Symplectic geometry, non-abelian localization and path integrals

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Disclaimer

These are notes from a course given by Brendan McLellan during the fall of 2011.[1] They have been written and TeX'ed during the lecture and some parts have not been completely proofread, so there's bound to be a number of typos and mistakes that should be attributed to me rather than the lecturer. Also, I've made these notes primarily to be able to look back on what happened with more ease, and to get experience with TeX'ing live. That being said, feel very free to send any comments and or corrections to fuglede@imf.au.dk.

The most recent version of these notes is available at http://home.imf.au.dk/pred.

1st lecture, August 26th 2011

The main goal of the course is to understand [Wit92] and [BW05].

1 Introduction to path integrals

1.1 The double slit experiment

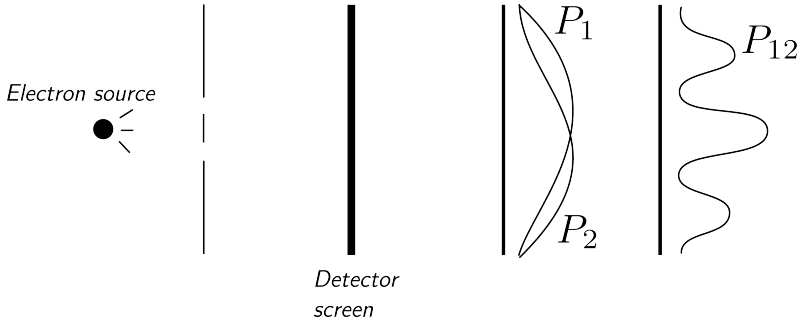

Figure 1: Interference patterns for the double slit experiment.

In the first lecture, we begin with physics motivation. Consider the *double slit experiment* (Fig. 1) where electrons pass through two slits to a detector screen. Closing either slit we obtain two probability distributions P_1 and P_2. Having both slits open looks one obtains a probability distribution P_{12} as in Fig. 1. Keeping both slits open but attaching detectors at the slits, the probability becomes $P_4 = P_1 + P_2$.

From this one concludes that in the first experiments, the electrons behave as particles, whereas in the latter ones, the electron behaves like an electron or a wave, depending on whether it is being observed or not.

Note that in the picture, P_{12} measures the intensity of waves at the detector screen D. We recall some basics of wave interference patterns. Waves can be represented as "phasors", $\varphi_1, \varphi_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$,

$$\varphi_j(x,t) = h_j(x)e^{i(\omega_j t + \delta_j)},$$

[1] The course homepage is located at http://aula.au.dk/courses/QGME11TKG/ – that probably won't be true forever though.

where $h_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2$, and w_j is a constant frequency. Assyme that $w_1 = w_2$. Now $\delta_j(x) : \mathbb{R} \to \mathbb{R}$ is called the phase. Physically, $\varphi_j(x, t)$ is the wave amplitude at time t at position x on D. So, $P_j(x) = |\varphi_j(x, t)|^2 = h_j^2(x)$.

The question is, what is P_{12} in terms of φ_1 and φ_2? It should be

$$P_{12} = |\varphi_1 + \varphi_2|^2 = |\varphi_1|^2 + |\varphi_2|^2 + 2\sqrt{|\varphi_1||\varphi_2|}\cos\delta,$$

where $\delta = \delta_1 - \delta_2$.

In summary, the probability distribution for waves is described by wave amplitudes $\varphi : \mathbb{R}^2 \to \mathbb{C}$, such that the probability amplitude is $P = |\varphi|^2$, and $P_{12} = |\varphi_1 + \varphi_2|^2$. In an experiment like the last one, $P = \varphi_1|^2 + |\varphi_2|^2$. Note also that the experiments are independent of the firing rate of the electron source. So what we can possibly conclude is that electrons interfere with themselves, i.e. they are somehow going through both slits.

1.2 The path integral

Consider the following experiment: If there are N slits, the probability amplitude without slit detectors becomes

$$P_{12\cdots N} = \Big|\sum_{j=1}^{N} \varphi_j\Big|^2.$$

A possible nice conclusion from this is that unobserved electrons take *all* possible paths. Taking the limit $N \to \infty$, corresponding to removing the slit wall entirely, should recover an electron in free space, and one obtains the probability distribution by summing over all paths.

Recall that the classical path can be obtained from the *Euler–Lagrange equation(s)* . I.e., for a path $x(t) : [a, b] \to M$ (where M is a space, manifold, ...), and a Lagrangian $L : TM \to \mathbb{R}$ (where $L = T - V$, T is kinetic energy, and V is the potential), we define the action functional

$$S[x(t)] = \int_a^b L(x, \dot{x})\, dt.$$

Then physical paths are given by stationary points $\frac{\delta S}{\delta x} = 0$. This is called *Hamilton's principle* . Here, by definition, $\frac{\delta S}{\delta x} = 0$, if and only if $\frac{d}{d\varepsilon}S(x + \varepsilon\eta) = 0$ for all smooth paths $\eta : [a, b] \to TM$, with $\eta(a) = \eta(b) = 0$ so that $x(a) = (x + \varepsilon\eta)(a)$, $x(b) = (x + \varepsilon\eta)(b)$.

To see this, we claim that that $\frac{\delta S}{\delta x} = 0$ if and only if

$$\frac{\partial L}{\partial x} - \frac{\partial}{\partial t}\frac{\partial L}{\partial v'} = 0,$$

where (x, v) are local coordinates on TM. To see this, let $\tilde{x} = x + \varepsilon\eta$, so that

$$\begin{aligned}
\frac{d}{d\varepsilon}S(x + \varepsilon\eta) &= \int_a^b L(\tilde{x}, \dot{\tilde{x}})\, dt = \int_a^b \frac{d}{d\varepsilon}L(\tilde{x}, \dot{\tilde{x}})\, dt \\
&= \int_a^b \left(\frac{\partial L}{\partial x}\frac{\partial \tilde{x}}{\partial \varepsilon} + \frac{\partial L}{\partial v}\frac{\partial \dot{\tilde{x}}}{\partial \varepsilon}\right) dt \\
&= \int_a^b \frac{\partial L}{\partial v}\frac{\partial}{\partial t}\frac{\partial}{\partial \varepsilon}\tilde{x}\, dt = \frac{\partial L}{\partial v}\eta\Big|_a^b - \int_a^b \frac{d}{dt}\frac{\partial L}{\partial v}\eta dt,
\end{aligned}$$

so

$$\frac{d}{d\varepsilon}S(x + \varepsilon\eta) = \int_a^b \frac{\partial L}{\partial x} - \frac{\partial}{\partial t}\frac{\partial L}{\partial v}\eta dt = 0.$$

By the "fundamental lemma of calculus of variations ",

$$\frac{\partial L}{\partial x} - \frac{\partial}{\partial t}\frac{\partial L}{\partial v} = 0.$$

Exercise 1.2.1. Make this fundamental lemma rigorous.

We return now to the experiment with the free electron. As noted, the probability distribution is given by the sum over all possible amplitudes of paths. So the question is: What are the amplitudes of these paths? A partial answer is the following: In quantum mechanics, the plane wave solution of the Schrödinger equation is

$$\psi(\overline{x}, t) = A(t)e^{-i(Et-\overline{p}\cdot\overline{x})/\hbar},$$

where $A(t) : \mathbb{R} \to \mathbb{C}$, E is the energy, t is time, $\overline{p} = m\overline{v}$ is momentum, $\overline{x}$ is position, and $\hbar$ is Planck's constant. Let $\delta = -(Et - \overline{p}\cdot\overline{x})/\hbar$ be the phase angle. The "wave/particle" speed is

$$\frac{d\delta}{dt} = -(E - \overline{p}\cdot\overline{v})/\hbar,$$

where $E = T + V$ (and here, $T = \frac{1}{2}m\|\overline{v}\|^2$, and $V = 0$). Also

$$\overline{p}\cdot\overline{v} = m\|\overline{v}\|^2 = 2T,$$

which implies that

$$\frac{d\delta}{dt} = \frac{T-V}{\hbar} = \frac{L}{\hbar},$$

and so

$$\delta(t) = \int_a^t \frac{L}{\hbar}\,dt = \frac{S}{\hbar}.$$

Putting all of this together,

$$\psi(x,t) = A(t)e^{i\int_0^t L/\hbar\,dt} = A(t)e^{iS/\hbar},$$

and the probability amplitude is $P = |\psi(x,t)|^2 = |A(t)|^2$. This will be our motivation for the definition of the path integral.

In Feynman's paper, [Fey48], there are two postulates :

I. If an "ideal" measurement is performed to determine if a "wave/particle" is in a region of space-time R, then the probability of an affirmative answer is absolute square of the sum of complex contributions, one from each path in R.

II. The paths contribute equally in magnitude, and the phase contribution is the classical action

$$S/\hbar = \int L\,dt/\hbar.$$

The two postulates say that the *partition function* , which is what one obtains by summing over the complex amplitudes, is given by

$$Z(R)\text{“}=\text{”}\lim_{\varepsilon\to 0}\text{“}(\prod\int_R)\text{”}\exp\left[\frac{i}{\hbar}\sum_{l=-\infty}^{\infty}S(x_{l+1},x_l)\right]\prod_{l=-\infty}^{\infty}\frac{dx_l}{A}.$$

Here, we picture space-time as being sliced into slices $t_0, t_1, \ldots, x_0, x_1, \ldots$, where x_l is the coordinate on the time slice t_l, and $\varepsilon = t_{l+1} - t_l$, and the action is given by classical paths between x_{l+1}, x_l (see Fig. 2), so $S(x_{l+1}, x_l)$ is to be thought of as the linear path between x_{l+1} and x_l at times t_{l+1} and t_l. Also, A is the "normalization factor", which we will describe in an example later.

We introduce some more notation. Let

$$Z(R) = \fint_{P(R)} \exp\left(\frac{iS}{\hbar}\right)[\mathcal{D}x],$$

where $P(R)$ is the "space of paths on R". There may not be a measure on this space, making sense of $\mathcal{D}x$, and we will occasionally remove in from the integral, instead letting it be part of the notation $\fint$.

Figure 2: Slicing space-time and a particular path therein.

1.3 From the path integral to the Schrödinger equation

Example 1.3.1 (Simple harmonic oscillator, SHO). We will heuristically derive equivalence of the path integral with the Schrödinger equation in the example of the 1-dimensional harmonic oscillator. Consider $Z(R) = Z(R; R'')$, where $R = R' \cup R''$ splitting space-time into past and future at some time t. Let

$$\psi(x_l; t) = \lim_{\varepsilon \to 0} \left(\int_{R'} \right) \exp\left[\frac{i}{\hbar} \sum_{j=1}^{l} S(x_{j+1}; x_j) \right] \frac{dx_{j+1}}{A} \cdots \frac{dx_j}{A} \cdots;$$

$$\psi(x_l; t) = \lim_{\varepsilon \to 0} \left(\int_{R''} \right) \exp\left[\frac{i}{\hbar} \sum_{j=l}^{N} S(x_{j+1}; x_j) \right] \frac{dx_{j+1}}{A} \cdots \frac{dx_j}{A} \cdots;$$

and so by definition,

$$Z(R) = \int \psi(x; t)\, \psi(x; t)\, dx.$$

To get the Schrödinger equation, we need to understand the behaviour of ψ (or ψ). From now on, we ignore ψ and look at ψ instead. We want to understand how ψ develops in time; i.e. we want to understand $\psi(x_{l+1}; t + \varepsilon)$ for ε small. Note that

$$\psi(x_{l+1}; t + \varepsilon) = \int_R \exp\left[\frac{i}{\hbar} S(x_{l+1}; x_l) \right] \psi(x_l; t) \frac{dx_l}{A}.$$

For the SHO (or more generally, other 1-dimensional potentials),

$$S(x_{l+1}; x_l) = \varepsilon \left[\frac{m}{2} \left(\frac{x_{l+1} - x_l}{\varepsilon} \right)^2 - V(x_{l+1}) \right].$$

2nd lecture, August 29th 2011

Last time, we went through path integrals, the double slit experiment, and the many paths interpretation. The classical path is given by the Euler Lagrange equation. The quantum path should have phase $e^{iS/\hbar}$, where $S = \int L\, dt$, $L = T - V$. We discussed the path integral $Z(R)$, and we left off showing that the path integral gives the Schrödinger equation for actions quadratic in $x(t)$ (including the simple harmonic oscillator).

Note also from last time that we discussed $\frac{S}{x} = 0$ if and only if $\frac{d}{d\varepsilon}(x + \varepsilon) = 0$ for all variations. Here, more precisely, for a variation $x : [a; b] \times (-\varepsilon; \varepsilon) \to M$, $\varepsilon > 0$, let $x_t(\varepsilon) = x(t; \varepsilon) : (-\varepsilon; \varepsilon) \to M$, and we put $\xi(t) = \frac{d}{d\varepsilon} x_t(0) : [a; b] \to TM$, and the requirement from last time that $\xi(a) = \xi(b) = 0$ becomes $\xi(t) = x(t; 0)$. Note also that the proof that the Euler Lagrange equations follow from Hamilton's principle that we gave last time assumed that TM was trivial.

Theorem 1.3.6. *If $I_p(\lambda) \equiv \int_{\mathbb{R}} \exp[-\lambda\xi^2/2]p(\xi)d\xi$, where $p \in \mathbb{R}[\xi], \lambda \in \mathbb{C}, \mathrm{Re}(\lambda) > 0$, then*

1. *$I_p(\lambda)$ converges.*
2. *$I_p(\lambda)$ is holomorphic and $\lim_{\lambda\to\pm it} I_p(\lambda)$ exists for $t \in \mathbb{R}^{>0}$.*

We proved convergence last time, left holomorphicity as an exercise, and we will see that the last part follows from analytic continuation. Recall the following theorem from complex analysis.

Theorem 1.3.7. *If $f, g : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ open, are holomorphic (and hence analytic), and $f(z_n) = g(z_n)$ for some sequence $\{z_n\}$ of complex numbers, converging to $z_0 \in D$, then $f = g$ in D.*

Consider the function

$$\tilde{I}_{p_n}(\lambda) \equiv \begin{cases} \left(\frac{2\pi}{\lambda}\right)^{1/2} \frac{(n-1)!!}{\lambda^{n/2}}, & n \geq 2 \text{ even} \\ 0, & n \text{ odd} \\ \left(\frac{2\pi}{\lambda}\right)^{1/2}, & n = 0 \end{cases}$$

for $\lambda \neq 0$. Here, if $\lambda = me^{i\theta}$, $\theta \in [0, 2\pi)$, $\sqrt{\lambda} = \sqrt{me^{i\theta}} \equiv \sqrt{m}e^{i\theta/2}$; i.e. we define the branch cut to be the positive real axis. Then, clearly, $\tilde{I}_{p_n}(\lambda)$ is holomorphic. Also, $\tilde{I}_{p_n}(x) = I_{p_n}(x)$ for $x \in \mathbb{R}^{>0}$, and by Theorem 1.3.7, $I_{p_n}(\lambda) = \tilde{I}_{p_n}(\lambda)$ for all $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda > 0)$. We extend $I_{p_n}(\lambda)$ to $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) \geq 0$, and we use $\lim_{\lambda\to it} I_{p_n}(\lambda)$ to define

$$\int_{\mathbb{R}} \exp[it\xi^2]d\xi := \sqrt{\frac{2\pi i}{t}} = \sqrt{\frac{2\pi}{t}}e^{i\pi/4},$$
$$\int_{\mathbb{R}} \exp[it\xi^2]\xi^2 d\xi := \sqrt{\frac{2\pi i}{t}}\left(\frac{i}{t}\right) = \sqrt{\frac{2\pi}{t}}\left(\frac{i}{t}\right)e^{i\pi/4}.$$

This matches the formulas of [Fey48].

Note that for other powers of ξ,

$$\int_{\mathbb{R}} \exp\left[\frac{im\xi^2}{2\varepsilon\hbar}\right]\xi^{2n}d\xi = O(\varepsilon^{n+1/2}).$$

We are only interested in keeping terms to first order in ε. Expanding the left hand sign of (2) in ε and the right hand side in ξ, using the above calculations, we obtain

$$\psi + \frac{\partial\psi}{\partial t}\varepsilon + \cdots = \exp\left[\frac{i\varepsilon V}{\hbar}\frac{(2\pi\hbar\varepsilon i/m)^{1/2}}{A}[\psi + \frac{\hbar\varepsilon i}{m}\frac{\partial^2\psi}{\partial x^2} + \cdots\right].$$

Equating powers of ε we find the following:

0) The 0th power gives $A = (2\pi\hbar\varepsilon i/m)^{1/2})$

1) The 1st power gives

$$-\frac{\hbar}{i}\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi,$$

which is exactly the Schrödinger equation for a wave/particle in a 1-dimensional potential.

1.4 Solving the Schrödinger equation with a quadratic potential

Example 1.4.1. Let $V(x) \equiv \frac{1}{2}m\omega^2x^2$ for $\omega \in \mathbb{R}^{>0}$ be an angular frequency, and $m \in \mathbb{R}^{>0}$ a mass. For example, this is the potential for the simple harmonic oscillator. We want to solve the

Schrödinger equation for this potential by using separation of variables. I.e., we assume that our solution splits as $\Psi(x,t) = \psi(x)f(t)$. Then we obtain from the Schrödinger equation two equations

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = E\psi, \quad E \in \mathbb{R}^{>0} \tag{3}$$

$$f(t) = e^{iEt/\hbar} \tag{4}$$

Here, the equation (3) is called the time independent Schrödinger equation.

