|(2\pi)^n B_k - \sum_{j=0}^{N} B_j k^{n-j}\|_{C^l} \leq C(l, N, h, H) k^{n-N-1}$$

Remark 7.3.3. If we have a family of metrics, (h_t, H_t), with bounded derivatives of order less than or equal to $2n + 2k + l + 6$ in the C^l-norm defined as above, and the ω_t are bounded from below, then C is independent of t (in [MM07] it is shown only that C is independent of ω_t).

In the theorem,

$$B_1 = n(F_h \wedge \omega^{n-1} + \frac{\rho_\omega}{2} \wedge \omega^{n-1}\,\mathrm{Id})/\omega^n,$$

where ρ_ω is the Ricci form. This is what related Bergman kernels to the Hermite–Einstein condition.

7.3.2 Outline of the proof

We break the proof of this Kodaira's embedding theorem into 3 steps:

1. There is a holomorphic map $\varphi : X \to G_r(\mathbb{C}^{N_k})$, where $N_k = \dim H^0(X, EL^k)$.
2. φ is a local embedding (i.e. $d\varphi \neq 0$).
3. φ is injective.

From these three points it follows that φ is a holomorphic embedding. When $E = X \times \mathbb{C}$, this implies that if L is positive, then L is ample.

7.3.3 Step 1

Lemma 7.3.4. *If E is holomorphic, L positive and X compact, the bundle $E \otimes L^k$ is generated by global sections for $k \gg 0$.*

Proof. If $E = X \times \mathbb{C}$ and $\{s_j\}$ an orthonormal basis of $H^0(X, L^k)$, then $B^k = \sum_j |s_j|^2$. Applying the Catlin–Zelditch theorem with $l = 0$, $N = 0$, we get

$$\|(2\pi)^n \sum_j |s_j|^2 - k^n\|_{C^0} < C \cdot k^{n-1},$$

and so

$$\sup_{x\in X} \left| \frac{(2\pi)^n}{k^n} B^k - 1 \right| < C' \cdot k^{-1}$$

for every k. Then for $k \gg 0$, we have $\sum |s_j|^2 = B^k > 0$ for all $x \in X$. This implies that for $k \gg 0$ for all $x \in X$ there exists d_j such that $s_j(x) \neq 0$, and we can write $x \mapsto [s_1(x) : \cdots : s_{N_k}(x)]$. In general, $B^k = \sum_j s_j(\cdot, s_j)_{H\otimes h^k}$. By the same argument,

$$\sup_{x\in X} \left| \frac{(2\pi)^n}{k^n} B^k - \mathrm{Id} \right|_{\mathrm{End} E} < C''/k,$$

and taking determinants,

$$\sup_{x\in X}\left|\frac{(2\pi)^n}{k^n}\det B^k - 1\right|_{\operatorname{End}E} < C'''/k,$$

so for $k \gg 0$, B^k is invertible for all $x \in X$. Therefore,

$$s'(x) = B^k(x)(s''(x)) = \sum_j s_j(s'', s_j)_{H\otimes h^k}(x),$$

which implies that E_x is generated by $\{s_j\}$, so E is generated by global sections. □

25th lecture, December 5th 2011

Last time we saw that whenever we had a holomorphic vector bundle $E \to X$ over a compact complex manifold, and a positive line bundle $L \to X$, then for all $k \gg 0$, the bundle $E \otimes L^K$ is generated by global sections. The important thing for us is that this implies that we can construct maps $\varphi_k : X \to G_r(\mathbb{C}^{N_k})$, where $N_k = \dim H^0(X, E \otimes L^k)$, and $r = \operatorname{rk} E$. The second step in the proof of Kodaira's embedding theorem is to see that φ_k is locally injective, and the final step, which is going to be a bit more involved, is to see that φ_k is globally injective.

7.3.4 Step 2

To see the second step, let h be a hermitian metric on L, such that $\omega = iF_h$ is Kähler. Recall that to construct φ_k, we fix a basis $\{s_j^k\}_{j=1}^{N_k}$ of $H^0(X, E \otimes L^k)$. Given $x \in X$, we fix a trivialization around x, $E|_U \cong U \times \mathbb{C}^r$ and put

$$\varphi_k(x) = [(s_1^k(x), \dots, s_{N_k}^k(x))] \in \mathrm{GL}(\mathbb{C}^r)\backslash M_{r\times N_k}(\mathbb{C}),$$

where the $s_i^k(x)$ are column vectors in the trivialization. Recall that without choosing a basis, we can construct $\varphi_k(x) : H^0(X, E\otimes L^k) \twoheadrightarrow E \otimes L^k|_x$ which gives maps $\varphi_k : X \to G_r(H^0(X, E \otimes L^k))$.

We need to compare $\varphi_k^* h_{\mathrm{FS}}$ in $G_r(\mathbb{C}^{N_k})$ with respect to the standard hermitian metric on $\mathbb{C}^{N_k}$ with the metric $H \otimes h^k$ on $E \otimes L^k$, where H is a hermitian metric on E, and h^k a hermitian metric on L^k. To do so, note that $\varphi_k^* U_r \cong E \otimes L^k$. We want to this in order to compare $i \operatorname{tr} F_{\varphi_k^* h_{\mathrm{FS}}} = \varphi_k^* \omega_{\mathrm{FS}}$, where ω_{FS} is a Kähler form on $G_r(\mathbb{C}^{N_k})$ with the Kähler form ω.