To motivate this a bit more, recall that the Hamiltonian by definition is $H(x,p) = \frac{p^2}{2m} + V(x)$, where $p = mv$. In quantum mechanics it is standard to do canonical quantization. That is, we replace the variable p with the operator $\hat{p} := \frac{\hbar}{i}\frac{\partial}{\partial x}$ on $L^2(\mathbb{R})$. Similarly, x gets replaced by the operator $\hat{x}$ which is just multiplication by x. Plugging these into H, the Hamiltonian becomes

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(\hat{x}).$$

Note that (3) then becomes $\hat{H}\psi = E\psi$, where E gets identified as the energy of the state ψ.

It is a general fact, that *every* solution to the time-dependent Schrödinger equation is of the form

$$\Psi(x,t) = \sum_{n=1}^{\infty} \psi_n(x)e^{-iE_n(t)/\hbar}.$$

We will see this after solving (3).

We can actually solve (3) and letting $\eta = \sqrt{\frac{m\omega}{\hbar}}x$, we find that

$$\psi_l(\eta) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^l l!}} H_l(\eta)e^{-\eta^2/2}, H_l(\eta) \equiv (-1)^l e^{\eta^2}\left(\frac{d}{d\eta}\right)^l e^{-\eta^2}.$$

The H_l are the Hermite polynomials. We give now the derivation. Write $\hat{H}\psi = E\psi$ as

$$\frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 + (m\omega x)^2\right]\psi = E\psi.$$

Define the *ladder operators*

$$a^\dagger \equiv \frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d}{dx} + im\omega x\right),$$
$$a \equiv \frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d}{dx} - im\omega x\right).$$

Here, $a^\dagger$ is called the raising operator, and a the lowering operator. Then $(aa^\dagger - \frac{1}{2}\hbar\omega)\psi = E\psi$, and $[a, a^\dagger] = \hbar\omega$, and so $(a^\dagger a + \frac{1}{2}\hbar\omega)\psi = E\psi$. We claim now that if ψ solves (3) with energy E, then $a^\dagger\psi$ solves the equation with energy $E + \hbar\omega$ and $a\psi$ with $E - \hbar\omega$. This is easy to see: One considers

$$(a^\dagger a + \frac{1}{2}\hbar\omega)a^\dagger\psi = \cdots = (E + \hbar\omega)a^\dagger\psi.$$

Note that we can not have ψ with energy $E < V_{\min}$ since

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar}[V - E]\psi$$

would otherwise imply that $\int_{\mathbb{R}}|\psi|^2\,dx$ is divergent (this is left as an exercise).

We are now in a position to describe the solutions. Let ψ_0 be a state with $a\psi_0 = 0$. This means that

$$\frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d\psi_0}{dx} - im\omega x\psi_0\right) = 0$$

or in other words

$$\frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar}x\psi_0,$$

which we solve to find $\psi_0(x) = A_0 e^{-\frac{m\omega}{2\hbar}x^2}$. Also $(a^\dagger a + \frac{1}{2}\hbar\omega)\psi_0 = E_0\psi$, which implies that the energy is $E_0 = \frac{1}{2}\hbar\omega$. Now, let

$$\psi_n(x) = A_n(a^\dagger)^n e^{-\frac{n\omega}{2\hbar}x^2},$$

that have energies $E_n = (n + \frac{1}{2})\hbar\omega$. The constants A_n can be calculated using $\int_{\mathbb{R}}|\psi|^2\,dx = 1$ to find

$$A_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{(-i)^n}{\sqrt{n!(\hbar\omega)^n}}.$$

2 The principle of stationary phase

2.1 Preliminaries and motivation

Summarizing the introduction, we are interested in computing path integrals

$$Z_R = \fint_{D(R)} e^{iS[\varphi]/\hbar}[\mathcal{D}\varphi]$$

This should correspond to the probability of finding a wave/particle in a region R of spacetime. Formally, as $\hbar \to 0$, Z_R should receive its main contributions from paths that satisfy the Euler–Lagrange equations $\frac{\delta S}{\delta\varphi} = 0$. That is, Z_R should "localize" to classical paths as $\hbar \to 0$. The limit $\hbar \to 0$ is called the *semi-classical limit* .

We now consider finite-dimensional integrals of the same form as the path integral and consider how the semi-classical limit works out in this case. Define

$$I_{\nu,\delta}(\hbar) \equiv \int_M e^{i\delta/\hbar\nu}, \quad \hbar \in \mathbb{R}^{>0}, \nu \in \Gamma_c(M, |\wedge|^1 M),$$

where $\delta : M \to \mathbb{R}$ is some function, and $\Gamma_c(M, |\wedge|^1 M)$ denotes the space of compactly supported densities on M (see below). We want to discuss when such integrals converge, and what their general form will be.

Definition 2.1.1. Given a manifold M (smooth, $\dim M = n$), the *bundle of α-densities on M* , denoted $|\wedge|^\alpha$, for $\alpha \in \mathbb{R}$, is the associated bundle to the principal frame bundle $\mathrm{GL}(M)$ (of the tangent bundle) for the character representation $\rho : \mathrm{GL}_(\, n, \mathbb{F}) \to \mathbb{F}$ (for $\mathbb{F}$ either $\mathbb{R}$ or $\mathbb{C}$) defined by

$$\rho(A) = |\det A|^{-\alpha}.$$

I.e. $|\wedge|^\alpha M = \mathrm{GL}(M) \times_\rho \mathbb{F}$.

An element of the fibre $\psi \in |\wedge|^\alpha_x M$ may be viewed as a map $\psi : \wedge^n T_x M \setminus \{0\} \to \mathbb{R}$ such that $\psi(\lambda x) = |\lambda|^\alpha\psi(x)$ for all $\lambda \in \mathbb{F}^*$.

Example 2.1.2. For $M = \mathbb{R}^n$, the standard α-density is $|dx|^\alpha \in |\wedge|^\alpha\mathbb{R}^n$ defined by

$$|dx|^\alpha(\partial_1 \wedge \cdots \wedge \partial_n) = 1$$

Exercise 2.1.3. Prove that $|\wedge|^\alpha M$ is a trivial line bundle.

Definition 2.1.4. Let $\Gamma_c(M, |\wedge|^\alpha M)$ denote the space of compactly supported sections of $|\wedge|^\alpha M$.

Proposition 2.1.5 (1.23 of Berline–Vergne)**.** *Let M be a manifold of dimension N. There is a unique linear form, called* the integral*, denoted by*

$$\int_M : \Gamma_c(M, |\wedge|^1 M) \to \mathbb{R},$$

such that on a chart $U \cong \mathbb{R}^n$ on M,

$$\int_U f|dx| = \int_{\mathbb{R}^n} f\, dx_1 \cdots dx_n.$$

Definition 2.1.6. Any real bilinear form Q on a real vector space W (with $\dim W = n$) induces a 1-density on W, denoted $\|Q\| : \wedge^n W \to \mathbb{R}$ by

$$\|Q\|(v_1 \wedge \cdots \wedge v_n) \equiv |\det\langle Qv_i, v_j\rangle|^{1/2}.$$

Note that we abuse notation slightly here, and let Q denote the map $W \to W^*$ associated to Q, so $Qv_i \in V^*$.

Definition 2.1.7. For Q a real non-degenerate bilinear form on W, and $\nu \in |\wedge|^1 W$, define $|\det_\nu Q| \in \mathbb{R}^{>0}$ by the relation

$$|\det{}_\nu Q|^{1/2}\nu = \|Q\|.$$

Example 2.1.8. Let $Q \in \mathrm{GL}(n, \mathbb{R})$, and let $e = (e_1, \ldots, e_n)$ be the standard basis for $\mathbb{R}^n$. Then

$$|\det{}_{|dx|} Q|^{1/2}|dx| = \|Q\|.$$

Here, $\|Q\|(e) \equiv |\det\langle Qe_i, e_j\rangle|^{1/2} = |\det Q|^{1/2}$. Also, if $|dx|(e) = 1$, then $|\det_{|dx|} Q|^{1/2} = |\det Q|^{1/2}$.

Before we study the general integrals $I_{\nu,\delta(\hbar)}$, we study the case of $\mathbb{R}^n$, letting

$$I_{u,\delta}(\hbar) = \int_{\mathbb{R}^n} e^{i\delta/\hbar} u|dx|,$$

where $u \in C_c^\infty(\mathbb{R}^n, \mathbb{C})$, $\delta \in C^\infty(\mathbb{R}^n, \mathbb{R})$, and we look at the behaviour of this integral as $\hbar \to 0$. Recall the principle of non-stationary phase in case:

Lemma 2.1.9. *Let $I_{u,\delta}(\hbar)$ be as above and assume that $d\delta \neq 0$ on* $\mathrm{supp}(u)$*, i.e. all points on* $\mathrm{supp}(u)$ *are non-stationary. Then $I_{u,\delta}(\hbar) = O(\hbar^\infty)$, when $\hbar \to 0$, i.e. $|I_{u,\delta}(\hbar)| \leq C_{N,u,\delta}\hbar^N$ when $\hbar > 0$ for some constants $C_{N,u,\delta} \in \mathbb{R}^{\geq 0}$.*

The lesson to learn from this is that the main contribution for the integral should come from stationary points of the function δ.

4th lecture, September 5th 2011

We are in the process of studying integrals of the form

$$I_{\nu,\delta}(\hbar) \equiv \int_M e^{i\delta/\hbar}\nu,$$

where M is a (smooth) manifold, $\nu \in \Gamma_c(|\wedge|^1 M)$, $\delta \in C^\infty(M, \mathbb{R})$, and $\hbar \in \mathbb{R}^{>0}$. In particular, we want to study the limit $\hbar \to 0$. First, we look at the subcase,

$$I_{u,\delta}(\hbar) = \int_{\mathbb{R}^n} e^{i\delta/\hbar} u|dx|,$$

where $u \in C_c^\infty(\mathbb{R}^n, \mathbb{C})$, $\delta \in C^\infty(\mathbb{R}^n, \mathbb{R})$, and $\hbar \in \mathbb{R}^{>0}$, and $|dx|$ denotes the standard density on $\mathbb{R}^n$.

Claim 2.1.10. *The integrals $I_{u,\delta}(\hbar)$ "localize" to* $\mathrm{Crit}(\delta)$ *in a "nice way" as $\hbar \to 0$. If* $\mathrm{Crit}(\delta)$ *are "non-degenerate" points, then δ has a quadratic expression near these critical points.*

Definition 2.1.11. Given a (smooth) function $\delta : M \to \mathbb{R}$, the *Hessian* of δ at a point $p \in \mathrm{Crit}(\delta)$, i.e. $d\delta_p = 0$, denoted $H_p\delta \in \mathrm{Sym}^2 T_p^* M$, is defined by

$$H_p\delta(X, Y) := X(Y(\delta))|_p = Y(X(\delta))|_p$$

for any $X, Y \in \Gamma(TM)$.

Note that $H_p\delta$ only depends on X_p, Y_p (which is an easy exercise). Also, $[X, Y](\delta)|_p = d\delta([X, Y])|_p = 0$ since $p \in \mathrm{Crit}(S)$, so $H_p\delta$ is well-defined.

Definition 2.1.12. A point $p \in \mathrm{Crit}(\delta)$ is called *non-degenerate* , when $H_p\delta$ is non-degenerate. In this case, the Hessian defines an isomorphism $H_p\delta : T_pM \to T_p^*M$.

Definition 2.1.13. Let M, N be smooth manifolds. A function $f : M \to N$ is called an *embedding* (and M is called an *(embedded) submanifold of* N) , if f is smooth, $\mathrm{rank}(f) = m = \dim M$ (i.e. f is an *immersion*), and f is a homeomorphism onto its image $f(M) \subseteq N$ with the subspace topology.

Exercise 2.1.14. Let $f : M \to \mathbb{R}$ be a smooth function, and $df \in \Gamma(M, T^*M)$, then $df : M \to T^*M$ is an embedding.

Taking $f : M \to \mathbb{R}$ to be constant (say $f(M) = 0$), then $df : M \to T^*M$ is just the zero section $z : M \to T^*M$ given by $z(p) = (p, 0)$.

Definition 2.1.15. Let $f : M_1 \to N$, $g : M_2 \to N$ be smooth functions. We say that f *is transverse to g at a point $x \in f(M_1) \cap g(M_2)$* if when $f(p) = g(q) = x$, then $df_p(T_pM_1) + dg_q(T_qM_2) = T_xN$. We write $f \pitchfork_x g$. We say that f *is transverse to g* and write $f \pitchfork g$, if $f \pitchfork_x g$ for all $x \in f(M_1) \cap g(M_2)$.

Note that if $f(M_1) \cap g(M_2) = \emptyset$, then $f \pitchfork g$ vacuously.

Definition 2.1.16. A *Morse function* $f \in C^\infty(M, \mathbb{R})$ is a function such that all critical points are non-degenerate.

Example 2.1.17. Let $f_A : \mathbb{R}^n \to \mathbb{R}$ defined by $f_A(x) = \langle x, Ax \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$, and $A \in \mathrm{GL}(n, \mathbb{R})$ is symmetric. Then f_A is Morse with $\mathrm{Crit}(f_A) = \{0\}$.

Proposition 2.1.18. *Let $z : M \to T^*M$ be the zero section of T^*M. Then $\delta \in C^\infty(M, \mathbb{R})$ is Morse if and only if $z \pitchfork d\delta$.*

Proof. Let $M_\delta \equiv \mathrm{Im}\, d\delta \subset T^*M$ and let $M_0 \equiv \mathrm{Im}\, z \subseteq T^*M$. Then clearly $x \in M_\delta \cap M_0$ if and only if $x = (p, 0)$ for $p \in \mathrm{Crit}(\delta)$. By definition, $z \pitchfork_x d\delta$ if and only if $T_{(p,0)}M_\delta + T_{(p,0)}M_0 = T_{(p,0)}T^*M$. Let $I : T_{(p,0)}T^*M \to T_pM \oplus T_p^*M$ be the natural isomorphism, where locally $T^*M|_U \cong U \times (\mathbb{R}^m)^*$, so

$$T_{(p,0)}T^*M \cong T_pU \oplus T_0(\mathbb{R}^M)^* \cong T_pM \oplus T_p^*M.$$

Use local coordinates to see that under I,

$$T^*\delta M := T_{(p,0)}M_\delta \cong \{v \oplus H_p\delta(v) \in T_pM \oplus T_p^*M \mid v \in T_pM\},$$

where $H_p\delta$ is the Hessian of δ at p. Also, define $T^0M := T_{(p,0)} \cong T_pM \oplus \{0\}$ under I. Thus $z \pitchfork_x d\delta$ if and only if $T^\delta M + T^0M = T_pM \oplus T_p^*M$, which is true if and only if $H_p\delta : T_pM \to T_p^*M$ is surjective. This is equivalent to $H_p\delta$ is an isomorphism, which again is equivalent to p being non-degenerate which finally tells us that $z \pitchfork d\delta$ if and only if δ is Morse. □

Theorem 2.1.19. *Let M_1, M_2, N be smooth manifolds of dimensions m_1, m_2, n respectively, and let $f : M_1 \to N$ be a smooth function and $g : M_n \to N$ an embedding. If $f \pitchfork g$, then $f^{-1}(f(M_1) \cap g(M_2))$ is an embedded submanifold of M_1 of dimension $m_1 + m_2 - n$.*

Proof. Let $x \in f(M_1) \cap g(M_2)$. By the immersion theorem, there is a chart around $x \in U \subseteq N$, $\psi : U \to \mathbb{R}^n$ such that $\psi(g(M_2) \cap U) \cong \mathbb{R}^{m_2} \times \{0\} \subseteq \mathbb{R}^n$. Identifying $d\psi(T_pN) \cong T_0\mathbb{R}^{m_2} \oplus T_0\mathbb{R}^{n-m_2} \cong T_0\mathbb{R}^n$, let $\pi : \mathbb{R}^n \cong \mathbb{R}^{m_2} \times \mathbb{R}^{n-m_2} \to \mathbb{R}^{n-m_2}$ be the projection onto the second factor. Then the condition that $f \pitchfork_x$ implies $d\pi \circ d\psi \circ df(T_pM_1) = T_0\mathbb{R}^{n-m_2}$, i.e. the map is a surjection, and in particular 0 is a regular value of $\pi \circ \psi \circ f$. The regular value theorem implies that $(\pi \circ \psi \circ f)^{-1}(\{0\})$ is an embedded submanifold of M_1 of dimension $m_1 - (n - m_2) = m_1 + m_2 - n$. Since $(\pi \circ \psi \circ f)^{-1}(\{0\})$ is a local description of $f^{-1}(f(M_1) \cap g(M_2))$, we are done. □

Proposition 2.1.20. *If $\delta \in C^\infty(M, \mathbb{R})$ is Morse, then $\mathrm{Crit}(\delta)$ consists of isolated points. In particular, if M is compact, there are finitely many critical points.*

Proof. By the previous theorem, δ is Morse if and only if $z \pitchfork d\delta$. As before, let $M_0 = \operatorname{Im} z \subseteq T*M$, $M_\delta = \operatorname{Im} d\delta \subseteq T^*M$. Then $\dim M_0 = \dim M_\delta = \dim M =: m$. The previous theorem implies that

$$\begin{aligned}\dim[z^{-1}(z(M) \cap d\delta(M))] &= \dim[z^{-1}(M_0 \cap M_\delta)] = \dim M_0 + \dim M_\delta - \dim T^*M \\ &= m + m - 2m = 0.\end{aligned}$$

So $\mathrm{Crit}(\delta)$ is a 0-dimensional embedded submanifold of M. Given $p \in \mathrm{Crit}(\delta)$ there exists a chart $\tilde{U}_p \subset \mathrm{Crit}(\delta)$ such that $\tilde{U}_p \cong \{0\}$. That $\tilde{U}_p$ is open implies that there is an open set $U_p \subseteq M$ such that $\tilde{U}_p = U_p \cap \mathrm{Crit}(\delta)$ by definition of the subspace topology. Then $U_p \cap \mathrm{Crit}(\delta) = \{p\}$ and so p is isolated in M. □

Theorem 2.1.21 (The Morse lemma)**.** *Let $\delta \in C^\infty(U, \mathbb{R})$ be a Morse function on a neighbourhood of $\overline{0} \in W$, where W is some vector space, and let $\overline{0} \in \mathrm{Crit}(\delta)$. Then there exists a neighbourhood V of $\overline{0}$ and a diffeomorphism $\psi : V \to \psi(V) \subseteq U$ with $\psi(\overline{0}) = \overline{0}$ and $\psi^*\delta(\overline{x}) = \frac{1}{2}H_{\overline{0}}\delta(\overline{x}, \overline{x})$. Diagonalizing $\frac{1}{2}H_0\delta(\overline{x}, \overline{x})$ (in the case $\dim W < \infty$), we can write*

$$\psi^*\delta = -x_1^2 - \cdots - x_u^2 + x_{u+1}^2 + \cdots + x_n^2.$$

Note that the following proof, from [Pal69], works for any $\delta : C^{l+2}(U, \mathbb{R})$, where U is a neighbourhood of $\overline{0}$ in a Banach space W yielding a diffeomorphism $\psi \in C^l(V, \tilde{U})$.