Recall that $h_{\mathrm{FS}}([v_1], [v_2])|_{[A]} = \overline{v_2}^T A^*(AA^*)^{-1} A v_1$, where $[A] \in \mathrm{GL}(\mathbb{C}^r)\backslash M_{r\times N_k}(\mathbb{C})$ and $[v_i] \in \mathbb{C}^{N_k}/\ker A$. For $s_1, s_2 \in E \otimes L^k|_x$, we have

$$\varphi_k^* h_{\mathrm{FS}}(s_1, s_2) = h_{\mathrm{FS}}(\underline{\varphi_k} s_1, \underline{\varphi_k} s_2),$$

where $\underline{\varphi_k} : E \otimes L^k|_x \to U_r|_{[\varphi_k(x)]}$ is the natural map given by the isomorphism $E \otimes L^k \cong \varphi_k^* U_r$, covering φ_k. This is given explicitly by $\underline{\varphi_k}(s_1) = [A]^{-1}(s_1) \in U_r|_{[\varphi_k(x)]}$, where $A = \varphi_k(x) : \mathbb{C}^{N_k} \to \mathbb{C}^r$ induces $[A] : \mathbb{C}^{N_k}/\ker A \xrightarrow{\cong} \mathbb{C}^r$ and considering s_1 as $s_1 \in E \otimes L^k|_x \cong \mathbb{C}^r$. Thus,

$$\begin{aligned}\varphi_k^* h_{\mathrm{FS}}(s_1, s_2) &= \overline{([A]^{-1}s_2)}^T A^*(AA^*)^{-1}A([A]^{-1}s_1) = \overline{s_2}^T (A^*)^{-1}A^*(AA^*)^{-1}AA^{-1}s_2 \\ &= \overline{s_2}^T (AA^*)^{-1} s_1.\end{aligned}$$

We want to relate $(AA^*)^{-1}$ with B_k, the density of states of the Bergman kernel. Assume now that $\{s_j^k\}$ is an orthonormal basis with respect to the L^2 metric on $H^0(X, E \otimes L^k)$ coming from H, h, and ω. We find that, locally,

$$B^k(x) = \sum_j s_j^k(\cdot, s_j^k)_{H_k} = AA^* H_k,$$

where $H_k = H \otimes h^k$ is the $r \times r$-matrix of the hermitian metric, and $A = (s_1^k(x), \dots, s_{N_k}^k(x))$. Recall for this that for any hermitian metric H, we use the notation $(v_1, v_2)_H = \overline{v_2}^T H v_1$. Therefore, we have the local expression

$$\varphi_k^* h_{\text{FS}}(s_1, s_2) = \overline{s_2}^T (AA^*)^{-1} s_1 = \overline{s_2}^T H_k (AA^* H_k)^{-1} s_1 = ((B^k)^{-1} s_1, s_2)_{H_k},$$

where the last equality gives a global expression; recall here that $B^k \in C^\infty(\text{End}E \otimes L^k)$. To summarize, assuming that $\{s_j\}$ is L^2-orthonormal, then B^k measures the "difference" between H_k and the pullback metric.

The next step is to compare the pullback of the Fubini–Study metric on $G_r(\mathbb{C}^{N_k})$ with ω on X. Let $\omega_{\text{FS}} \in \Omega^2(G_r(\mathbb{C}^{N_k}))$ be the Fubini–Study metric (or Kähler form) on $G_r(\mathbb{C}^{N_k})$, defined by $\omega_{\text{FS}} = iF_{\det h_{\text{FS}}}$, where $\det h_{\text{FS}}$ is the hermitian metric on $\omega^r U_r \to G_r(\mathbb{C}^{N_k})$ induced by h_{FS} on U_r.

Recall that $\wedge^r U^r \to G_r(\mathbb{C}^{N_k})$ is an ample line bundle over $G_r(\mathbb{C}^{N_k})$ and hence positive, and ω_{FS} is precisely the Kähler form arising from the pull-back by the embedding $G_r(\mathbb{C}^{N_k}) \hookrightarrow \mathbb{P}(\mathbb{C}^M)$. Locally, we have

$$\begin{aligned}
\varphi_k^* \omega_{\text{FS}} &= \varphi_k^* i F_{\det h_{\text{FS}}} = iF_{\varphi_k^* \det h_{\text{FS}}} = iF_{\det \varphi_k^* h_{\text{FS}}} \\
&= i\overline{\partial}\partial \log \det \varphi_k^* h_{\text{FS}} = i\overline{\partial}\partial \log \det H_k (B^k)^{-1} \\
&= i\overline{\partial}\partial \log \det H_k - i\overline{\partial}\partial \log \det B^k = ikF_h + i \operatorname{tr} F_H - i \operatorname{tr}(\overline{\partial}_E (B^k)^{-1} \partial_E (B^k)) \\
&= k\omega + i \operatorname{tr} F_H - i \operatorname{tr} \overline{\partial}_E ((B^k)^{-1} \partial_E (B^k)),
\end{aligned}$$

where $\partial_E \equiv D_H^{1,0} : C^\infty(\text{End}E) \to \Omega^{1,0}(\text{End}E)$, and similarly $\overline{\partial}_E \equiv D_H^{0,1} : \Omega^{1,0}(\text{End}E) \to \Omega^{1,1}(\text{End}E)$. We will now use Bergman kernel asymptotics to see that the φ_k tend to an isometry as k grows.

Now, we compare $\frac{1}{k}\varphi_k^* h_{\text{FS}}$ with $\omega_k' := \omega + \frac{1}{k} i \operatorname{tr} F_H$.

Claim 7.3.5. *The form ω_k' is Kähler for $k \gg 0$ (since X is compact).*

Proof of Claim. For all $k \gg 0$ and all $0 \neq v \in TX_0$, we have to show that $\omega_k'(v, Jv) > 0$. If this is not the case, there exist $\{v_{x_k}\}_{k=1}^\infty$, $p(v_{x_k}) = x_k$ such that $\omega(v_{x_k}, Jv_{x_k}) \leq \frac{i}{k} \operatorname{tr} F_H(v_{x_k}, Jv_{x_k})$, and we can assume that v_{x_k} have norm 1. Since X is compact, there is a limit v_∞ with $\omega(v_\infty, Jv_\infty) \leq 0$ which contradicts that $\|v_\infty\| = 1$. □

Given $l \gg 0$, we compare $\frac{1}{k}\varphi_k^* \omega_{\text{FS}}$ with ω_k' in C^l-norm and find

$$\|\frac{1}{k}\varphi_k^* \omega_{\text{FS}} - \omega_k'\|_{C^l} = \frac{1}{k}\|\operatorname{tr} \overline{\partial}_E ((B^k)^{-1} \partial_E B^k)\|_{C^l}.$$

We claim that this is less than C'''/k^2 for some $C''' = C'''(l, H, h) \in \mathbb{R}$ independent of k. Taking $N = 0$ and $l \geq 0$ in the Catlin–Zelditch theorem, we find that

$$\|\frac{(2\pi)^n}{k^n} B^k - \text{Id}\|_{C^l} \leq C(h, H, l) k^{-1}.$$

We will only check the claim for $l = 0$. Here

$$\begin{aligned}
\|\operatorname{tr} \overline{\partial}_E ((B^k)^{-1} \partial_E B^k)\|_{C^0} &\leq \|\partial_E (B^k)^{-1} \partial_E B^k\|_{C^0} \leq \|(B^k)^{-1} \partial_E B^k\|_{C^1} \\
&\leq \|(B^k)^{-1}\|_{C^1} \|\partial_E B^k\|_{C^1} \leq C'' \|\partial_E (\frac{(2\pi)^n}{k^n} B^k)\| C^1 \\
&= C'' \|\partial_E (\frac{(2\pi)^n}{k^n} B^k - \text{Id})\| C^1 \leq C'' \|\frac{(2\pi)^n}{k^n} B^k - \text{Id}\|_{C^2} \\
&\leq C'' C(H, h, 2) \cdot k^{-1},
\end{aligned}$$

which proves the claim. Here, in the third to last inequality, we use that we know that $\|\frac{(2\pi)^n}{k^n} B^{k-1}\|_{C^0} \to 1$ for $k \to \infty$.

Then as $\frac{1}{k}\varphi_k^*\omega_{\mathrm{FS}} \to \omega$ in $\|\cdot\|_{C^0}$ as $k \to \infty$, we have – locally – that

$$\frac{1}{k}\det(d\varphi_k)\omega_{\mathrm{FS}}^n = \frac{1}{k^n}\left(\varphi_k^*\omega_{\mathrm{FS}}\right)^n \to \omega^n$$

in $\|\cdot\|_{C^0}$ for $k \to \infty$, and since ω is non-degenerate, this implies that $\det(d\varphi_k) \neq 0$ for all $k \gg 0$. Hence $d\varphi_k \neq 0$ for $k \gg 0$, so φ_k is a local embedding for all $k \gg 0$, which completes the second part of the proof of the Kodaira theorem.

7.3.5 Step 3

The tool we need to work out the global injectivity of the φ_k is an asymptotic expansion outside the diagonal. Recall that $B^k = B^k(x,y) \in C^\infty(X \times X, E \otimes (L^k)^* \boxtimes E \otimes L^k)$. So far we have only used $B(x,x) = \sum s_j(x)(\cdot, s_j(x))_{H_k}$. We will not state the full theorem concerning this asymptotic expansion but only give the key steps that make it work.