The integer u is called the *Morse index* of δ at $\overline{0}$.

Proof of Theorem. Let $Q \equiv \frac{1}{2}H_{\overline{0}}\delta = d^2\delta_{\overline{0}}$, the second derivative of δ at $\overline{0}$, and define

$$\delta^t(\overline{x}) \equiv Q(\overline{x}, \overline{x}) + t(\delta(\overline{x}) - Q(\overline{x}, \overline{x}))$$

for $t \in [0, 1]$. Then $\delta^1 = \delta$, $\delta^0 = Q$, and $\frac{d}{dt}\delta^t = \dot{\delta}^t = \delta - Q$.

The goal is to find a 1-paramter family of diffeomorphisms $\psi^t : V \to \tilde{U}$, where $V, \tilde{U} \subseteq W$ are open such that

$$(\psi^t)^*\delta^t = Q, \quad t \in [0, 1]. \tag{5}$$

Then ψ^1 satisfies $(\psi^1)^*\delta = Q$ and will be the ψ in the statement of the Theorem. Let X^t be the vector field generated by the flow of (so far non-existing) ψ^t so that $\frac{d}{dt}\psi^t(\overline{x}) = X^t(\psi^t(\overline{x}))$. Differentiating (5) we see that

$$(\psi^t)^*\left(\frac{d\delta^t}{dt} + X^t\delta^t\right) = 0.$$

In particular, if we can find $X^t \in \Gamma(M)$, $t \in [0, 1]$ such that $\frac{d\delta^t}{dt} + X^t\delta^t = 0$, then we can integrate X^t (using the Picard–Llindelöf theorem from ordinary differential equations) to find ψ^t in (5). So we want X^t such that $\delta - Q + X^t\delta^t = 0$ or, rewriting this, X^t such that

$$d\delta^t(X^t) = Q - \delta.$$

For any $\overline{x}$ near $\overline{0}$ and $\overline{w} \in W$ we have

$$d\delta^t_{\overline{x}} = \int_0^1 \frac{d}{ds}(d\delta^t_{s\cdot\overline{x}}(\overline{w}))ds,$$

since $d\delta^t_{\overline{0}} = 0$. This follows from integration by parts. Thus

$$d\delta^t_{\overline{x}} = \int_0^1 \frac{d}{ds}(d\delta^t_{s\cdot\overline{x}}(\overline{w}))ds = \int_0^1 d^2\delta^t_{s\cdot\overline{x}}(\overline{x},\overline{w})ds.$$

Therefore, defining the quadratic form $B^t_{\overline{x}}(\overline{v},\overline{w}) := \int_0^1 d^2\delta^t_{s\cdot\overline{x}}(\overline{v},\overline{w})ds$, then since $\delta^t = Q + t(\delta - Q)$, we obtain

$$B^t_{\overline{x}} = B^0_{\overline{x}} 0 t(B^1_{\overline{x}} - B^0_{\overline{x}}),$$

where $B^0_{\overline{x}} = 2Q$ is independent of $\overline{x}$. Thus, at $\overline{x} = \overline{0}$, since $B^t_{\overline{0}} = 2Q$ is non-degenerate for all $t \in [0,1]$, we have $B^t_{\overline{x}}$ is non-degenerate for all $\overline{x}$ in a small enough neighbourhood of $\overline{0}$ for all $t \in [0,1]$. Write $B^t_{\overline{x}}(\overline{x}, X^t) = d\delta^t_{\overline{x}}(X^t) = Q - \delta$. To obtain X^t satisfying this, observe that $(\delta^0 - \delta^1)(\overline{0}) = 0$, and $(d\delta^0 - d\delta^1)(\overline{0}) = 0$. So if $g = \delta^0 - \delta^q$, we have

$$g = \int_0^1 \frac{d}{ds} g(s\overline{x})ds = \int_0^1 dg_{s\overline{x}}(\overline{x})ds = \int_0^1 \int_0^1 d^2 g_{rs\overline{x}}(s\overline{x},\overline{x})\, dr\, ds.$$

We use this to define a quadratic form $C_{\overline{x}}$ by

$$C_{\overline{x}}(\overline{u},\overline{v}) = \int_0^1 \int_0^1 d^2 g_{rs\overline{x}}(s\overline{u},\overline{v})\, dr\, ds$$

satisfying $g(\overline{x}) = C_{\overline{x}}(\overline{x},\overline{x})$. Since $B^t_{\overline{x}}$ is non-degenerate, we can solve for X^t to be unique solution to the equation

$$B^t_{\overline{x}}(\overline{u}, X^t) = C_{\overline{x}}(\overline{u},\overline{x})$$

for all $\overline{u} \in \tilde{V}$ some open set. It is easy to see that X^t is smooth, and $X^t(\overline{0}) = \overline{0})$ by definition, and the Picard–Lindelöf theorem there is a small enough open set $V \subseteq \tilde{V} \subseteq U$ and a solution ψ^t to $\frac{d}{dt}\psi^t = X^t(\psi^t)$ for all $t \in [0,1]$. □

5th lecture, September 9th 2011

Last time we saw the Morse lemma for subsets of Euclidean space, and we begin today's lecture by extending this result to manifolds.

Theorem 2.1.22 (Morse Lemma; Global version)**.** *Let $\delta \in C^\infty(M;\mathbb{R})$ be a Morse function for M a smooth manifold, and let $p \in \mathrm{Crit}(\delta)$. Then there exists a non-degenerate quadratic form $Q \in \mathrm{Sym}^2(\mathbb{R}^m)^*$, and an embedding $\psi : U \subseteq \mathbb{R}^m \to M$, where $\overline{0} \in U$, and $\psi(\overline{0}) = p$ such that*

$$\psi^*\delta(\overline{x}) = \delta(p) + Q(\overline{x}).$$

Proof. Let $\varphi : U_0 \to \mathbb{R}^m$ be a chart around p, $\varphi(p) = \overline{0}$ and let $\tilde{U}_0 := \varphi(U_0) \subseteq \mathbb{R}^m$. Let $\tilde{\delta} : \tilde{U}_0 \to \mathbb{R}$ be a map satisfying $\tilde{\delta}(\overline{x}) = (\varphi^{-1})^*\delta(\overline{x}) - \delta(p)$. Then $\tilde{\delta}(\overline{0}) = 0$ and $\overline{0} \in \mathrm{Crit}(\tilde{\delta})$. Then we can apply the (local) Morse lemma which tells us that there exists a map $\tilde{\psi} : U \to \psi(U) \subseteq \tilde{U}_0$ such that $\tilde{\psi}(\overline{0}) = \overline{0}$ and $Q \equiv \frac{1}{2}H_{\overline{0}}\tilde{\delta}$ satisfies $\tilde{\psi}^*\tilde{\delta} = Q$. Let $\psi := \varphi^{-1} \circ \tilde{\psi} : U \to M$. Then $\psi(\overline{0}) = p$ and

$$\begin{aligned}\psi^*\delta(\overline{x}) &= (\varphi^{-1})^*\tilde{\delta}(\tilde{\psi}(\overline{x})) = \delta(p) + \tilde{\delta}(\tilde{\psi}(\overline{x}))\\ &= \delta(p)\tilde{\psi}^*\tilde{\delta}(\overline{x}) = \delta(p) + Q(\overline{x}).\end{aligned}$$

□

Recall that we are studying $I_{\nu,\delta}(\hbar) \equiv \int_M e^{i\delta/\hbar}\nu$ for $\delta \in C^\infty(M,\mathbb{R})$ a Morse function, $\hbar \in \mathbb{R}^{>0}$ and $\nu \in \Gamma_c(|\wedge|^1 M)$ for M smooth. Since $\nu \in \Gamma_c(|\wedge|^1 M)$ has compact support, we have $|\operatorname{Crit}\delta \cap \operatorname{supp}\nu| < \infty$. Let $\tilde{\theta}$ denote the indexing for $\operatorname{Crit}\delta \cap \operatorname{supp}\nu$, and let $p_l \in \operatorname{Crit}\delta \cap \operatorname{supp}\nu$, $l \in \tilde{\theta}$, and let $L := |\tilde{\theta}|$. Let $p_l \in \tilde{U}_l$, let $l \in \tilde{\theta}$ be open sets in M given as $\psi_l(U_l) =: \tilde{U}_l$ from the Morse lemma for maps $\psi_l : (U_l \subset \mathbb{R}^m) \to M$. Let $U = \{U_\alpha, \theta_\alpha\}_{\alpha\in\theta}$ containing $\{\tilde{U}_l, \psi_l^{-1}\}_{l\in\tilde{\theta}\subset\theta}$, such that for any $U_\alpha \in U$ where $\alpha \notin \tilde{\theta}$, we have $p_l \notin U_\alpha$ for any $l \in \tilde{\theta}$ (which can be constructed taking closed $K_l \subseteq \tilde{U}_l$ and $M \setminus (\bigcup K_l)$ open). Let $\varphi = \{\varphi_\alpha\}_{\alpha\in\theta}$ be a partition of unity subordinate to U. Then

$$\begin{aligned} I_{\nu,\delta}(\hbar) &= \int_M e^{i\delta/\hbar}\nu = \sum_{\alpha\in\theta}\int_M e^{i\delta/\hbar}\varphi_\alpha\nu \\ &= \sum_{l\in\tilde{\theta}}\int_M e^{i\delta/\hbar}\varphi_l\nu = \sum_{\alpha\in\theta\setminus\tilde{\theta}}\int_M e^{i\delta/\hbar}\varphi_\alpha\nu. \end{aligned}$$

Here the last term is $O(\hbar^\infty)$ by the principle of non-stationary phase, so we are really interested in expressions like $\int_M e^{i\delta/\hbar}\varphi_l\nu$.

On $\tilde{U}_l = \psi_l(U_l)$ write $\psi_l^*\nu = \tilde{u}|dx|$ for some $\tilde{u} : (U_l \subseteq \mathbb{R}^m) \to \mathbb{C}$. Let $\hat{\delta}(\overline{x}) \equiv \psi_l^*\delta(x) = \delta(p_l) + Q(\overline{x})$ (by the Morse lemma), and we define $u(\overline{x}) \equiv (\varphi_\alpha \circ \psi_l)(\overline{x}) \cdot \tilde{u}(\overline{x})$. Then

$$\int_M e^{i\delta/\hbar}\varphi_l\nu = \int_{U\subseteq\mathbb{R}^m} e^{i\hat{\delta}/\hbar}u|dx| = e^{i\delta(p)/\hbar}\int_{\mathbb{R}^m} e^{iQ/\hbar}u|dx|,$$

where the last equality follows from the fact that $\operatorname{supp} u \subset U_l$. Thus, we end up studying

$$I_{Q,u}(\hbar) \equiv \int_{\mathbb{R}^m} e^{iQ/\hbar}u|dx|,$$

for $Q \in \operatorname{Sym}^2(\mathbb{R}^m)^*$ some non-degenerate form, $u \in C^\infty(\mathbb{R}^m,\mathbb{C})$.

First, we need to understand

$$I_Q(\hbar) \equiv \int_{\mathbb{R}^m} e^{iQ/\hbar}|dx|. \tag{6}$$

Note that we need to be somewhat careful here, since this integral is analagous to the integrals $\int_{\mathbb{R}} e^{it\xi^2}\,d\xi \equiv \sqrt{\frac{2\pi}{t}}e^{i\pi/4}$.

We will do something slightly more general and consider

$$I_A(\hbar) \equiv \int_{\mathbb{R}^m} e^{-\langle Ax,x\rangle/\hbar}|dx|,$$

where $A \in \operatorname{Sym}^2(\mathbb{R}^m)^* \otimes_{\mathbb{R}} \mathbb{C}$ is non-degenerate, and $\langle(\operatorname{Re}A)x, x\rangle \geq 0$ for $x \in \mathbb{R}^m$. Note that $\operatorname{Sym}^2(\mathbb{R}^m)^* \otimes_{\mathbb{R}} \mathbb{C} \cong \operatorname{Sym}(m,\mathbb{C})$, where $\operatorname{Sym}(m,\mathbb{F}) \equiv \{A \in M(m,\mathbb{F}) \mid A = A^T\}$. Note also, that $\operatorname{Re}A$ makes sense, since we can always write $A = \operatorname{Re}A + i\operatorname{Im}A$. Note that $A \in \operatorname{Sym}(m,\mathbb{C})$ is not necessarily diagonalizable. For instance

$$A = \begin{pmatrix} 2i & 1 \\ 1 & 0 \end{pmatrix}$$

is not. and we obtain the special case (6) by letting $A = iQ$.

If $A \in \operatorname{Sym}(m,\mathbb{R})$ is positive-definite then A is diagonalizable by a orthogonal matrix Θ, i.e. $A = \Theta^T D\Theta$, with $D = \operatorname{diag}\{\lambda_1,\ldots,\lambda_m\}$ with $\lambda_j > 0$ for $1 \leq j \leq m$. Let $y = \theta x$. Then

$$\begin{aligned} I_A(\hbar) &= \prod_{j=1}^m \left(\int_{\mathbb{R}} e^{-\lambda_j y_j^2/\hbar}dy_j\right) \\ &= \prod_{j=1}^m \left(\frac{\hbar\pi}{\lambda_j}\right)^{1/2} = (\hbar\pi)^{m/2}(\det A)^{-1/2}. \end{aligned}$$

We conclude that for $A \in \mathrm{Sym}(m, \mathbb{R})$ we have $I_A(\hbar) = (\hbar\pi)^{m/2}(\det A)^{-1/2}$. This is the basic result we will use to extend this to complex matrices. Let

$$H = \{A \in \mathrm{Sym}(m, \mathbb{C}) \mid \langle \mathrm{Re}\, Ax, x\rangle > 0 \ \forall x \in \mathbb{R}^m\}.$$

For $A \in H$ we have $\det A \neq 0$, sinec if $Ax = 0$ for some $x \in \mathbb{C}^m$, then

$$0 = \mathrm{Re}\langle Ax, \overline{x}\rangle = \langle \mathrm{Re}\, Ax, \overline{x}\rangle$$

which implies that $x = 0$. The second equality follows from the fact that $A^T = A$.

Claim 2.1.23. *There is a unique analytic branch of $F : H \to \mathbb{C}$ defined by $F(A) = (\det A)^{1/2}$ such that $(\det A)^{1/2} > 0$ for $A \in \mathrm{Re}\, H$ (by which we mean that A is real-valued).*

We will use this claim without proof; it can be proved using that H is convex in the space of complex matrices.

Since $I_A(\hbar) = \int_{\mathbb{R}^m} e^{-\langle Ax,x\rangle/\hbar}|dx|$ for $A \in H$, since this integral agrees with

$$\tilde{I}_A(\hbar) = (\hbar\pi)^{m/2}(\det A)^{-1/2}$$

for $A \in \mathrm{Re}(\hbar)$, and since they are both analytic on H, then $I_A(\hbar) = \tilde{I}_A(\hbar)$ for all $A \in H$. We define $I_A(\hbar)$ for $A \in \overline{H}$, the closure of H, by continuity of $I_A(\hbar)|_H$.

We focus now on $A = iB$, for $B \in \mathrm{Sym}(m, \mathbb{R})$ non-singular, so $A \in \overline{H}$. We can assume that B is diagonal, since transformations $A \mapsto \Theta^T A\Theta$ leave $(\det A)^{1/2}$ unchanged for $A \in \mathrm{Sym}(m, \mathbb{R})$. By analytic continuation and continuity, we can assume that the same holds for B. Write $B = \mathrm{diag}\{b_1, \ldots, b_m\}$ for $b_l \neq 0$, $1 \leq l \leq m$. Then

$$\det(iB + \varepsilon\mathbf{1}) = \prod_{l=1}^{m}(\varepsilon + ib_l),$$

and, using the principal branch cut of the square root, we find

$$\mathrm{Arg}\,\sqrt{\det(iB + \varepsilon\mathbf{1})} = \frac{1}{2}\sum_{l=1}^{m}\mathrm{Arg}(\varepsilon + ib_l).$$

When $\varepsilon \to 0$,

$$\mathrm{Arg}\,\sqrt{\det(iB + \varepsilon\mathbf{1})} \to \frac{1}{2}\sum_{l=1}^{m}\frac{\pi}{2}\,\mathrm{sgn}(b_l),$$

where $\mathrm{sgn}\, b_l \equiv b_l/|b_l|$.

Definition 2.1.24. The *signature* of $B \in \mathrm{Sym}(m, \mathbb{R})$ is

$$\mathrm{sgn}(B) := \sum_{l=1}^{m}\mathrm{sgn}(b_l).$$

In conclusion, by the continuity definition, we obtain

$$(\det iB)^{1/2} = |\det B|^{1/2}\exp(\pi i\,\mathrm{sgn}(B)/4).$$

2.2 Basics of Fourier transforms

We recall some basics on Fourier transforms, that will be useful.