26th lecture, December 6th 2011

Remark 7.3.6. Before going on with step 3, we begin with some comments about the calculation of $\varphi_k^* h_{\mathrm{FS}}$. We will do this base free; i.e. consider the map $\varphi_k : X \to G_r(H^0(X, EL^k))$ given by

$$\varphi_k(x) \equiv [H^0(X, EL^k) \twoheadrightarrow E|_x : s \mapsto s|x] = [A].$$

Recall then that

$$U_r|_{\varphi_k(x)} = H^0(X, EL^k)/\ker A.$$

We considered the map $\underline{\varphi_k} : E \otimes L^k \cong \varphi_k^* U_r \to U_r$. We never wrote down the isomorphism explicitly. We can do that as $\underline{\varphi_k}(s_k) = f_1 f_2(s|_x)$, where $s|_x \in EL^k|_x, x \in X$, $f_2 : EL^k \to \varphi_k^* U_r : s_x \mapsto (x, [A]^{-1}s|_x)$, and $f_1 : \varphi_k^* U_r \to U_r : (x, v_{\varphi_k(x)}) \mapsto v_{\varphi_k(x)}$. Picking now an orthornormal basis $\{s_j\}$ of $H^0(X, EL^k) \cong \mathbb{C}^{N_k}$. Also $\mathbb{C}^r \cong E|_x$, and we have the formula from the previous lecture,

$$\varphi_k^* h_{\mathrm{FS}}(s_1, s_2) = h_{\mathrm{FS}}(\underline{\varphi_k} s_1, \underline{\varphi_k} s_2) = \overline{[A][A]^{-1}}^T s_2 (AA^*)^{-1}[A][A]^{-1} s_1.$$

We return now to the proof of the Kodaira theorem concerning the global injectivity of φ_k, which is the most involved part, and which we only sketch. The proof relies on the following key fact:

$$\forall (x,y) \in X \times X \setminus \mathrm{Diag},\ \exists k_0 \geq 0 : \forall k \geq k_0\ \varphi_k(x) \neq \varphi_k(y). \tag{5}$$

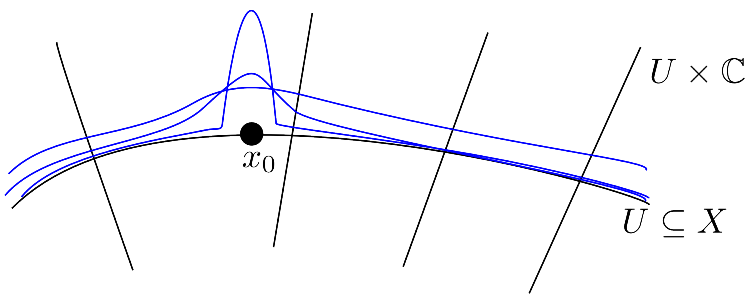

Figure 5: Peak sections of a trivial line bundle over $U \subset X$ localizing around a point $x_0 \in U \subseteq X$ as k grows.

To prove (5), we need so-called peak sections which as $k \to \infty$ peaks at a point $x_0 \in X$ (see Fig. 5). Let $x_0 \in X$ and take $e \in E_{x_0}$, $v \in L|_{x_0}$, both with with unit norm. Consider $e \otimes v^k \in EL^k|_{x_0}$. Consider $s^{x_0,k} \in H^0(X, EL^k)$ defined by

$$s^{x_0,k}(x) = B^k(x, x_0)e \otimes v^k = \sum_j s_j^k(x)(e \otimes v^k, s_j(x_0))_{H^k},$$

given an orthonormal basis $\{s_j\}$. Recall here that $B^k(x,y) \in C^\infty(X \times X, E \otimes (L^k)^* \boxtimes E \otimes L^k)$. Assume that $\{s_j^k\}$ is such that $\{s_1^k, \dots, s_r^k\}$ is a basis of $\ker \varphi_k(x_0)^\perp$ and $\{s_{r+1}, \dots, s_{N_k}\}$ a basis of $\ker \varphi_k(x_0)$. Then

$$s^{x_0,j} = \sum_{j=1}^r s_j(e \otimes v^k, s_j),$$

so this section concentrates some information of sections not vanishing at x_0.

Claim 7.3.7. *Rescaling $s^{x_0,k}$ so they have unit L^2-norm, we have*

$$\lim_{k\to\infty} \int_{B(x_0,p_k)} |s^{x_0,k}(x)|^2_{H_k} \frac{\omega^n}{n!}(x) = 1,$$

where $B(x_0,p_k)$ are geodesic balls centered at x_0 with radius p_k, and $\{p_k\}_{k=1}^\infty$ is a sequence such that $\lim_{k\to\infty} p_k = 0$ and $\lim_{k\to\infty} p_k\sqrt{k} = \infty$.

The idea is the following: For $E = X \times \mathbb{C}$, note that

$$|s^{x_0,k}(x_0)|^2_{H_k} = B^k(x_0,x_0)$$

because the $s^{x_0,k}$ have norm 1 and $s^{x_0,k}(x_0) \neq 0$. The other thing we are going to use is that we can write

$$s^{x_0,k}(x) = \frac{1}{B^k(x_0,x_0)} B^k(x,x_0) s^{x_0,k}(x_0).$$

We find that

$$\begin{aligned} 1 = \int_X |s^{x_0,k}(x)|^2_{H_k} \frac{\omega^n}{n!}(x) &= \int_{B(x,p_k)} |s^{x_0,k}(x)|^2_{H_k} \frac{\omega^n}{n!} + \int_{X\setminus B(x_0,p_k)} |s^{x_0,k}(x)|^2_{H_k} \frac{\omega^n}{n!} \\ &= \int_{B(x_0,p_k)} \frac{|B^k(x,x_0)|^2}{|B(x_0,x_0)|^2} \frac{\omega^n}{n!} + \int_{X\setminus B(x_0,p_k)} \frac{|B^k(x,x_0)|^2}{|B^k(x_0,x_0)|^2} \frac{\omega^n}{n!} \end{aligned}$$

for some $x' \in B(x_0,p_k)$. We are now going to apply the off-diagonal asymptotics of the Bergman kernel.

Theorem 7.3.8. *For every l and every $\varepsilon > 0$, there exists $C_{l,\varepsilon} > 0$ such that for every $k \geq 1$, we have*

$$\|B^k\|_{C^0(\{(x,y)|d(x,y)>\varepsilon\})} \leq C_{\varepsilon,l} k^{-l}.$$

That is, if x, x' are points with $d(x,x') > \varepsilon$, we have

$$|B^k(x,x')|_{H_k} \leq C_{l,\varepsilon} k^{-l}.$$

This proves the claim. To see (5), assume that it does not hold. Then there exist $x \neq y$ and $\{k_n\}_{n=1}^\infty$ with $k_n \to \infty$ as $n \to \infty$ with $\varphi_{k_n}(y) = \varphi_{k_n}(x)$. Take peak sections $\{s^{x,k}\}_{k=1}^\infty$ on x. By construction, $s^{x,k}$ depend only on $\varphi_k(x)$ because $s^{x,k}$ is just a unitary element in $\varphi_{k_n}(x)^\perp = \varphi_{k_n}(y)^\perp$, and for large enough k_n,

$$1 = \|s^{x,k_n}\|_{L^2} \geq \int_{B(x,p_{k_n})} |s^{x,k_n}|^2 \frac{\omega^n}{n!} + \int_{B(y,p_{k_n})} |s^{y,k_n}|^2 \frac{\omega^n}{n!} \to 2,$$

as $n \to \infty$, which is a contradiction.

Take $(x_0, y_0) \in X \times X \setminus \text{Diag}$, and let $k_0 = k_0(x_0, y_0)$ given by (5). For $k \geq k_0$, define

$$A_k = \{(x, y) \in X \times X \mid \varphi_k(x) = \varphi_k(y)\}.$$

Our goal is to see that this set tends to the diagonal for large enough k. By (5), there are points $(x_0, y_0) \notin A_k$. We want to construct an increasing sequence $\{k_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$, $k_n \geq k_0$, such that

1. $\bigcap_{n \geq 1} A_{k_n} = \text{diag}(X \times X)$, and
2. $A_{k_{n+1}} \subseteq A_{k_n}$ for all n.

Given such a sequence, we apply the following Lemma

Lemma 7.3.9 (Noetherian property). *Every descending sequence of analytic sets on a compact complex manifold is stationary.*

Here, an *analytic set* is a set cut out locally by zeros of holomorphic functions. By construction of φ_k, the sets A_k are analytic by construction. Thus from 1. and 2. above it follows that

$$\bigcap_{n \geq 1} A_{k_n} = A_{k_{n_0}} = \text{Diag}(X \times X),$$

and $A_{k_n} = A_{k_{n_0}}$ for all $n \geq n_0$, which implies that the φ_{k_n} are globally injective, completing the proof of the Kodaira theorem.

To see the existence of the sequence, note that by (5), 1. holds independently of the specific $\{k_n\}$: If $(x, y) \notin \text{Diag}$ there exists $k_0' \geq 0$ such that for all $k \geq k_0'$, we have $\varphi_k(x) \neq \varphi_k(y)$, so $(x, y) \notin A_k$.

To construct $\{k_n\}$ satisfying 2., we consider two cases. First, if $E = X \times \mathbb{C}$, we let $k_n = k^n$. We have to prove that $A_{k^n} \subseteq A_{k^{n+1}}$. For $(x, y) \notin A_{k^n}$, $\varphi_{k^n}(x) \neq \varphi_{k^n}(y)$. Then there exist sections $s_1, s_2 \in H^0(X, L^{k^n})$ separating x and y in the sense that $s_1(x) \neq 0$, $s_1(y) = 0$, $s_2(y) \neq 0$, and $s_2(x) = 0$. For this, consider $\varphi_{k^n} : X \to \mathbb{P}(\mathbb{C}^{N_{n^k}})$, and we can find two sections of $\mathcal{O}(1) \to \mathbb{P}(\mathbb{C}^{N_{n^k}})$ seperating x and y in the above sense for $\varphi_{k^n}(x)$ and $\varphi_{k^n}(y)$, and pulling back such sections to $L^{k^n} \to X$, we get sections with the desired property. Then $s_j^k \in H^0(X, L^{k^{n+1}})$ will have the same property, separating x and y, and just by definition of the φ_k, we find that $(x, y) \in A_{k^{n+1}}$.