Definition 2.2.1. For $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_m) \in (\mathbb{N}_0)^m$, $x \in \mathbb{R}^m$, define

$$x^\alpha \equiv x_1^{\alpha_1} \cdots x_m^{\alpha_m}, \ \partial^\beta \equiv \left(\frac{\partial}{\partial x_1}\right)_1^\beta \cdots \left(\frac{\partial}{\partial x_m}\right)^{\beta_m}.$$

For $\varphi \in C^\infty(\mathbb{R}^m, \mathbb{C})$, define a semi-norm by

$$\|\varphi\|_{\alpha,\beta} \equiv \sup_{x \in \mathbb{R}^m} |x^\alpha \partial^\beta \varphi(x)|.$$

Definition 2.2.2. The *Schwartz space* (of rapidly decreasing functions) on $V = \mathbb{R}^m$ or $V = (\mathbb{R}^m)^*$, is the function space

$$\mathcal{S}(V) = \mathcal{S} \equiv \{\varphi \in C^\infty(V, \mathbb{C}) \mid \|\varphi\|_{\alpha,\beta} < \infty \, \forall \alpha, \beta \in (\mathbb{N}_0)^m\}.$$

Example 2.2.3. Any smooth function $\varphi \in C_c^\infty(\mathbb{R}^m, \mathbb{C})$ with compact support is in $\mathcal{S}$.

Any function of the form $\varphi(x) \equiv p(x)e^{-|x|^2}$, for $p \in \mathbb{R}[x]$, is in $\mathcal{S}$.

Definition 2.2.4. Given $u \in \mathcal{S}(\mathbb{R}^m)$, $\hbar > 0$, then the *asymptotic Fourier transform* (*AFT*) of u is the function $\mathcal{F}_\hbar u : (\mathbb{R}^m)^* \to \mathbb{C}$ defined by

$$(\mathcal{F}_\hbar u)(\xi) \equiv (2\pi\hbar)^{-m/2} \int_{\mathbb{R}^m} e^{-i\langle x,\xi\rangle/\hbar} u(x)\, dx,$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing on $\mathbb{R}^m \times (\mathbb{R}^m)^*$. If $w \in \mathcal{S}((\mathbb{R}^m)^*)$, then the *inverse AFT* of w is the map $\mathcal{F}_\hbar^{-1} w : \mathbb{R}^m \to \mathbb{C}$ defined by

$$(\mathcal{F}_\hbar^{-1} w)(x) \equiv (2\pi\hbar)^{-m/2} \int_{(\mathbb{R}^m)^*} e^{i\langle x,\xi\rangle/\hbar} w(\xi)\, d\xi.$$

Note that the topology on $\mathcal{S}$ defined by the semi-norms (i.e. the coarsest topology such that the semi-norms are continuous) make $\mathcal{S}$ a "Fréchet space". We leave the following result as an exercise.

Theorem 2.2.5 ([Hör83], 7.1.3)**.** *The asymptotic Fourier transform maps $\mathcal{S}(\mathbb{R}^m) \to \mathcal{S}((\mathbb{R}^m)^*)$ and does so continuously, and if $D_l \equiv -i\hbar\partial_l$, then*

$$\mathcal{F}_\hbar(D_l u) = \xi_l \mathcal{F}_\hbar u, \ \mathcal{F}_\hbar(x_l u) = -D_l \mathcal{F}_\hbar u.$$

Lemma 2.2.6 ([Hör83], 7.1.5)**.** *The asymptotic Fourier transform $\mathcal{F}_\hbar : \mathcal{S}(\mathbb{R}^m) \to \mathcal{S}((\mathbb{R}^m)^*)$ is an isomorphism with inverse $\mathcal{F}_\hbar^{-1}$ defined above. I.e. $\mathcal{F}_\hbar \circ \mathcal{F}_\hbar^{-1} = \mathbf{1}_{\mathcal{S}((\mathbb{R}^m)^*)}$, $\mathcal{F}_\hbar^{-1} \circ \mathcal{F}_\hbar = \mathbf{1}_{\mathcal{S}(\mathbb{R}^m)}$.*

Theorem 2.2.7 ([Hör83], 7.1.6)**.** *If $u, v \in \mathcal{S}(\mathbb{R}^m)$, then we have the* asymptotic Parseval formula *(*APF *)*

$$\int_{\mathbb{R}^m} u\overline{v}\, dx = \int_{(\mathbb{R}^m)^*} \mathcal{F}_\hbar u \cdot \overline{\mathcal{F}_\hbar v}\, d\xi,$$

and if $\psi \in \mathcal{S}((\mathbb{R}^m)^*)$, $\varphi \in \mathcal{S}(\mathbb{R}^m)$, then

$$\int_{\mathbb{R}^m} (\mathcal{F}_\hbar^{-1}\psi)(\varphi)\, dx = \int_{(\mathbb{R}^m)^*} \psi(\mathcal{F}_\hbar \varphi)\, d\xi.$$

Definition 2.2.8. A continuous linear form $\tau : \mathcal{S}(V) \to \mathbb{C}$ on $\mathcal{S}(V)$, ($V = \mathbb{R}^m$ or $V = (\mathbb{R}^m)^*$) is called a *tempered distribution* . The set of all tempered distributions is denoted by $\mathcal{S}'(V)$.

Note that given $f \in C^0(V, \mathbb{C})$, we identify f with the tempered distribution $\tau_f \in \mathcal{S}'$ defined by

$$\tau_f(\varphi) := \int_V f\varphi\, dv,$$

where dv denotes the appropriate measure on V. The map τ is injective (we will se why this should be the case later).

Definition 2.2.9. Given $\tau \in \mathcal{S}'$ the *asymptotic Fourier transform* is defined by

$$(\mathcal{F}_\hbar \tau)(\varphi) \equiv \tau(\mathcal{F}_\hbar^{-1}\varphi)$$

for $\tau \in \mathcal{S}'(\mathbb{R}^m)$, $\varphi \in \mathcal{S}((\mathbb{R}^m)^*)$, and we define the *inverse asymptotic Fourier transform* is

$$(\mathcal{F}_\hbar^{-1} \tau)(\varphi) \equiv \tau(\mathcal{F}_\hbar \varphi)$$

for $\tau \in \mathcal{S}'((\mathbb{R}^m)^*)$, $\varphi \in \mathcal{S}(\mathbb{R}^m)$.

Theorem 2.2.10 ([Hör83], 7.1.10)**.** *The asymptotic Fourier transform of tempered distributions $\mathcal{F}_\hbar$ is an isomorphism $\mathcal{S}'(\mathbb{R}^m) \to \mathcal{S}'((\mathbb{R}^m)^*)$ with inverse $\mathcal{F}_\hbar^{-1} : \mathcal{S}'((\mathbb{R}^m)^*) \to \mathcal{S}'(\mathbb{R}^m)$.*

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Definition 2.2.11. Let $\tau \in \mathcal{S}'(V)$, $D_l \equiv i\overline{\partial}_l$, $f \in C^\infty(V, \mathbb{C})$, where $f \cdot \varphi \in \mathcal{S}$ for all $\varphi \in \mathcal{S}$. Define the following:

1. The derivative of a tempered distribution is $D_l\tau(\varphi) \equiv -\tau(D_l\varphi)$.
2. Define a new tempered distribution $(f \cdot \tau)(\varphi) \equiv \tau(f \cdot \varphi)$.

The following is left as an exercise.

Proposition 2.2.12. *Let $\tau \in \mathcal{S}'$. Then*

1. $\mathcal{F}_\hbar(D_l\tau) = \xi_l(\mathcal{F}_\hbar\tau)$,
2. $\mathcal{F}_\hbar(x_l\tau) = -D_l(\mathcal{F}_\hbar\tau)$.

2.3 The principle of stationary phase

Theorem 2.3.1. *Let W be a vector space, whose topology is induced by a family of semi-norms $\{\rho_\alpha\}_{\alpha\in J}$ (i.e. the coarsest topology such that the semi-norms are continuous), and let Y be a normed linear space. If $L : W \to Y$ is linear, then L is continuous if and only if there is an $N \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_N \in J$, and $K \in \mathbb{R}^{>0}$ such that*

$$\|Lw\| \leq K \cdot \sum_{l=1}^{N} \rho_{\gamma_l}(w)$$

for all $w \in W$.

The proof is left as an exercise. The main point of stating this theorem is the following corollary of it.

Corollary 2.3.2. *Let $f_A(x) \equiv \exp(-\langle Ax, x\rangle/\hbar)$, and let $\tau_A := \tau_{f_A} : \mathcal{S} \to \mathbb{C}$ be the linear functional defined by*

$$\tau_A(\varphi) \equiv \int_{\mathbb{R}^m} f_A(x)\varphi(x)\, dx.$$

Then $\tau_A \in \mathcal{S}'$ if and only if $\langle(\operatorname{Re} A)x, x\rangle \geq 0$ for all $x \in \mathbb{R}^m$.

Proof. We prove one half of this statement and leave the other as an exercise. By Theorem 2.3.1, since $\tau_A : \mathcal{S} \to \mathbb{C}$ is linear by definition, τ_A is continuous if and only if

$$|\tau_A(\varphi)| \left| \int_{\mathbb{R}^m} f_A(x)\varphi(x)\, dx \right| \leq K \sum_{l=1}^{N} \|\varphi\|_{\alpha_l,\beta_l},$$

for some $K \in \mathbb{R}^{>0}$ and $N \in \mathbb{N}$. We will show that this condition implies that $\langle (\operatorname{Re} A)x, x\rangle \geq 0$ for all x. Assume that $\langle (\operatorname{Re} A)x, x\rangle < 0$ for some x and let us show that $|\tau_A(\varphi)|$ is unbounded in φ. Let $\varphi(x) \equiv \exp(-\langle B_\varepsilon x, x\rangle/\hbar)$, where $B_\varepsilon \equiv \varepsilon \mathbf{1} - i \operatorname{Im} A$, and $0 < \varepsilon < -\lambda_0$, where $\lambda_0 = \max(\operatorname{spec}(\operatorname{Re} A))$ is negative, since $\langle (\operatorname{Re} A)x, x\rangle < 0$. Then $\|\varphi\|_{\alpha,\beta} < \infty$ for all $\alpha, \beta \in (\mathbb{N}_0)^m$, since by construction $\langle (\operatorname{Re} B_\varepsilon)x, x\rangle > 0$ for all $x \in \mathbb{R}^m$. We have

$$\tau_A \varphi = \int_{\mathbb{R}^m} f_A(x)\varphi x \, dx = \int_{\mathbb{R}^m} \exp(-\langle (\operatorname{Re} A + \varepsilon \mathbf{1})x, x\rangle/\hbar) \, dx.$$

Diagonalizing $\operatorname{Re} A + \varepsilon \mathbf{1}$ by Θ and changing variables to $x = \Theta y$, we can write this as

$$\tau_A \varphi = \int_{\mathbb{R}^m} \exp\left(-\sum_{j=1}^m \lambda_j y_j^2/\hbar\right) dy_1 \cdots dy_m,$$

where for some j, $\lambda_j < 0$, and this integral diverges. □

Fact 2.3.3. *For $f, g \in \mathcal{S}$ we have $f = g$ if and only if $\tau_f = \tau_g \in \mathcal{S}'$*

Proof. This follows from the fundamental theorem of calculus of variations. □

The following result will help us derive stationary phase.

Theorem 2.3.4 ([Hör83], 7.6.1)**.** *Let $A \in \operatorname{Sym}(m, \mathbb{C})$ be a non-singular, and $\langle \operatorname{Re} Ax, x\rangle \geq 0$ for all $x \in \mathbb{R}^m$.*

(A) If $\operatorname{Re} A \neq 0$, let $B = A^{-1}$, and define

$$g_B(\xi) \equiv 2^{-m/2} (\det B)^{1/2} \exp(-\langle B\xi, \xi\rangle/4\hbar).$$

(B) In the special case $\operatorname{Re} A = 0$, so $A = -iA_0$ for $A_0 \in \operatorname{Sym}(m, \mathbb{R})$ non-singular, let $B = A^{-1} = iA_0^{-1}$ and

$$g_B(\xi) \equiv 2^{-m/2} |\det A_0|^{-1/2} \exp\left(\frac{-i\langle A_0^{-1}\xi, \xi\rangle}{4\hbar}\right) \exp\left(\frac{\pi i \operatorname{sgn} A_0}{4}\right).$$

Let τ_A be the tempered distribution

$$\tau_A(\varphi) \equiv \int_{\mathbb{R}^m} \exp(\langle -Ax, x\rangle/\hbar)\varphi(x) \, dx,$$

and define $\mu_B \in \mathcal{S}'$ by

$$\mu_B(\varphi) \equiv \int_{(\mathbb{R}^m)^*} g_B(\xi)\varphi(\xi) \, d\xi.$$

Then

$$\mathcal{F}_\hbar \tau_A = \mu_B$$

Proof. Consider $D_l \tau_A = 2i(Ax)_l \tau_A$ for $1 \leq l \leq m$. Taking the asymptotic Fourier transform on both sides, we obtain

$$\xi_l \mathcal{F}_\hbar \tau_A = -2i(AD)_l \mathcal{F}_\hbar \tau_A.$$

Multiplying by $B = A^{-1}$ on both sides,

$$(B\xi)_l (\mathcal{F}_\hbar \tau_A) = -2iD_l (\mathcal{F}_\hbar \tau_A).$$

Defining $\mu_B \equiv \mathcal{F}_\hbar \tau_A$, we find

$$D_l \mu_B = \frac{i}{2}(B\xi)_l \mu_B. \tag{7}$$

Let $\tilde{g}_B(\xi) = \exp(\langle B\xi, \xi\rangle/4\hbar)$. Then if $\tilde{\mu}_B(\varphi) \equiv \mu_{\tilde{g}_B}(\varphi) = \int \tilde{g}_B \varphi$. Then we observe that

$$D_l \tilde{\mu}_B = \frac{-i}{2}(B\xi)_l \tilde{\mu_B}. \tag{8}$$

Note that given $\mu_B, \tilde{\mu}_B \in \mathcal{S}'$ of the form $\mu_B = \mu_{g_B}$, $\tilde{\mu}_B = \mu_{\tilde{g_B}}$, we can define

$$(\mu_B \cdot \tilde{\mu}_B) \equiv \int g_B(\xi)\tilde{g_B}(\xi)\varphi(\xi)\, d\xi = \mu_{g_B \tilde{g_B}}.$$

Observe that the product rule $D_l(\mu_B \tilde{\mu}_B) = (D_l \mu_B)\tilde{\mu}_B + \mu_B D_l \tilde{\mu}_B$ holds. Then by (7) and (8) we conclude $D_l(\mu_B \tilde{\mu}_B) = 0$. We claim that in general, if $D_l \tau_f = 0$, then $f = c$ for some constant c. Thus,

$$g_B = C\left(\frac{1}{\tilde{g}_B}\right) = C\exp(-\langle B\xi, \xi\rangle/4\hbar)$$

for some C that we need to find. Clearly $\tilde{g}_B(0) = 1$ so

$$C = g_B(0) = \mathcal{F}_\hbar f_A(0) = (2\pi\hbar)^{-m/2} \int_{\mathbb{R}^m} \exp(-\langle Ax, x\rangle/\hbar)\, dx.$$

Thus, in the case (A), if $\operatorname{Re} A \neq 0$, then by the previous result,

$$C = 2^{-m/2}(\det A)^{-1/2}$$

In the case (B), if $\operatorname{Re} A = 0$, then taking $A \mapsto A + \varepsilon \mathbf{1}$ and letting $\varepsilon \to 0$, we find

$$C = 2^{-m/2}|\det A_0|^{-1/2} \exp\left(\frac{\pi i \operatorname{sgn} A_0}{4}\right).$$

□

Lemma 2.3.5. *If $Q = H/2 \in \operatorname{Sym}^2(\mathbb{R}^m)^*$ is non-degenerate, then for each $N \in \mathbb{N}$,*

$$\begin{aligned} I_{Q,u}(\hbar) &= \int_{\mathbb{R}^m} e^{i\langle Hx,x\rangle/2} u|dx| = (2\pi\hbar)^{m/2} \\ &= (2\pi\hbar)^{m/2} \frac{e^{i\pi \operatorname{sgn} H/4}}{|\det_{|dx|} H|^{1/2}} \sum_{q=0}^{N} \left(\frac{1}{q!}(D^q u)(0)\hbar^q\right) + O(\hbar^{N+1+m/2}), \end{aligned}$$

where $D \equiv \frac{i}{2}\sum_{j,k} H^{-1}_{j,k} \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}$.