In the case of arbitrary E, by the previous case there exists $l_0 \gg 0$ such that L^{l_0} is very ample and we define $k_n = l_0 n + k_0$. Again, if $(x, y) \notin A_{l_0 n + k_0}$ there exist $s_1, s_2 \in H^0(X, E \otimes L^{l_0 n + k_0})$ such that $s_1(x) \neq 0$, $s_1(y) = 0$, $s_2(y) \neq 0$, and $s_2(x) = 0$. The argument is the same as before, using the Grassmannian instead of projective space. Since L^{l_0} is very ample, we also have $s_1', s_2' \in H^0(X, E \otimes L^{l_0})$ separating x and y. Then $s_1 \otimes s_1', s_2 \otimes s_2' \in H^0(X, E \otimes L^{l_0(n+1)+k_0})$ separating x and y. So $(x, y) \notin A_{(n+1)l_0 + k_0}$.

7.4 Summary

Summarizing, we have proved the following two theorems.

Theorem 7.4.1. *If $E \to X$ is a holomorphic vector bundle over a compact complex manifold X and $L \to X$ a positive line bundle, then there exists $k \gg 0$ such that*

1. *$E \otimes L^k$ is generated by global sections, and*
2. *$\varphi_k : X \to G_r(H^0(X, E \otimes L^k))$ is an embedding.*

Theorem 7.4.2. *A compact complex manifold is projective if and only if it can be endowed with a positive line bundle. Moreover, a line bundle over X is positive if and only if it is ample.*

By Chow's theorem, we obtain the following corollary, bridging the worlds of complex geometry and algebraic geometry:

Corollary 7.4.3. *A compact complex manifold is algebraic if and only if it can be endowed with a positive line bundle.*

7.5 Differential operators

We turn now to the question whether or not the spaces $H^0(X, E \otimes L^k)$ are actually finite-dimensional, which we used in our above proof of the Kodaira theorem.

27th lecture, December 12th 2011

The goal for the last two lectures will be to prove that if $E \to X$ is a holomorphic vector bundle over a compact complex manifold X, and $L \to X$ a positive line bundle, then $\dim_{\mathbb{C}} H^0(X, E \otimes L^k) < \infty$.

If $E \to M$, $F \to M$ are smooth compelx vector bundles over a smooth manifold M.

Definition 7.5.1. A $\mathbb{C}$-linear map $L : C^\infty(E) \to C^\infty(F)$ is called a *differential operator of order* k if for every local coordinate patch $U \subseteq M$, so that $E|_U \cong U \times \mathbb{C}^r$, $F|_U \cong U \times \mathbb{C}^q$ and $s \in C^\infty(E)$, we have a local expression

$$Ls|_U = ((Ls)_1, \ldots, (Ls)_q),$$

where for $i = 1, \ldots, q$,

$$(Ls)_i = \sum_{j=1}^{r} a_\alpha^{ij} D^\alpha s_j,$$

where $|\alpha| \leq k$, $a_\alpha^{ij} \in C^\infty(U) \otimes \mathbb{C}$, $s = (s_1, \ldots, s_r)$, and $D^\alpha = (-i)^{|\alpha|} D^{\alpha_1} \ldots D^{\alpha_n}$, where $D^j = \frac{\partial}{\partial x_j}$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, $n = \dim M$, and $|\alpha| = \sum \alpha_i$.

Example 7.5.2. If $E = M \times \mathbb{C}$, $F = T^*M \otimes \mathbb{C}$, then $d : C^\infty(M, \mathbb{C}) \to \Omega^1(M) \otimes \mathbb{C}$ mapping $f \mapsto \sum_{j=1}^n \frac{\partial}{\partial x_j}(f) dx_j$ is a differential operator.

Example 7.5.3. If X is a complex manifold, consider $E = X \times \mathbb{C}$ and $F = (T^*X)^{0,1}$ the operator $\overline{\partial} : C^\infty(M, \mathbb{C}) \to \Omega^{0,1}(X)$ mapping $f \mapsto \overline{\partial} f$ is a differential operator. Here, $\ker \overline{\partial}$ consists of the holomorphic functions on X.

Example 7.5.4. The kind of differential operators we will be interested in are the following: If $E \to X$ is a smooth complex vector bundle over a complex manifold, and D a connection on E, consider $\overline{\partial}_D : C^\infty(E) \to \Omega^{0,1}(E)$ mapping $s \mapsto \overline{\partial}_D s = (Ds)^{0,1}$. If $F_D^{0,2} = 0$, then $\overline{\partial}_D$ induces a holomorphic structure on E. We want to compute $\dim \ker(\overline{\partial}_D) = \dim H^0(X; E)$. As an aside, note that in local holomorphic coordinates for the holomorphic bundle $(E, \overline{\partial}_D)$, we have $\overline{\partial}_D = \overline{\partial}$.

We want to compute $L^{-1}(0)$ for a differential operator L. The idea is that given L, we want to associate to L a *symbol* σ, which will be a homomorphism which is invertible, when L satisfies a ellipticity operator. Then from σ^{-1}, we can associate an operator $L(\sigma^{-1})$ which in some sense is “close” to inverting the operator L.