Proof. By the previous theorem,

$$\begin{aligned} \mathcal{F}_\hbar f_Q(\xi) &= \int_{\mathbb{R}^m} e^{-i\langle x,\xi\rangle/\hbar} e^{iQ/\hbar} dx \\ &= 2^{-m/2}|\det Q|^{-1/2} \exp\left(\frac{-i\langle Q^{-1}\xi, \xi\rangle}{4\hbar}\right) \exp\left(\frac{\pi i \operatorname{sgn} Q}{4}\right), \end{aligned}$$

where here

$$\begin{aligned} |\det Q|^{1/2} &= |\det{}_{|dx|} Q|^{1/2} = |\det{}_{|dx|} H/2|^{1/2} \\ &= 2^{-m/2}|\det{}_{|dx|} H|^{1/2} \end{aligned}$$

under the identification $\mathrm{Sym}^2(\mathbb{R}^m)^* \cong \mathrm{Sym}(m,\mathbb{R})$. Substituting $Q^{-1} = 2H^{-1}$, we find

$$\mathcal{F}_\hbar f_Q(\xi) = \frac{\exp\left(\frac{\pi i \operatorname{sgn} H}{4}\right)}{|\det_{|dx|} H|^{1/2}} \exp\left(\frac{-i\langle H^{-1}\xi,\xi\rangle}{2\hbar}\right).$$

We can then write

$$\begin{aligned} I_{Q,u}(\hbar) &= \int_{\mathbb{R}^m} e^{iQ/\hbar} u |dx| \\ &= \frac{e^{i\pi \operatorname{sgn} H/4}}{|\det_{|dx|} H|^{1/2}} \int_{(\mathbb{R}^m)^*} \exp(-i\langle H^{-1}\xi,\xi\rangle/2\hbar)\overline{\mathcal{F}_\hbar \overline{u}}(\xi)|d\xi| \end{aligned}$$

by the asymptotic Parseval formula. It is easy to show that $\overline{\mathcal{F}_\hbar \overline{u}}(\xi) = \mathcal{F}_\hbar u(-\xi)$, and the transformation $\xi \mapsto -\xi$ leaves $I_{Q,u}(\hbar)$ unchanged. Thus we are left with

$$I_{Q,u}(\hbar) = \frac{e^{i\pi \operatorname{sgn} H/4}}{|\det_{|dx|} H|^{1/2}} \int_{(\mathbb{R}^m)^*} \exp\left(\frac{-i\langle H^{-1}\xi,\xi\rangle}{2\hbar}\right) \mathcal{F}_\hbar u(\xi)|d\xi|.$$

Let $H^*(\xi) = \langle H^{-1}\xi,\xi\rangle$ and use the Taylor series for $\exp(-i\langle H^*/(2\hbar))$ to rewrite the integral above as

$$\sum_{q=0}^{\infty} \frac{1}{q!}\left(\frac{-i}{2\hbar}\right)^q \int_{(\mathbb{R}^m)^*} (H^*(\xi))^q \mathcal{F}_\hbar u(\xi)|d\xi|.$$

The integral in this last expression, Z say, is the inverse Fourier transform of a function evaluated at 0. That is,

$$Z = (2\pi\hbar)^{m/2}\mathcal{F}_\hbar^{-1}((H^*(\xi))^q \mathcal{F}_\hbar u)(0),$$

where $H^*(\xi) = \sum_{j,k} H^{-1}_{j,k}\xi_j\xi_k$. By the previous result regarding multiplying Fourier transforms by polynomials, we obtain

$$\mathcal{F}_\hbar^{-1}((H^*(\xi))^q \mathcal{F}_\hbar u) = \mathcal{F}_\hbar^{-1}(\mathcal{F}_\hbar(D_\hbar^q u))(0) = (D_\hbar^q u)(0),$$

where here

$$\begin{aligned} D_\hbar^q u &\equiv \sum_{j,k} H^{-1}_{j,k}\left(i\hbar\frac{\partial}{\partial x_j}\right)\left(-i\hbar\frac{\partial}{\partial x_k}\right) = (-i\hbar)^2\tilde{D}, \\ \tilde{D} &\equiv \left(\sum_{j,k} H^{-1}_{j,k}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\right). \end{aligned}$$

Putting this together, we find

$$\begin{aligned} I_{Q,u} &= \frac{e^{i\pi \operatorname{sgn} H/4}}{|\det_{|dx|} H|^{1/2}}\left(\sum_{q=0}^{N}\frac{(2\pi\hbar)^{m/2}}{q!}\left(\frac{-i}{2\hbar}\right)^q(-\hbar)^{2q}(\tilde{D}^q u)(0)\right) + O(\hbar^{N+1+m/2}) \\ &= (2\pi\hbar)^{m/2}\left(\sum_{q=0}^{N}\frac{1}{q!}(D^q u)(0)\hbar^q\right) + O(\hbar^{N+1+m/2}), \end{aligned}$$

where $D = \frac{i}{2}\sum_{j,k} H^{-1}_{j,k}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}$. □

Recall that we were considering

$$\begin{aligned} \int_M e^{i\delta/\hbar}\nu &= \sum_{l\in\tilde{\theta}} I_{Q_l,u_l}(\hbar) + O(\hbar^\infty), \\ I_{Q_l,u_l} &\equiv e^{i\delta(p_l)}/\hbar \int_{\mathbb{R}^m} e^{iQ_l/\hbar}u_l|dx|, \end{aligned}$$

where $p_l \in \operatorname{Crit}\delta \cap \operatorname{supp}\nu$, for $u_l(x)|dx| \equiv \psi_l^*(\varphi_l \cdot \nu)$ and $Q_l(x) = H_0(\psi_l^*\delta)/2$.

By previous results,

$$I_{Q_l,u_l} = (2\pi\hbar)^{m/2}\frac{e^{i\delta(p_l)/\hbar}e^{i\pi \operatorname{sgn} H_l/4}}{|\det_{|dx|} H_l|^{1/2}}\sum_{q=0}^{N}\frac{1}{q!}(D^q u_l)(0)\hbar^q + O(\hbar^{N+1+m/2}).$$

Notice that the term $q = 0$ is simply $u_l(0)$.

Claim 2.3.6. *We have the following:*

1. $|\det_{u_l|dx|} H_0(\psi_l^*\delta)|^{1/2} = (u_l(0))^{-1}|\det_{|dx|} H_0(\psi_l^*\delta)|^{1/2}$,
2. $|\det_{u_l|dx|} H_0(\psi_l^*\delta)|^{1/2} = |\det_{\nu} H_p\delta|^{1/2}$.

We leave this as an exercise. It follows that

$$u_l(0)|\det{}_{|dx|}H_l|^{-1/2} = |\det{}_{\nu}H_p\delta|^{-1/2}.$$

Writing only the contributions from $q = 0$, we obtain:

Theorem 2.3.7 (Principle of stationary phase)**.** *Let M be a smooth m-manifold, $\nu \in \Gamma_c(|\wedge|^1 M)$, $\delta \in C^\infty(M,\mathbb{R})$ a Morse function, $p_l \in \operatorname{Crit}\delta \cap \operatorname{supp}\nu$, $1 \le l \le L$. Then*

$$\begin{aligned} I_{\nu,\delta}(\hbar) &\equiv \int_M e^{i\delta/\hbar}\nu \\ &= (2\pi\hbar)^{m/2}\sum_{l=1}^{L}\frac{e^{i\pi\delta(p_l)/\hbar}e^{i\operatorname{sgn} H_{p_l}\delta/4}}{|\det_{\nu} H_{p_l}\delta|^{1/2}} + O(\hbar^{1+m/2}). \end{aligned}$$

7th lecture, September 16th 2011

3 The Duistermaat–Heckman theorem

3.1 Motivation

Recall from last the time the following theorem.

Theorem 3.1.1 (Principle of stationary phase)**.** *Let M be a (smooth) m-manifold, $\nu \in \Gamma_c(|\wedge|^1 M)$, $\delta \in C^\infty(M,\mathbb{R})$ a Morse function, $p_l \in \delta \cap \operatorname{supp}\nu$, $1 \le l \le L$. Then*

$$\begin{aligned} I_{\nu,\delta}(\hbar) &\equiv \int_M e^{i\delta/\hbar}\nu \\ &= (2\pi\hbar)^{m/2}\sum_{l=1}^{L}\frac{e^{i\pi\delta(p_l)/\hbar}e^{i\operatorname{sgn} H_{p_l}\delta/4}}{|\det_{\nu} H_{p_l}\delta|^{1/2}} + O(\hbar^{1+m/2}). \end{aligned}$$

Our next goal is to study special triples $(\delta,\nu) \in C^\infty(M) \times \Gamma_c(|\wedge|^1 M)$, where the remainder term from the stationary phase formula truncates, i.e.

$$I_{\nu,\delta}(\hbar) = \sum_{l=1}^{L} e^{i\delta(p_l)/\hbar}F_l(\hbar)$$

where $F_l \in \mathbb{C}[\hbar]$. We follow inspired analysis of [DH82]. In order to do so, we need to develop some terminology. In fact, our case (i.e. the Duistermaat–Heckman case) implies that

$$I_{\nu,\delta}(\hbar) = \sum_{l=1}^{L} e^{i\delta(p_l)/\hbar}F_l(\hbar),\ F_l \in \mathbb{C}[\hbar],\ \deg F_l = m/2,\ 1 \le l \le L,$$

so in particular for this to make sense, $\dim M = m$ will have to be even, $m = 2n$. We try to motivate the theorem by the following instructive example (even if it doesn't quite do the theorem justice).

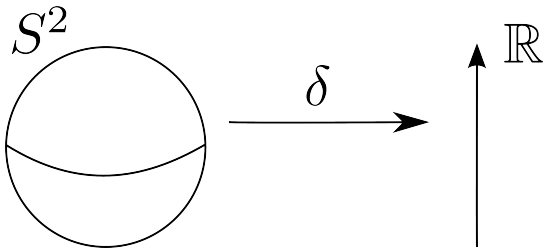

Figure 3: The height function of Example 3.1.2.

Example 3.1.2. Consider $M = S^2 = \{p \in \mathbb{R}^3 \mid \|p\| = 1\}$ and $\nu \equiv d\theta \wedge dz$ on $(\theta, z) \in (0, 2\pi) \times (-1, 1)$, where (θ, z) are the usual cylindrical coordinates on $\mathbb{R}^3 = \{(x, y, z)\}$. Let $\delta(x, y, z) = z$, i.e. the height function on the sphere (see Fig. 3). Then we want to consider the integral

$$\begin{aligned} I_{\nu,\delta}(\hbar) &= \int_{S^2} e^{i\delta/\hbar}\nu = \int_0^{2\pi}\int_1^1 e^{iz/\hbar}\, d\theta\, dz \\ &= \frac{2\pi\hbar}{i}\left(e^{i/\hbar} - e^{-i/\hbar}\right). \end{aligned}$$

Notice first of all that the right hand side looks like a sum over the critical points of the integrand, since clearly $d\delta = dz = 0$ at $z = \pm 1$. In other words, we can write

$$I_{\nu,\delta}(\hbar) = \sum_{p_l \in \operatorname{Crit}\delta} e^{i\delta(p_l)/\hbar} F_l(\hbar),$$

where $F_l(\hbar) = (2\pi/i)\hbar$ has degree $1 = 2/2$.

Note that in this example we get an *exact* stationary phase result (i.e. $\deg F_l(\hbar) = m/2$), and clearly δ was Morse. Furthermore, ν was a volume form on S^2 and also a symplectic form (of course, all volume forms on 2-manifolds are symplectic), so we can look at the *Hamiltonian vector field* associated to δ, i.e. the vector field V_δ given by $d\delta = \iota_{V_\delta}\nu$, or in other words $d\delta(w) = \nu(V_\delta, w)$ for all $w \in \Gamma(TS^2)$.

For $\delta(x, y, z) = z$, we can take $V_\delta = \frac{\partial}{\partial\theta}$, and observe that V_δ has *periodic flow* in this case. This is the key property we will need for the Duistermaat–Heckman theorem.

3.2 The theorem

Theorem 3.2.1 (Duistermaat–Heckman, version 1)**.** *Let M be a compact, smooth manifold of even dimension $\dim M = m = 2n$ with a symplectic form $\omega \in \Omega^2(M, \mathbb{R})$ (see below) and Liouville form $\beta = \omega^n/n!$, and let $\delta \in C^\infty(M, \mathbb{R})$ be a Morse function such that the Hamiltonian vector field V_δ has periodic flow. Then for any $\hbar \in \mathbb{R}^{>0}$,*

$$\int_M e^{i\delta/\hbar}\beta = (2\pi\hbar)^n \sum_{p_l \in \operatorname{Crit}\delta} \frac{e^{i \operatorname{sgn} H_p\delta/4}}{|\det {}_\beta H_p\delta|^{1/2}}.$$

Definition 3.2.2. A *symplectic manifold* is a pair (M, ω), where M is a (smooth) manifold, and $\omega \in \Omega^2(M, \mathbb{R})$ is closed and non-degenerate form.

Note that the existence of a non-degenerate form $\omega \in \Omega^2(M, \mathbb{R})$ implies that $\dim M$ is even.

Definition 3.2.3. Let (M, ω) be a symplectic manifold with an Lie group action $G \circlearrowright M$. Let $\mu : M \to \mathfrak{g}^*$ (where $\mathfrak{g}^*$ is the dual Lie algebra of G) such that

1. $d\mu^Y = \iota_{Y^\sharp}\omega$ for all $Y \in \mathfrak{g}$, where $\mu^Y(p) \equiv \langle \mu(p), Y\rangle$, and $Y^\sharp(p) \equiv \frac{d}{dt}|_{t=0}(\exp(t \cdot Y) \cdot p)$, and
2. $g^*\mu = \mathrm{Ad}^*_g \circ \mu$, where $g^*\mu(x) = \mu(g \cdot x)$.

Then the action $G \circlearrowright M$ is called a *Hamiltonian group action* , μ is called a *moment map* , and (M, ω, G, μ) is called a *Hamiltonian G-space* .

We would now like to state, without proof, a general Duistermaat–Heckman theorem. In particular, we are interested in looking at $\delta = \mu^Y : M \to \mathbb{R}$ for some particular choice $Y \in \mathfrak{g}$.

Exercise 3.2.4. Show that $(M, \omega, G, \mu) = (S^2, \omega_{\text{st}}, \mathrm{U}(1), \delta)$ from Example 3.1.2 is a Hamiltonian G-space.

Notice that in this example,

$$\mu^Y(p) \equiv \delta^Y(p) \equiv \langle \delta(p), Y \rangle = Y \cdot z,$$

and we used the particular case $Y = 1$. In general, the integral would have been

$$\int_M e^{i\delta^Y(p)/\hbar}\beta = \int_M e^{i(Y\cdot z)/\hbar}\beta.$$

Also, $\operatorname{Crit} \delta^Y = \operatorname{Crit} \delta$ for $Y \neq 0$. In general, we will be interested in

$$\operatorname{Crit} \mu^Y = M^Y := \{p \in M \mid Y^\sharp(p) = 0\},$$

where the equality follows from the facts that $d\mu^Y = \iota_{Y^\sharp}\omega$, and that ω is non-degenerate.

Definition 3.2.5. A vector field $Y \in \mathfrak{g}$ is called an *elliptic element* of $\mathfrak{g}$ when the closure of the one-parameter subgroup generated by Y, i.e. $T_Y = \overline{\langle \exp(\mathbb{R}Y) \rangle}$ is a compact torus.

Exercise 3.2.6. Given an elliptic element $Y \in \mathfrak{g}$, we have $\operatorname{Crit} \mu^Y = M^{T_Y} := \{p \in M \mid T_Y \cdot p = p\}$.

Of course, if G is compact, then all $Y \in \mathfrak{g}$ will be elliptic. Since we are interested in $\int_M e^{i\mu^Y/\hbar}\beta$, and since the result (i.e. the right-hand side of the eventual Duistermaat-Heckman theorem) will depend on $\operatorname{Crit} u^Y$, we may assume that $G = T$ is a torus.

An important ingredient in our study connects the Hessian of μ^Y to the group action $T_Y \circlearrowright M$; This is the isotropy representation $L_p(Y) : T_pM \to T_pM$, i.e. a linear map for $p \in \operatorname{Crit} \mu^Y$, defined as follows.

Definition 3.2.7. Let $G \circlearrowright M$ be a Lie group action on a (smooth) manifold M, let $Y \in \mathfrak{g}$, and let $p \in M^Y := \{p \in M \mid Y^\sharp(p) = 0\}$. The *isotropy representation* of Y at p is the map $L_p(Y) : T_pM \to T_pM$ defined by

$$L_p(Y)(v_p) \equiv (\mathcal{L}_{Y^\sharp} v)_p$$

for $v \in \Gamma(TM)$. Here $\mathcal{L}$ denotes the Lie derivative.

One can check that the above definition depends only on v in the point p.

Let $h_G \in \operatorname{Sym}^2 T^*M$ be a G-invariant Riemannian metric on M (we can assume that G is either compact, a torus, or semi-simple). Then $Y^\sharp \in \Gamma(TM)$ is a *Killing vector field* for h_G, i.e. $\mathcal{L}_{Y^\sharp} h_G = 0$.

Let $\overline{v}, \overline{w} \in \Gamma(TM)$ be two vector fields on M. Then using a property of the Lie derivative,

$$\mathcal{L}_{Y^\sharp}(h_G(\overline{v}, \overline{w})) = (\mathcal{L}_{Y^\sharp} h_G)(\overline{v}, \overline{w}) + h_G(\mathcal{L}_{Y^\sharp}\overline{v}, \overline{w}) + h_G(\overline{v}, \mathcal{L}_{Y^\sharp}\overline{w}).$$

Since $\mathcal{L}_{Y^\sharp} h_G = 0$, the equation above implies that

$$Y^\sharp(h_G(\overline{v}, \overline{w})) = h_G([Y^\sharp, \overline{v}], \overline{w}) + h_G(\overline{v}, [Y^\sharp, \overline{w}]),$$

and at a point $p \in M^Y$, we have

$$h_G(L_p(Y)(\overline{v}_p), \overline{w}_p) = -h_G(\overline{v}_p, L_p(Y)(\overline{w}_p)).$$

In other words, $\mathcal{L}_p(Y)$ is skew-symmetric with respect to the metric h_G. For now, we will focus on the case of isolated points. We prove the following proposition later.

Proposition 3.2.8. *Let* $\exp_p : T_pM \to M$ *be the Riemannian exponential map defined by* h_G*. Then* $v_p \in \ker(L_p(Y))$ *if and only if* $\exp_p(v_p) \in M^Y$*. In particular,* $p \in M^Y$ *is isolated, if and only if* $L_p(Y)$ *is invertible.*

Assuming now that $p \in M^Y$ is isolated, then since $L_p(Y)$ is skew-symmetric, there exists an oriented h_G-orthonormal basis $e = \{e_1, \dots, e_{2n}\}$ of T_pM such that

$$h_G(L_p(Y)e_i, e_j) = \begin{pmatrix} \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix} \end{pmatrix}$$

for $\lambda_i \in \mathbb{R}^{\neq 0}$, $1 \leq i \leq n$. Recall that the *Pfaffian* of the skew-symmetric matrix $L_p(Y)$ is

$$\mathrm{Pf}(L_p(Y)) = \prod_{i=1}^{n} \lambda_i.$$

Theorem 3.2.9 (Duistermaat–Heckman, version 2)**.** *Let* (M, ω, T, μ) *be a compact, smooth Hamiltonian* T*-space of dimension* $\dim M = m = 2n$ *with Liouville form* $\beta = \omega^n/n!$*, and let* $Y \in \mathrm{Lie}\, T$*, such that* M^Y *consists of isolated points. Then for any* $\hbar \in \mathbb{R}^{>0}$*,*

$$\int_M e^{i\mu^Y/\hbar}\beta = (2\pi\hbar)^n \sum_{p_l \in M^Y} i^n \frac{e^{i\mu^Y(p)/\hbar}}{\mathrm{Pf}(L_p(Y))}.$$

We want to prove this theorem directly from stationary phase; there are other ways to do it that do not use stationary phase approximation at all. Our proof will show explicitly the conditions of a Hamiltonian G-action coming into play. Later, we will see a proof using equivariant differential forms when we prove a more general statement.

Looking at the formulas, we have one the one hand

$$(2\pi\hbar)^n \sum_{p \in M^Y} \frac{e^{\pi i \operatorname{sgn} H_p\mu^Y/4} e^{i\mu^Y(p)/\hbar}}{|\det{}_\beta H_p\mu^Y|^{1/2}}$$

and on the other

$$(2\pi\hbar)^n \sum_{p \in M^Y} \frac{i^n e^{i\mu^Y(p)/\hbar}}{\mathrm{Pf}(L_p(Y))}.$$

Claim 3.2.10. *We have*

$$\frac{e^{\pi i \operatorname{sgn} H_p\mu^Y/4}}{|\det{}_\beta H_p\mu^Y|^{1/2}} = \frac{i^n}{\mathrm{Pf}(L_p(Y))}$$

This claim follows from the following fact.

Exercise 3.2.11. For all $v_p, w_p \in T_pM$,

$$H_p\mu^Y(v_p, w_p) = -\omega(L_p(Y)v_p, w_p).$$

Proof of Claim. By definition

$$|\det{}_\beta H_p\mu^Y|^{1/2}\beta(e) = |\det\langle H_p\mu^Y e_i, e_j\rangle|^{1/2}$$

for some orthonormal basis e which we can choose to satisfy $\beta(e) = 1$. In this basis

$$h_G(L_p(Y)e_i, e_j) = \begin{pmatrix} \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix} \end{pmatrix}.$$

We find that

$$|\det {}_\beta H_p\mu^Y|^{1/2} = |\det\langle H_p\mu^Y e_i, e_j\rangle|^{1/2},$$

and by the exercise, one can show that

$$H_p\mu^Y(e_i, e_j) = \operatorname{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n).$$

Let $\#(\lambda_i^-)$ denote the number of negative eigenvalues. The signature of $H_p\mu^Y$ then becomes

$$\operatorname{sgn} H_p\mu^Y = 2n - 4\#(\lambda_i^-),$$

and so

$$e^{\pi \operatorname{sgn} H_p\mu^Y/4} = e^{\frac{\pi i(2n-4\#(\lambda_i^-))}{4}} = i^n(-1)^{\#(\lambda_i^-)}.$$

On the other hand, the Pfaffian becomes

$$\operatorname{Pf}(L_p(Y)) = |\operatorname{Pf}(L_p(Y))|(-1)^{\#(\lambda_i^-)} = \prod_{i=1}^n \lambda_i,$$

which proves the claim. □

8th lecture, September 19th 2011

Recall from last time the following theorem.

Theorem 3.2.12 (Duistermaat–Heckman, version 2)**.** *Let (M, ω, T, μ) be a compact, smooth Hamiltonian T-space (T a compact torus) of dimension $\dim M = m = 2n$ with Liouville form $\beta = \omega^n/n!$, and let $Y \in \operatorname{Lie} T$, such that M^Y consists of isolated points. Then for any $\hbar \in \mathbb{R}^{>0}$,*

$$\int_M e^{i\mu^Y/\hbar}\beta = (2\pi\hbar)^n \sum_{p_l \in M^Y} i^n \frac{e^{i\mu^Y(p)/\hbar}}{\operatorname{Pf}(L_p(Y))}.$$

We want to derive this, following the original approach by Duistermaat and Heckman. This method shows how one would go about doing so "from the beginning", and it shows where in the process higher order terms vanish.

4 Proof of the theorem

Recall that since M^Y is isolated, the isotropy representation $L_p(Y)$ is invertible. Hence $H_p\mu^Y$ is non-degenerate. Recall here that $H_p\mu^Y(u_p, v_p) = -\omega(L_p(Y)u_p, v_p)$. Thus we are in the same situation as in our stationary phase result, i.e. μ^Y is Morse, the volume form satisfies $\beta \in \Gamma_c(|\wedge|^1 M)$, since M is compact. The stationary phase theorem, applied to this situation, tells us that

$$I_{\beta,\mu^Y}(\hbar) = \int_M e^{i\mu^Y/\hbar}\beta = (2\pi\hbar)^n \sum_{p \in M^Y} \frac{i^n e^{i\mu^Y/\hbar}}{\operatorname{Pf}(L_p(Y))} + O(\hbar^{1+n}).$$

Recall that this was derived using the Morse Lemma to give coordinates where μ^Y is quadratic near critical points $p \in \operatorname{Crit} \mu^Y =: M^Y$. That is, there exists a local diffeomorphism $\psi^* \mu^Y(x) = \mu^Y(p) + Q(x)$ for $p \in M^Y$.

The key idea of Duistermaat and Heckman is to use the symplectic structure of the Hamiltonian T-space M together with the symmetry of the T-action to construct "Morse coordinates" naturally. These coordinates are related to the Darboux coordinates (see below) for which the higher order terms will vanish, since the volume form corresponds to the standard one on Euclidean space. Recall the following theorem which tells us that in symplectic geometry there are no local invariants:

Theorem 4.0.13 (Darboux's theorem)**.** *Let (M, ω) be a symplectic manifold of dimension $\dim M = 2n$, and let $p \in M$. There exist coordinates $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ around p such that*

$$\omega|_U = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

This theorem is not quite what we need, since it does not tell us anything about the structure of the moment map μ^Y near p.

Theorem 4.0.14 (Darboux–Weinstein, [GS90] Thm. 22.1)**.** *Let N be an embedded submanifold of M, and let $\omega_0, \omega_1 \in \Omega^2(M)$ be symplectic forms on M such that $\omega_0|_N = \omega_1|_N$. Then there exists a neighbourhood U of N and a map $\psi : U \to M$, diffeomorphic onto its image, such that $\psi|_N = \mathbf{1}_N$ and $\psi^* \omega_1 = \omega_0$.*

If G is a compact group acting on M with ω_0 and ω_1 invariant under the action, and if N is invariant under the action, then we can take ψ to be G-equivariant, i.e. $\psi(g \cdot p) = g \cdot \psi(p)$ for all $p \in U$.

Note that in particular, if $N = \{p\}$ is a point, then the Darboux–Weinstein theorem says that if $\omega_0|_p = \omega_1|_p$, then up to a diffeomorphism, they agree on some neighbourhood of that point.

Choosing $\exp_p : T_pM \to M$ for some Riemannian metric on M, identify a neighbourhood of $0 \in T_pM$ with a neighbourhood of $p \in M$. Also, recall that any two symplectic forms on a vector space are linearly symplectomorphic, i.e. for symplectic vector spaces (V_0, ω_0), (V_1, ω_1) of the same dimension, there is an isomorphism $\psi : V_0 \to V_1$ such that $\psi^* \omega_1 = \omega_0$.

The Darboux–Weinstein also says that if ω_0, ω_1 are *any* symplectic forms defined near p in M, there exists a map $\psi : U \to M$ near p such that $\psi(p) = 0$, such that ψ is diffeomorphism onto its image, and such that $\psi^* \omega_1 = \omega_0$. To see this, one uses the fact that $(d \exp_p)_0$ is the sum of id_{T_pM} and the linear symplectomorphism above.

If a compact Lie group G acts on M with fixed point p, we can choose a G-invariant Riemannian metric and a G-equivariant linear symplectomorphism (which is not obvious, but we will see a special case) to have ψ be G-equivariant. These observations together with the Darboux–Weinstein theorem give the following.

Theorem 4.0.15 (Equivariant Darboux theorem, [GS90] Thm. 22.2)**.** *Let $G \circlearrowright M$ be the action of a compact group, and let $p \in M$ be a fixed point of the action. Let $\omega_0, \omega_1 \in \Omega^2(M)$ be G-invariant symplectic forms on M. Then there exists a G-invariant neighbourhood U of p and a G-equivariant map $\psi : U \to M$, diffeomorphic onto its image, such that $\psi(p) = p$ and $\psi^* \omega_1 = \omega_0$.*

From now on, assume that $G = T$ is a compact torus.

Example 4.0.16. The following example is the model, we will use locally for Hamiltonian T-spaces. Let $T = (\mathrm{U}(1))^d$ act on $\mathbb{C}^n$ via

$$\exp(\eta)(z_1, \ldots, z_n) = (e^{2\pi i \alpha_1(\eta)} z_1, \ldots, e^{2\pi i \alpha_n(\eta)} z_n),$$

where $\alpha_i : \operatorname{Lie} T \to \mathbb{R}$, $1 \leq i \leq n$ are linear, i.e. $\alpha_i \in (\operatorname{Lie} T)^*$ such that if $\Lambda \subseteq \operatorname{Lie} T$ denotes the integral lattice, then $\alpha_i \in \Lambda^*$; i.e. α_i maps $\Lambda \to \mathbb{Z}$.

Let $\omega_0 \in \Omega^2(\mathbb{C}^n)$ be the standard symplectic form, i.e.

$$\omega_0 = \frac{i}{2}\sum_{j=1} dz_j \wedge d\overline{z}_j = \sum_{j=1} dx_j \wedge dy_j.$$

Let $\mu : \mathbb{C}^n \to (\operatorname{Lie} T)^*$ be defined as

$$\mu(z_1, \ldots, z_n) \equiv \frac{\pi}{2}\sum_{j=1}^{n} |z_j|^2 \alpha_j = \frac{\pi}{2}\sum_{j=1}^{n}(x_j^2 + y_j^2)\alpha_j.$$

We claim that $(\mathbb{C}^n, \omega_0, (\mathrm{U}(1))^d, \mu)$ is a Hamiltonian T-space. Note that the moment map μ is unique up to a choice of constant, since $d\mu^Y = \iota_{Y^\sharp}\omega$ for all $Y \in (\operatorname{Lie} T)$. We leave these claims as an exercise (use polar coordinates $\omega = \sum r_j dr_j \wedge d\theta_j$).

Recall that any unitary T-representation V spllits into a sum of 1-dimensional representations, i.e.

$$V \cong \bigoplus_{j=1}^{n} \mathbb{C}_{\alpha_j},$$

where $\alpha_j \in (\operatorname{Lie} T)^*$ are called the *weights* of the representation (this follows from Schur's Lemma and the fact that T is abelian).

Given a symplectic vector space V with a symplectic T-representation, one chooses an invariant complex structure $J : V \to V$, i.e. $J^2 = \mathbf{1}$, which makes V into a unitary T-representation, i.e. $\psi(g)\psi(g)^* = \psi(g)^*\psi(g) = \mathbf{1}$.

Fact 4.0.17. *The weights α_j are independent of the choice of complex structure J, since any two complex structures are deformation equivalent.*

Thus, $\alpha_j \in (\operatorname{Lie} T)^*$ are called the weights of the symplectic T-representation V.

Given $T \circlearrowright (M, \omega)$, let $p \in M$ be a fixed point of the action. We want to construct coordinates around p such that the moment map μ^Y is quadratic. By the equivariant Darboux theorem, we find $p \in U \subseteq M$ that is identified with a neighbourhood of $0 \in T_pM$, with a linear T-action on T_pM (which we can think of as the isotropy representation on T_pM). Since the moment map is unique on $T_pM = V$, we get the following theorem:

Theorem 4.0.18. *Let (M, ω, T, μ) be a Hamiltonian T-space with a fixed point $p \in M^T$. Then there exists a chart $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p such that*

$$\omega|_U = \sum_{j=1}^{n} dx_j \wedge dy_j$$

and such that

$$\mu^Y|_U(x) = \mu^Y(p) + \frac{\pi}{2}\sum_{j=1}^{n} \alpha_j(Y)(x_j^2 + y_j^2)$$

where the $\alpha_{p,j} \in (\operatorname{Lie} T)^*$ are the weights of the associated T-action on T_pM.

Note that the associated T-action on T_pM is exactly the isotropy representation $L_p(Y) : T_pM \to T_pM$. One can show, similarly to what we did last time, that

$$\frac{e^{i\pi \operatorname{sgn} H_p\mu^Y/4}}{|\det {}_\beta H_p\mu^Y|^{1/2}} = \frac{i^n}{\prod_{j=1}^{n} \alpha_{p,j}(Y)}.$$

In fact, $\alpha_{p,j}(Y) = \lambda_j$, are identified as in

$$L_p(Y) = \begin{pmatrix} \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix} \end{pmatrix}$$

with the particular choice of basis made in the previous lecture.

Our goal is to prove the following from [DH82]:

Theorem 4.0.19 (Duistermaat–Heckman, version 3, [DH82] Thm. 4.1)**.** *Let (M, ω, T, μ) be a compact, smooth Hamiltonian T-space of dimension $\dim M = 2n$, let $\beta = \omega^n/n!$, and let $Y \in \operatorname{Lie} T$, such that M^Y consists of isolated points. Then*

$$\int_M e^{i\mu^Y/\hbar}\beta = (2\pi\hbar)^n \sum_{p \in M^Y} \frac{e^{i\mu^Y(p)/\hbar}}{\prod_{j=1}^n \alpha_{p,j}(Y)}.$$

Note that in [DH82], this is proved for non-isolated points; we assume that points are isolated for simplicity and indicate generalizations later. Before going into the proof, we make some simplifying observations.

First observe that if $Y \in \operatorname{Lie}(T)$ such that $T^Y := \overline{\langle \exp(\mathbb{R}Y)\rangle} \neq T$, we may replace T by T^Y. Define $F_T : \operatorname{Lie} T \to \mathbb{R}$ by $F_T(Z) := \prod_{j=1}^n \alpha_j(Z)$. This is clearly continuous since $\alpha_j \in (\operatorname{Lie} T)^*$. There exists a neigbourhood $U^Y \subseteq \operatorname{Lie} T$ around Y such that $F_T|_{U^Y} \neq 0$ since $F_T(Y) \neq 0$.

Claim 4.0.20. *The set of all $Z \in U^Y$ such that $\langle \exp(\mathbb{R}Z)\rangle \cong \mathrm{U}(1)$ is dense in U^Y.*

Sketch of proof. This follows from the fact that if we write $\operatorname{Lie} T \cong \mathbb{R}^d$, for $T \cong (\mathrm{U}(1))^d$, then $\langle \exp(\mathbb{R}Z)\rangle \cong \mathrm{U}(1)$ if and only if $Z \in \mathbb{Q}^d$. □

Now consider the stationary phase formula. Let

$$F^Y(p_l) \equiv \frac{i^n e^{i\mu^Y(p_l)/\hbar}}{\prod_{j=1}^n \alpha_{p_l,j}(Y)}, \quad p_l \in M^Y.$$

Writing the stationary phase approximation more carefully, we can write

$$I_{\mu^Y,\beta}(\hbar) = \int_M e^{i\mu^Y/\hbar}\beta = (2\pi\hbar)^n \sum_{p_l \in M^Y} F^Y(p_l)(1 + O(\hbar)) + O(\hbar^\infty),$$

where the $O(\hbar^\infty)$ term come from non-stationary contributions. The $O(\hbar)$ term we found to be proportional to

$$\sum_{q=1}^{\infty} \frac{1}{q!}(D^q u_l)(0)\hbar^q,$$

where $u_l|dx| \equiv \psi_l^*(\varphi_l\beta)$, $\varphi_l :\to [0,1]$ a partition of unity, and

$$D \equiv \frac{i}{2}\sum_{j,k} H^{-1}_{j,k} \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}.$$

In our case where (M, ω, T, μ) is a Hamiltonian T-space, the diffeomorphism ψ_l was chosen such that $\psi_l^*(\varphi_l\beta) = \omega_0^n$, the standard symplectic volume. By definition, this implies that $u_l(x)$ is constant. Overall, $(D^q u_l)(0) = 0$ for all $1 \leq l \leq L$.

9th lecture, September 23th 2011

Recall the following theorem

Theorem 4.0.21 ([DH82] Thm 4.1, Version 3)**.** *Let (M, ω, T, μ) be a compact smooth Hamiltonian T-space of dimension $\dim M = 2n$, let $\beta = \omega^n/n!$, $Y \in \operatorname{Lie} T$ such that M^Y is isolated. Then*

$$\int e^{i\mu^Y/\hbar} = (2\pi\hbar)^n \sum_{p \in M^Y} \frac{i^n e^{i\mu^Y(p)/\hbar}}{\prod_{i=1}^n \alpha_{p,j}(Y)}.$$

We want to derive

$$I_{\mu^Y,\beta}(\hbar) = \int_M e^{i\mu^Y/\hbar}\beta = (2\pi\hbar)^n \sum_{p_l \in M^Y} F_\hbar^Y(p_l)(1 + O(\hbar)) + O(\hbar^\infty)$$

from the principle of stationary phase, where here

$$F_\hbar^Y(p_l) = \frac{i^n e^{i\mu^Y(p_l)/\hbar}}{\prod_{j=1}^n \alpha_{p_l,j}(Y)}.$$

Last time we saw that $O(\hbar) = 0$; more precisely we saw that $u_l|dx| = \psi_l^*(\varphi_l\beta))$ for partition of unity-type functions φ_l. A subtlety here is that our proof $D^j u_l(0) = 0$ depends on the φ_l, so the equality really follows from the existence of so-called G-invariant partions of unity. We will not go into more detail with this.