7.5.1 Symbols of differential operators

If M is a differential manifold, write $T'M = T^*M \setminus$ zero section with projection $p : T'M \to m$. Given bundles $E \to M$, $F \to M$, we can consider bundles $p^*E \to T'M$, $p^*F \to T'M$. Given $k \in \mathbb{Z}$, we define the *space of k-symbols*,

$$\mathrm{Smbl}_k(E, F) = \{\sigma \in \mathrm{Hom}(p^*E, p^*F) \mid \sigma(\lambda v) = \lambda^k \sigma(v),\, \lambda \in \mathbb{R}_{>0}, v \in T'M\}.$$

Let $\mathrm{Diff}_k(E, F)$ denote the space of differential operators from E to F of order k. We construct a map

$$\begin{aligned} \mathrm{Diff}_k(E, F) &\to \mathrm{Smbl}_k(E, F) \\ L &\mapsto \sigma_k(L), \end{aligned}$$

where $\sigma_k(L)$ is the *k'th symbol of* L, given by

$$\sigma_k(L)(v,e) = L\left(\frac{i^k}{k!}(g-g(x))^k s\right)|_x$$

for $(v,e) \in p^*E$, $x = p(v)$, and $g \in C^\infty(M)$ satisfies $dg|_x = v \in T_x^*M$, and $s \in C^\infty(M)$ satisfies $s|_x = e$.

Example 7.5.5. Let $E \to X$ be a smooth complex vector bundle over a complex manifold with connection D. We want to compute the symbol of $\overline{\partial}_D$. We find(?)

$$\begin{aligned}\sigma_k(\overline{\partial}_D)(v,e) = \overline{\partial}_D\left(\frac{i^k}{k!}(g-g(x))^k s\right)_x &= \frac{i^k}{(k-1)!}(g-g(x))^{k-1}\overline{\partial}g \otimes s|_x + \frac{i^k}{k!}(g-g(x))^k\overline{\partial}_D s|_x \\ &= \frac{i^k}{(k-1)!}(g-g(x))^{k-1}\overline{\partial}g \otimes s|_x = \begin{cases} iv^{0,1}\otimes e, & k=1, \\ 0, & k>1, \\ \text{ill-defined}, & k<1. \end{cases}\end{aligned}$$

Remark 7.5.6. This example generalizes to an exact sequence

$$0 \to \mathrm{Diff}_{k-1}(E,F) \to \mathrm{Diff}_k(E,F) \overset{\sigma_k}{\to} \mathrm{Smbl}_k(E,F) \to 0.$$

Namely, given an abritrary L, with $k > \mathrm{ord}(L)$, the order of L, then $\sigma_k(L)$ becomes something like $(g-g(x))L(\cdot)|_x$. On the other hand, of $k < \mathrm{ord}L$, then $\sigma_k(L)$ is ill-defined.

We note that, locally, $L \in \mathrm{Diff}_k(E,F)$ can be written

$$L = \sum_{|\alpha|\le k} A_\alpha D^\alpha,$$

where $A^\alpha \in C^\infty(U, M_{r\times q}(\mathbb{C}))$. Then

$$\sigma_k(L)(v) = \sum_{|\alpha|=k} A_\alpha v^\alpha,$$

where $v \in T'M$, $v = (v_1,\dots,v_n)$, $v^\alpha = v_1^{\alpha_1}\cdots v_n^{\alpha_n}$.

Proposition 7.5.7. *If* $L_1 \in \mathrm{Diff}_k(E,F)$, $L_2 \in \mathrm{Diff}_m(F,G)$, *then* $L_2\circ \in \mathrm{Diff}_{k+m}(E,G)$, *and* $\sigma_{k+m}(L_2\circ L_1) = \sigma_m(L_2)\cdot\sigma_k(L_1)$.

7.5.2 Elliptic operators

We want to define a class of "good" operators that have "good" symbols.

Definition 7.5.8. An element $\sigma \in \mathrm{Smbl}_k(E,F)$ is called *elliptic* if for all $v \in T'M$, $\sigma(v) : E|_{p(v)} \to F|_{p(v)}$ is invertible.

Definition 7.5.9. A differential operator $L \in \mathrm{Diff}_k(E,F)$ is called an *elliptic operator* if $\sigma_k(L)$ is elliptic.

The following example shows that we are on the right way.

Example 7.5.10. If $L \in \mathrm{Diff}_0(E,F)$, then $\Sigma_0(L) = p^*L$, and if p^*L is invertible, then L is invertible.

The following example shows that the ellipticity condition fails for the case we are interested in.

Example 7.5.11. If $E \to X$ is a smooth complex vector bundle over a complex manifold, then $\overline{\partial}_D : C^\infty(E) \to \Omega^{0,1}(E)$, then $\sigma_1(\overline{\partial}_D)(v,e) = iv^{0,1}\otimes e$. Now if $\dim_{\mathbb{C}} X > 1$, then $\dim E|_{p(x)} < \dim T^*X^{0,1}\otimes E|_{p(v)}$, so σ_1 is *not* elliptic in this case.

We need something slightly more general, which is the notion of elliptic complexes.