We consider now the $O(\hbar^\infty)$ terms. We change variables to $t = 1/\hbar$ and define

$$I_{\mu^Y,\beta}(t) = \int_M e^{it\mu^Y}\beta = \left(\frac{2\pi}{t}\right)^n \sum_{p_l \in M^Y} F_t^Y(p_l) + s_Y(t), \tag{9}$$

where $s_Y(t)$ is $O(t^{-\infty})$, and

$$F_t^Y(p_l) \equiv \frac{i^n e^{i\mu^Y(p_l)t}}{\prod_{j=1}^n \alpha_{p_l,j}(Y)}.$$

We want to argue that $s_Y(t) = 0$. To see this, we make a series of simplifying remarks. Let $U_p^Y \subseteq \operatorname{Lie} T$ be open such that for $Z \in U_p^Y$ we have $G_p^T(Z) := \prod_{j=1}^n \alpha_{p,j}(Z) \neq 0$. Let $U^Y = \bigcap_{p \in M^Y} U_p^Y$. Let

$$U_{\mathbb{Q}}^Y := \{Z \in U^Y \mid T^Z := \langle \exp \mathbb{R}Z \rangle \cong \mathrm{U}(1)\}.$$

We saw last time that $U_{\mathbb{Q}}^Y \subseteq U^Y$ is dense. Since both sides of (9) are smooth in Y (check) and since $U_{\mathbb{Q}}^Y \subseteq U^Y$ is dense, it is sufficient to show that $s_Z(t) = 0$ for $Z \in U_{\mathbb{Q}}^Y$. We want to show that

$$\int_M e^{i\mu t}\beta = \left(\frac{2\pi}{t}\right)^n \sum_{p_l \in \operatorname{Crit}\mu} F_t(p_l),$$

where since $T^Z \cong \mathrm{U}(1)$, we abuse notation and write $\mu^Z = \mu$ and $F_t^Z(p_l) = F_t(p_l)$. There is a subtlety here: Critical points of μ^Z and μ^Y may not coincide. In fact, the set of $Z \in U_{\mathbb{Q}}^Y$ with $\operatorname{Crit}\mu^Z = \operatorname{Crit}\mu^Y = M^Y$ is dense in $U_{\mathbb{Q}}^Y$, as the following proposition shows, and so we assume that $\operatorname{Crit}\mu^Z = \operatorname{Crit}\mu^Y$.

Proposition 4.0.22. *Write $\mathfrak{g} = \operatorname{Lie} T$. Let $M^Z := \{p \in M \mid Z^\sharp(p) = 0\}$ and let*

$$M^Y = M^T = \{p \in M \mid Y^\sharp(p) = 0\}.$$

Then the set

$$D^Y := \{Z \in \mathfrak{g} \mid M^Z = M^Y\} \cap U_{\mathbb{Q}}^Y$$

is dense in $U_{\mathbb{Q}}^Y$

Proof. This follows from the slice theorem below. □

Let $G \circlearrowright M$ be a smooth, proper (which is automatic when M is compact, but the following holds generally) action, and let $G_p = \mathrm{Stab}_G(p)$. For $g \in G_p$, the map $g : M \to M$ defined by $g(q) = g \cdot q$, $q \in M$, has differential $dg_p : T_pM \to T_pM$, and we call dg_p the *isotropy action* of G_p on T_pM. Since the action is proper, G_p is compact, and we get a G_p-invariant decomposition

$$T_pM = T_p\mathcal{O}_p \oplus N, \tag{10}$$

where N is normal to the orbit (and can be taken to be the orthogonal complement of $T_p\mathcal{O}_p$ with respect to a G_p-invariant metric, which can be chosen by averaging since G_p is compact).

Definition 4.0.23. The space $T_pM/T_p\mathcal{O}_p$ is called the *normal to the orbit at* p and has a natural G_p-action induced by the isotropy action on T_pM (i.e. given by the identification $T_pM/T_p\mathcal{O}_p \cong N$ from (10)).

Let $G \times_{G_p} N$ be the associated bundle over G/G_p, and let Q be some small disc around $0 \in N$ with respect to some G_p-invariant metric. Now the slice theorem, which we will not prove, says a tubular neighbourhood of an orbit of the action can be modelled in a concrete way.

Theorem 4.0.24 (Slice theorem, [CdS01], Thm. 23.5)**.** *There exists a G-equivariant diffeomorphism from the disc bundle $G \times_{G_p} Q$ onto a neighbourhood of the orbit $\mathcal{O}_p$ in M with restriction to the zero section $G \times_{G_p} \{0\} = G/G_p$ given by the map $G/G_p \to M$ mapping $[g \cdot G_p] \mapsto g \cdot p$.*

We can assume that G acts on M *effectively* in general, i.e. that $T_M = \bigcap_{p \in M} G_p = \{e\}$ (if not, replace G by G/T_M), and we also assume that M is connected. We now need the following technical lemma.

Lemma 4.0.25 ([DH82], Lemma 3.1)**.** *Let (M, ω, T, μ) be a Hamiltonian T-space. Then if M is connected, and $T \circlearrowright M$ effectively, then the set $\tilde{M}$ on which T acts freely, is the complement of a locally finite union of closed symplectic submanifolds of codimension ≥ 2 (locally finite in the sense that at every point, there is a neighbourhood intersecting only finitely many of the submanifolds). In particular, $\tilde{M}$ is open, connected, dense and $M \setminus \tilde{M}$ has (Lebesgue) measure 0. Also, $d\mu$ is surjective for any $p \in \tilde{M}$.*

Proof. For any closed subspace $S \subseteq T$, one can show that $M^S \subseteq M$ is a closed symplectic submanifold of M (this is a slightly involved exercise using the equivariant Darboux theorem). Using the slice theorem, take all M^S not open to get a locally finite collection of submanifolds of codimension ≥ 2 with complement $\tilde{M}$. If M^S is open, then since it is also closed, and since M is connected, we get that $M^S = M$. Thus $S = \{e\}$ since $\bigcap_{p \in M} G_p = \{e\}$. That $d\mu_p$ is surjective follows from the next lemma. □

Lemma 4.0.26. *Let (M, ω, G, μ) be a Hamiltonian G-space for a general group G. For any $p \in M$, the kernel and image of the map $d\mu_p : T_p \to T_{\mu(p)}\mathfrak{g}^*$ are given by*

$$\ker(d\mu_p) = (T_p\mathcal{O}_p)^\omega := \{w \in T_pM \mid \omega_p(v, w)\, \forall w \in T_p\mathcal{O}_p\}$$
$$im(d\mu_p) = \mathrm{ann}(\mathfrak{g}_p),$$

where $\mathfrak{g}_p = \mathrm{Lie}(G_p)$, and $\mathcal{O}_p = G \cdot p$ is the orbit through p.

Proof. By the moment map condition, $\iota_v \iota_{Y^\sharp} \omega = \iota_v d\mu^Y$: At p we have

$$\omega(Y^\sharp(p), v_p) = \iota_v d\langle \mu, Y \rangle|_p = \iota_v \langle d\mu, Y \rangle|_p = \langle d\mu_p(v_p), Y \rangle. \tag{11}$$

From this we see that $v_p \in \ker d\mu_p$ if and only if $\omega_p(Y^\sharp(p), v_p) = 0$ for all $Y \in \mathfrak{g}$. This implies that

$$\ker d\mu_p = \{Y^\sharp(p) \in T_p\mathcal{O}_p \mid Y \in \mathfrak{g}\}^\omega = (T_p\mathcal{O}_p)^\omega.$$

Secondly, by (11), for $Y \in \mathrm{Lie}(G_p) = \mathfrak{g}_p$ we have $\langle d\mu_p(v_p), Y\rangle = 0$ for all v, since $Y \in \mathfrak{g}_p$ if and only if $Y^\sharp(p) = 0$. Thus, $\mathrm{im} d\mu_p \subseteq \mathrm{ann}(\mathfrak{g}_p)$. The second claim of the Lemma now follows from the following dimension count:

$$\begin{aligned}\dim \mathrm{im} d\mu_p &= \dim M - \dim \ker d\mu_p = \dim M - \dim(T_p\mathcal{O}_p)^\omega \\ &= \dim T_p\mathcal{O}_p = \dim G - \dim G_p = \dim \mathfrak{g} - \dim \mathfrak{g}_p \\ &= \dim(\mathrm{ann}\, \mathfrak{g}_p),\end{aligned}$$

where we used the rank-nullity theorem, the first result, the non-degeneracy of ω, the orbit-stabilizer theorem, and definitions respectively. □

Thus, if G acts freely at p, then $\dim \mathfrak{g}_p = 0$, and so $\dim \mathrm{im} d\mu_p = \dim(\mathrm{ann}\, \mathfrak{g}_p) = \dim \mathfrak{g}^*$, so $d\mu_p$ is onto.

Consider again the integral

$$I_{\mu,\beta}(t) = \int_M e^{it\mu}\beta = \left(\frac{2\pi}{t}\right)^n \sum_{p_l \in \mathrm{Crit}_\mu} F_t(p_l) + s(t).$$

Since we are assuming that $\mathrm{Crit}\,\mu$ consists of isolated points, we focus on the regular values for μ (i.e. those where $d\mu_p \neq 0$) and the behaviour of the integral $I_{\mu,\beta}(t)$ for such values. Let

$$\xi_0 \in (\mathrm{Lie}\,\mathrm{U}(1))^*_{\mathrm{reg}} = \{\xi \in (\mathrm{Lie}\,\mathrm{U}(1))^* \mid \mu^{-1}(\xi) \subseteq M_{\mathrm{reg}}\} =: \mathbb{R}^*_{\mathrm{reg}},$$

where $M_{\mathrm{reg}} = \{p \in M \mid d\mu_p \neq 0\}$. We want to relate the integral $I_{\mu,\beta}(t)$ to the integral over an open subset $U_{\xi_0} \subseteq \mathbb{R}^*_{\mathrm{reg}}$ containing ξ_0, instead of one over $\mu^{-1}(U_{\xi_0})$. In order to do so, we need to construct a new measure on $(\mathrm{Lie}\,\mathrm{U}(1))^*$, but first we recall some definitions from measure theory.

Definition 4.0.27. A *σ-algebra* of sets on a topological space $X \neq 0$ is a nonempty collection $\mathcal{C}$ of subsets of X such that $\sum_{j\in\mathbb{Z}} E_j \in \mathcal{C}$ for $E_j \in \mathcal{C}$, and if $E \in \mathcal{C}$, then $E^c = X \setminus E \in \mathcal{C}$.

Definition 4.0.28. The σ-algebra $\mathcal{B}_X$ generated by the family of open subsets of a topological space is called the *Borel σ-algebra* on X.

Definition 4.0.29. Given a symplectic manifold (M, ω), the *Liouville measure* of a Borel set $U \in \mathcal{B}_M$ is

$$\beta_\omega(U) := \int_U \frac{\omega^n}{n!}.$$

Here, we implicitly restrict to measurable sets, as β_ω might be infinite.

Definition 4.0.30. Let (M, ω, G, μ) be a Hamiltonian G-space, such that $\mu : M \to \mathfrak{g}^*$ is proper. The *Duistermaat–Heckman measure* m_{DH} on $\mathfrak{g}^*$ is the pushforward of the Liouville measure β_ω by μ, i.e.

$$m_{\mathrm{DH}}(U) := (\mu_*\beta_\omega)(U) := \int_{\mu^{-1}(U)} \frac{\omega^n}{n!}$$

for any subset $U \subseteq \mathfrak{g}^*$ such that $\mu^{-1}(U)$ is measurable.

In this definition, properness is assumed to assure that compact sets are measurable. We write $m_{\mathrm{DH}}(U) = \int_U dm_{\mathrm{DH}}$.

Definition 4.0.31. For $h \in C^\infty_c(\mathfrak{g}^*)$, define

$$\int_{\mathfrak{g}^*} h\, dm_{\mathrm{DH}} = \int_M (h \circ \mu)\beta.$$

Let $d\xi$ denote the usual Lebesgue (or Euclidean) measure on $\mathfrak{g}^*$.

Definition 4.0.32. A measure $\nu : \mathcal{C} \to \mathbb{R}^{\geq 0}$ for a σ-algebra $(X, \mathcal{C})$ is called a *σ-finite measure*, if $X = \bigcup_{j\in\mathbb{Z}} E_j$ for $E_j \in \mathcal{C}$ with $\nu(E_j) < \infty$ for all $j \in \mathbb{Z}$.

Definition 4.0.33. If $\nu, \tilde{\nu}$ are measures on a σ-algebra $(X, \mathcal{C})$, we say that ν is *absolutely continuous* with respect to $\tilde{\nu}$ and write $\nu \ll \tilde{\nu}$ if $\tilde{\nu}(E) = 0$ implies that $\nu(E) = 0$ for $E \in \mathcal{C}$.

10th lecture, September 26th 2011

Definition 4.0.34. Let $X \neq \emptyset$ be a set and $\mathcal{C}$ a σ-algebra on X. Then $(X, \mathcal{C})$ is called a *measurable space* . If $\nu$is a measure on $(X, \mathcal{C})$, then $(X, \mathcal{C}, \nu)$ is called a *measure space* .

Definition 4.0.35. If $(X, \mathcal{C}), (\tilde{X}, \tilde{\mathcal{C}})$ are measurable spaces, then a function $f : X \to \tilde{X}$ is called a *measurable function* if $f^{-1}(E) \in \mathcal{C}$ for all $E \in \tilde{\mathcal{C}}$.

Definition 4.0.36. Let $(X, \mathcal{C})$ be a measurable space, and consider the measurable space $(\mathbb{C}, \mathcal{B}_{\mathbb{C}})$ consisting of the Borel σ-algebra on $\mathbb{C}$. Then a measurable function $\varphi : X \to \mathbb{C}$ is called a *simple function* if $\varphi(X) \subseteq \mathbb{C}$ is finite, say $\varphi(X) = \{z_1, \dots, z_n\}$. Let

$$\chi_E := \begin{cases} 1, & x \in E \\ 0, & x \in E^c \end{cases}$$

be the *characteristic function* on $E \subseteq X$. Then the *standard representation* of a simple function $\varphi : X \to \mathbb{C}$, $\varphi(X) = \{z_1, \dots, z_n\}$ is

$$\varphi = \sum_{j=1}^{n} z_j \chi_{F_j},$$

where $F_j := \varphi^{-1}(z_j) \subseteq X$.

Definition 4.0.37. Let $(X, \mathcal{C}, \nu)$ be a measure space, and $\varphi : X \to \mathbb{R}^{\geq 0}$ a simple function. The *integral* of φ with respect to ν on $A \subseteq X$ is

$$\int_A \varphi \, d\nu := \sum_{j=1}^{n} a_j \nu(F_j \cap A).$$

For any measurable function $f : X \to \mathbb{R}^{\geq 0}$, we define

$$\int_A f \, d\nu := \sup\{\int_A \varphi \, d\nu \mid 0 \leq \varphi \leq f \text{ on } A, \, \varphi \text{ simple}\},$$

where we write $\varphi \leq f$ on A, when $\varphi(x) \leq f(x)$ for all $x \in A$.

We extend integration to $f : X \to \mathbb{R}$ by writing $f = f^+ - f^-$ for $f^{\pm} : X \to \mathbb{R}^{\geq 0}$ and let

$$\int_A f \, d\nu := \int_A f^+ \, d\nu - \int_A f^- \, d\nu.$$

Further, we extend to functions $f : X \to \mathbb{C}$ by writing $f = \operatorname{Re} f + i \operatorname{Im} f$, where $\operatorname{Re} f, \operatorname{Im} f : X \to \mathbb{R}$, and we let

$$\int_A f \, d\nu := \int_A \operatorname{Re} f \, d\nu + i \int_A \operatorname{Im} f \, d\nu.$$

Definition 4.0.38. Given $(X, \mathcal{C}, \nu)$ be a measure space, a function $f : X \to \mathbb{C}$ is called *ν-integrable* on A if $\int_A \operatorname{Re} f \, d\nu$ and $\int_A \operatorname{Im} f \, d\nu$ are finite.

Theorem 4.0.39 (Lebesgue–Radon–Nikodym, [Fol99], Thm. 3.8)**.** *If $\nu, \tilde{\nu}$ are σ-finite measures on $(X, \mathcal{C})$ such that $\nu \ll \tilde{\nu}$, then there exists $f : X \to \mathbb{R}$ a $\tilde{\nu}$-integrable function such that*

$$\int_U d\nu = \int_U f \, d\tilde{\nu}$$

for $U \in \mathcal{C}$.

Definition 4.0.40. The function f in the Lebesgue–Radon–Nikodym theorem is called the *Radon–Nikodym derivative* of ν with respect to $\tilde{\nu}$, and we write

$$f =: \frac{d\nu}{d\tilde{\nu}} := \{g : X \to \mathbb{R} \mid g = f, \ \tilde{\nu}\text{-almost everywhere } \}.$$

Recall here that we say that $g = f$ *$\tilde{\nu}$-almost everywhere* if $g(x) = f(x)$ for all $x \in X \setminus E$ for some $E \in \mathcal{C}$ such that $\tilde{\nu}(E) = 0$.

Proposition 4.0.41 ([Fol99], Prop. 3.9)**.** *If ν and $\tilde{\nu}$ are σ-finite measures on $(X, \mathcal{C})$, such that $\nu \ll \tilde{\nu}$ and g is ν-integrable on X, then $g\left(\frac{d\nu}{d\tilde{\nu}}\right)$ is $\tilde{\nu}$-integrable on X, and*

$$\int_U g \, d\nu = \int_U \left(g \frac{d\nu}{d\tilde{\nu}}\right) d\tilde{\nu}$$

for all $U \in \mathcal{C}$.