7.5.3 Elliptic complexes

Example 7.5.12. We have the deRham complex

$$0 \to C^\infty(M,\mathbb{C}) \xrightarrow{d} \Omega^1(M)\otimes\mathbb{C} \xrightarrow{d} \Omega^2(M)\otimes\mathbb{C} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M)\otimes\mathbb{C}.$$

This is a complex since $d^2 = 0$. Consider as before $p : T'M \to M$. The above sequence induces a sequence

$$p^*(M\times\mathbb{C}) \xrightarrow{\sigma_1(d)} p^*(T^*M\otimes\mathbb{C}) \xrightarrow{\sigma_1(d)} p^*(\wedge^2 T^*M\otimes\mathbb{C}) \to \cdots .$$

In this case, for $v \in \sigma_1(d)$, $e \in \wedge^q T^*M|_{p(v)}\otimes\mathbb{C}$, we have $\sigma_1(d)(v)e = iv\wedge e$.

Example 7.5.13. Consider again $E \to X$ a smooth complex vector bundle over X complex with connection D. We have complexes

$$\Omega^{p,0}(E) \xrightarrow{\overline{\partial}_D} \Omega^{p,1}(E) \xrightarrow{\overline{\partial}_D} \cdots \xrightarrow{\overline{\partial}_D} \Omega^{p,q}(E) \xrightarrow{\overline{\partial}_D} \cdots ,$$

using the skew-symmetric extension of $\overline{\partial}_D$. We are interested in the case $p = 0$. To have a complex, we need that $\overline{\partial}_D^2 = 0$. This is equivalent to having $F_D^{0,2} = 0$, which we again recall is the condition for d to determine a holomorphic structure on $E \to X$.

Remark 7.5.14. If $F_D^{0,2} = 0$, then E is a complex manifold and it is indeed a holomorphic vector bundle over X

Suppose that $E \to X$ is a holomorphic vector bundle over X a complex manifold. Then without any choice of connection, the local $\overline{\partial}$-operator on holomorphic trivializations determines a complex

$$\cdots \to \Omega^{p,q}(E) \xrightarrow{\overline{\partial}_E} \Omega^{p,q+1}(E) \to \cdots .$$

The associated sequence of symbols is

$$0 \to \cdots \to p^*(\wedge^{p,q}T^*X\otimes E) \xrightarrow{\sigma_1(\overline{\partial}_E)} p^*(\wedge^{p,q+1}T^*X\otimes E) \to \ldots, \tag{6}$$

where, for $v \in T'M$,

$$\sigma_1(\overline{\partial}_E)(v)e = iv^{0,1}\wedge e.$$

Exercise 7.5.15. (6) is an exact sequence of bundles.

Definition 7.5.16. Let $E_0, E_1, \ldots, E_N$ be smooth complex vector bundles over M and $L_0, L_1, \ldots, L_{N-1}$ be differential operators mapping

$$C^\infty(E_0) \xrightarrow{L_0} C^\infty(E_1) \xrightarrow{L_1} C^\infty(E_2) \xrightarrow{L_2} \cdots \xrightarrow{L_{N-1}} C^\infty(E_N).$$

This is called a *complex* if $L_j \circ L_{j-1} = 0$ for all $j = 1, \ldots, N-1$.

Associated with a complex $(E^\bullet, L^\bullet)$, we have a sequence of symbols

$$0 \to p^*E_0 \xrightarrow{\sigma_{k_0}(L_0)} p^*E_1 \to \cdots \xrightarrow{\sigma_{k_{N-1}}(L_{N-1})} p^*E_N \to 0,$$

where $k_i = \mathrm{ord}L_i$. Note that this is always a complex.

Definition 7.5.17. A complex $(E^\bullet, L^\bullet)$ is called *elliptic* if the associated sequence of symbols is exact.

Example 7.5.18. The complexes arising from the operators d and $\overline{\partial}_E$ are both elliptic.

Remark 7.5.19. We will work as in [Wel08] with $k = k_j$ fixed for all j but this is not necessary. A reference for the more general case is Atiyah's collected works.

Given such a complex, we have a cohomology

$$H^q(E^\bullet) = \frac{\operatorname{Ker}(L_q : C^\infty(E_q) \to C^\infty(E_{q+1}))}{\operatorname{Im}(L_{q-1} : C^\infty(E_{q-1}) \to C^\infty(E_q))}.$$

Recall that we are interested in

$$E^\bullet : C^\infty(E) \overset{\overline{\partial}_E}{\to} \Omega^{0,1}(E) \to \cdots \overset{\overline{\partial}(E)}{\to} \Omega^{0,q}(E) \to \cdots,$$

which corresponds to $H^0(X, E)$, but we will obtain much more information than that, namely a family $H^{p,q}(E)$. These are invariants of the holomorphic structure of E.

We want to prove that if X is compact, then $H^q(E^\bullet)$ ar efinite dimensional for arbitrary elliptic complexes. For that we need Sobolev spaces.

7.5.4 Sobolev spaces

The idea is to include the spaces $C^\infty(E_i)$ in Banach spaces W_i and apply the machinery of functional analysis.

28th+29th lecture, December 13th 2011

—The last lectures were on Sobolev spaces. I lazied up a bit and missed this part, but those interested in the topic should see the notes on the course "Introduction to Gauge Theory" which dealt with this topic in detail. In essence, the lectures covered all of [Wel08, Ch. IV]. /Søren

Index

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