Theorem 4.0.42 (The Duistermaat–Heckman theorem, version 4, [DH82], 3.3)**.** *Let (M, ω, T, μ) be a Hamiltonian T-space with effective action, $\dim M = 2n$, $\dim T = d$. The Radon–Nikodym derivative of m_{DH} with respect to the Lebesgue measure $d\xi$ on $(\operatorname{Lie} T)^*$,*

$$f = \frac{dm_{\mathrm{DH}}}{d\xi},$$

is a polynomial of degree less than or equal to $n - d$ on each connected component of $\mathfrak{g}^*_{\mathrm{reg}} = \mathbb{R}^{*d}_{\mathrm{reg}}$ of μ.

We now see that the degree $n - d$ is actually non-negative.

Proposition 4.0.43. *Let (M, ω, T, μ) be a Hamiltonian T-space, $\dim M = 2n$, with effective action, then $n \geq d$, where $d = \dim T$.*

Proof. One uses the fact that for an effective action, there exists at least orbit $\mathcal{O}_p$ with $\dim \mathcal{O}_p = d$. See [Bre72]. □

Example 4.0.44. Consider the height function example $(S^2, \omega_{\mathrm{st}}, \mathrm{U}(1), \mu)$. Then $f(\xi) = a$ by Theorem 4.0.42 for some $a \in \mathbb{R}$ on $\mathbb{R}^*_{\mathrm{reg}}$. Also

$$\int_{S^2} \beta = 4\pi = \int_{(-1,1)} dm_{\mathrm{DH}} = \int_{(-1,1)} f(\xi) \, d\xi = \int_{(-1,1)} a \, d\xi,$$

so $a = 2\pi$, and thus

$$\frac{dm_{\mathrm{DH}}}{d\xi} = 2\pi$$

on $(-1, 1)$. This implies that

$$\int_{\mu^{-1}(c,d)} \beta = (d - c)2\pi,$$

for $-1 < c < d < 1$ so the spherical area between two horizontal circles (see Fig. 4) depends only on the vertical distance between them. This result was known to Archimedes around 230 BC.

Our next goal is to prove Thm. 4.0.42. In order to do so, we need to understand how the symplectic form ω changes as $\xi \in \mathfrak{g}^*_{\mathrm{reg}}$ varies, i.e. since we are interested in

$$\int_{\mu^{-1}(U_{\xi_0})} \beta = \int_{U_{\xi_0}} f(\xi) d\xi,$$

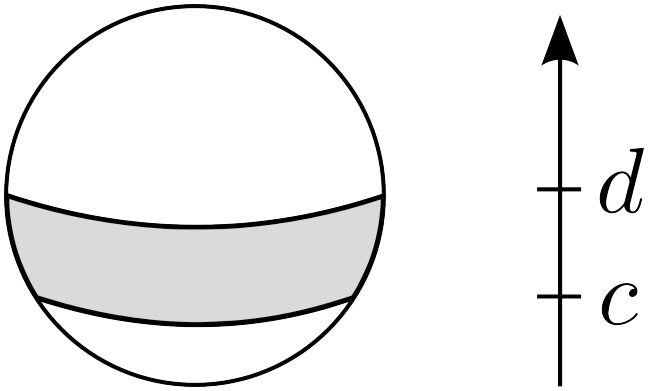

Figure 4: The area between two horizontal circles on a sphere.

given $\xi \in \mathfrak{g}^*_{\text{reg}}$, how does ω depend on $\mu^{-1}(\xi)$ for $\xi \in U_{\xi_0} \subseteq \mathfrak{g}^*_{\text{reg}}$. This leads to the whole new subject of symplectic reduction, which we will state in the most general sense.

Observe that if $\xi \in \mathfrak{g}^*_{\text{reg}}$, then $\mu^{-1}(\xi) := Z_\xi \subseteq M$ is a submanifold of dimension $\dim M - \dim G$ (which essentially boils down to the implicit function theorem). Let $i_\xi : Z_\xi \hookrightarrow M$ be the inclusion map. We want to look at $i^*\omega$ on Z_ξ. This will tell us how the symplectic form will vary with ξ. Clearly, $T_p Z_\xi = \ker d\mu_p$. Also, from a previous theorem, $\ker d\mu_p = (T_p\mathcal{O}_p)^\omega$. If $T_p\mathcal{O}_p \subseteq \ker d\mu_p$, we could restrict ω to the quotient $(T_p\mathcal{O}_p)^\omega / T_p\mathcal{O}_p$ and get a symplectic form on $Z_\xi/G|_p$.

Definition 4.0.45. Given a symplectic vector space (V, ω), a subspace $Q \subseteq V$ is called

1. *symplectic* if $\omega|_Q$ is non-degenerate,
2. *isotropic* if $Q \subseteq Q^\omega$ (which implies that $\dim Q \leq \frac{1}{2}\dim V$),
3. *coisotropic* if $Q^\omega \subset Q$ (which implies that $\dim Q \geq \frac{1}{2}\dim V$), or
4. *Lagrangian* if Q is isotropic $\dim Q = \frac{1}{2}\dim V$ (or we could say Q coisotropic and $Q = Q^\omega$).

Lemma 4.0.46. *Let (V, ω) be a symplectic vector space and $I \subseteq V$ an isotropic subspace of V (i.e. $I \subseteq I^\omega$). Then ω induces a canonical symplectic form $\tilde{\omega}$ on I^ω/I.*

Proof. Let $u, v \in I^\omega$ and $[u], [v] \in I^\omega/I$. Define $\tilde{\omega}([u], [v]) := \omega(u, v)$. This is well-defined since by definition of I^ω, for all $i, j \in I$ we have

$$\omega(u + i, v + j) = \omega(u, v) + \omega(u, j) + \omega(i, v) + \omega(i, j) = \omega(u, v).$$

The form $\tilde{\omega}$ is non-degenerate since if $u \in I^\omega$ such that $\omega(u, v) = 0$ for all $v \in I^\omega$, then $u \in (I^\omega)^\omega = I$, which implies that $[u] = 0 \in I^\omega/I$. □

Note that $T_p\mathcal{O}_p \subseteq \ker d\mu_p$ if and only if $\mathcal{O}_p \subseteq Z_\xi$. But this is the same as having $g^*\mu(p) = \text{Ad}_g\, \mu(p) = \xi$ for all $g \in G$. At this point, there are several ways to proceed: We could restrict to $\xi \in (\mathfrak{g}^*)^G = \{\xi \in \mathfrak{g}^* \mid \text{Ad}_g\, \xi = \xi\}$ which would then solve our problem; i.e. we could take $\xi = 0$. We could also assume that G is abelian (which we will eventually do anyway), or we could replace G by $G_\xi := \{g \in G \mid \text{Ad}_g\, \xi = \xi\}$. We will do the last thing, since it is the most general.

Definition 4.0.47. Let (M, ω) be a symplectic manifold and $i : Z \to M$ an embedded submanifold. Then Z is called a *constant rank submanifold* , if the dimension of the kernel of $(i^*\omega)|_z$ is independent of $z \in Z$.

Exercise 4.0.48. Come up with some examples of submanifolds that are not constant rank.

Definition 4.0.49. A *submersion* is a smooth map $f : M \to B$ such that df_p is surjective for all $p \in M$.

Recall that inverse images of points in B under a submersion are embedded submanifolds in M.

Definition 4.0.50. Let $\pi : M \to B$ be a submersion. Define the space of *vertical vector fields* by

$$\mathcal{X}_{\text{vert}}(M) := \{v \in \Gamma(TM) \mid v_p \in \ker d\pi_p \ \forall p \in M\},$$

the space of *horizontal forms* by

$$\Omega_{\text{hor}}(M) := \{\alpha \in \Omega(M) \mid \iota_v \alpha = 0 \ \forall v \in \mathcal{X}_{\text{vert}}(M)\},$$

and the space of *basic forms* by

$$\Omega_{\text{basic}}(M) := \{\alpha \in \Omega(M) \mid \mathcal{L}_v \alpha = 0, \iota_v \alpha = 0 \ \forall v \in \mathcal{X}_{\text{vert}}(M)\}.$$

Definition 4.0.51. A submersion $f : M \to B$ is called a *fibration* if it is surjective and there exists a manifold F (called a standard fibre) such that for all $p \in M$, there exists an open neighbourhood $U \ni p$ and a diffeomorphism $\varphi : U \to f(U) \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & f(U) \times F \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ f(U) & \xrightarrow{\mathbf{1}} & f(U) \end{array}$$

Here π_1 is the projection onto the first factor.

Recall that for any fibration $\pi : M \to B$, the pullback defines an isomorphism

$$\pi^* : \Omega^r(B) \to \Omega^r_{\text{basic}}(M).$$

One way to see this is to verify it in bundle charts $\pi : U \times F \to U$.

Definition 4.0.52. A vector subbundle $D \subseteq TM$ of rank r is called a *distribution* on M of rank r. An embedded submanifold $i : Z \hookrightarrow M$ is called an *integral submanifold* for D if $TZ = D|_Z$. A distribution D of rank r is called *integrable* if for all $p \in M$ there exists and r-dimensional integral submanifold of D through p. Integrable distributions are also called *foliations* .

One can show that $D \subseteq TM$ is integrable if very $p \in M$ has a neighbourhood U and a submersion $f_p : U \to \mathbb{R}^{m-r}$ such that $D|_U = \ker(df)$.

Definition 4.0.53. Given a rank r foliation D on M, the r-dimensional submanifold at p is called the *leaf of D through p* .

Note that the leaves of a foliation form a partition of M into disjoint submanifolds. Observe also that if a distribution D is integrable, then for any $v_1, v_2 \in \Gamma(D)$, the Lie bracket $[v_1, v_2]$ is also in $\Gamma(D)$. The converse is also true:

Theorem 4.0.54 (Frobenius)**.** *A distribution $D \subseteq TM$ is integrable if and only if for all $v_1, v_2 \in \Gamma(D)$, the Lie bracket $[v_1, v_2]$ is also in $\Gamma(D)$.*

Proposition 4.0.55. *Let Z be a manifold, and $\sigma \in \Omega^2(Z)$ a closed form of constant rank. Then $\ker(\sigma) \subseteq TZ$ is integrable.*

Proof. By Frobenius' theorem, if $W_1, W_2 \in \Gamma(TZ)$ such that $\iota_{W_j}\sigma = 0$, then since $d\sigma = 0$, we have

$$\mathcal{L}_{W_j}\sigma = d\iota_{W_j}\sigma + \iota_{W_j} d\sigma = 0,$$

and hence, using that in general $[\mathcal{L}_{W_1}, \iota_{W_2}]\alpha = \iota_{[W_1,W_2]}$, we find

$$\iota_{[W_1,W_2]}\sigma = \mathcal{L}_{W_1}\iota_{W_2}\sigma - \iota_{W_2}\mathcal{L}_{W_1}\sigma = 0.$$

□

Definition 4.0.56. Given a manifold Z, and $\sigma \in \Omega^2(Z)$ closed of constant rank, call $\ker(\sigma)$ the *null foliation* of (Z, σ).

Definition 4.0.57. We say that the null foliation of (Z, σ) is *fibrating* , if there exists a fibration $\pi : Z \to B$ such that $\ker(\sigma) = \ker(d\pi)$.

Proposition 4.0.58. *If the null foliation of (Z, σ) is fibrating with $\pi : Z \to B$ a fibration, then there exists a non-degenerate form $\omega_B \in \Omega^2(B)$ such that $\pi^*\omega_B = \sigma$.*

11th lecture, September 30th 2011

4.1 Background on orbifolds

Definition 4.1.1. Let $|B|$ be a paracompact Hausdorff space. An *orbifold chart* (also called a *local uniformizing system*) on $|B|$ is a triple $(\tilde{U}, \Gamma, \varphi)$ where $\tilde{U} \subseteq \mathbb{R}^b$ is a connected open subset, Γ is a finite group acting on $\tilde{U}$ and $\varphi : \tilde{U} \to U$ a continuous map onto an open set $U \subseteq |B|$ such that $\varphi \circ \gamma = \varphi$ for all $\gamma \in \Gamma$ and the induced natural map of $\tilde{U}/\Gamma \to U$ is a homeomorphism. An *injection* or *embedding* between charts $(\tilde{U}, \Gamma, \varphi), (\tilde{U}', \Gamma', \varphi')$ is a smooth (or holomorphic in the complex category) embedding $\lambda : \tilde{U} \to \tilde{U}'$ such that $\varphi' \circ \lambda = \varphi$. An *orbifold atlas* on $|B|$ is a family $\mathcal{U} = \{\tilde{U}_j, \Gamma_j, \varphi_j\}_{j \in I}$ of orbifold charts such that:

(i) $|B| = \bigcup_{j \in I} \varphi_j(\tilde{U}_j)$,

(ii) Given $(\tilde{U}_i, \Gamma_i, \varphi_j)$ and $(\tilde{U}_j, \Gamma_j, \varphi_j)$ in $\mathcal{U}$ with $U_i = \varphi_i(\tilde{U}_i), U_j = \varphi_j(\tilde{U}_j)$ and $p \in U_i \cap U_j$, there exists $U_l \ni p$ open, and a chart $(\tilde{U}_l, \Gamma_l, \varphi_l)$ with injections $\lambda_{il} : (\tilde{U}_l, \Gamma_l, \varphi_l) \to (\tilde{U}_i, \Gamma_i, \varphi_i)$ and $\lambda_{jl} : (\tilde{U}_l, \Gamma_l, \varphi_l) \to (\tilde{U}_j, \Gamma_j, \varphi_j)$.

Definition 4.1.2. An orbifold atlas $\mathcal{U}$ is called a *refinement* of an atlas $\mathcal{V}$ if there exists an injection of every chart of $\mathcal{U}$ into some chart of $\mathcal{V}$.

Two orbifold atlases are called *equivalent* if they have a common refinement.

Definition 4.1.3. A smooth (resp. complex) *orbifold* (also called a *V-manifold*) is a paracompact Hausdorff space $|B|$ with an equivalence class of smooth (resp. holomorphic) orbifold atlases, and we write $B = (|B|, \mathcal{U})$.

Remark 4.1.4. A number of different definitions or orbifolds exist in different contexts. One reason is that if we were to consider category of orbifolds, even if the objects make sense, morphisms will not, and one can prove that the notion of a 2-category becomes necessary. This will not be important for our purpose.

Definition 4.1.5. The finite groups Γ_i are called *local uniformizing groups* for an orbifold $B = (|B|, \mathcal{U})$

Note that we do not necessarily require that the Γ_i act effectively although some people will.

Example 4.1.6. Let (M, ω, T, μ) be a Hamiltonian T-space, $\dim M = m = 2n$, $\dim T = d$, with effective action, and write $G = T$. For any $\xi \in \mathfrak{g}^*_{\text{reg}}$, the level set $\mu^{-1}(\xi)$ is preserved b y the action, and $d\mu_p$ is surjective for all $p \in \mu^{-1}(\xi)$. Recall that this is equivalent to $\mathfrak{g}_p = 0$ for all $p \in \mu^{-1}(\xi)$, which again is equivalent to $|G_p| < \infty$. Write $\Gamma_p = G_p$.

We will identify Γ_p as local uniformizing groups for $\mu^{-1}(\xi)/G$ for some open sets $\tilde{U} \subseteq \mathbb{R}^b$. To do this we use the slice theorem which says that there exists a G-equivariant diffeomorphism $\psi : V \to G_{G_p}Q$ for some open set $V \subseteq \mu^{-1}(\xi)$ such that $\mathcal{O}_p \subseteq V$. Thus, we can identify $\mu^{-1}(\xi)/G \cong (G \times_{\Gamma_p} Q)/G \cong Q/\Gamma_p$.

Let $\tilde{U}_p \subseteq Q$ be some Γ_p-invariant open subset and $\varphi_p : \tilde{U}_p \to \tilde{U}_p/\Gamma_p =: U_p$ be the quotient map. We claim that $\mathcal{U} = \{(\tilde{U}_p, \Gamma_p, \varphi_p)\}_{p \in \mu^{-1}(\xi)}$ forms an orbifold atlas on $|B| := \mu^{-1}(\xi)/G$ and taking the equivalence class of $\mathcal{U}$, $(\mu^{-1}(\xi)/G, \mathcal{U})$ is a smooth orbifold (this is an exercise).

Example 4.1.7. Let $\mathrm{U}(1)$ act on $\mathbb{C}^2$ by $e^{i\theta}(z_1, z_2) = (e^{is\theta}z_1, e^{it\theta}z_2)$ for $2 \leq s, t \in \mathbb{N}$. Assume $(s, t) = 1$. Then if

$$\begin{aligned} S_1 &:= \{(z_1, z_2) \in \mathbb{C} \mid z_2 = 0\}, \\ S_2 &:= \{(z_1, z_2) \in \mathbb{C} \mid z_1 = 0\}, \\ S &:= \mathbb{C}^2 \setminus (S_1 \cup S_2). \end{aligned}$$

Then $\mathrm{Stab}_{\mathrm{U}(1)} S_1 \cong \mathbb{Z}_s$, $\mathrm{Stab}_{\mathrm{U}(1)} S_2 \cong \mathbb{Z}_t$, and $\mathrm{Stab}_{\mathrm{U}(1)} S = \{e\}$.

Let $\mu : \mathbb{C}^2 \to \mathbb{R}$ be the unique moment map with $\mu(0) = 0$ for this action, i.e.

$$\mu(z_1, z_2) := \frac{1}{2}(s|z_1|^2 + t|z_2|^2).$$

Then for any $0 < \xi \in \mathrm{Lie}(\mathrm{U}(1))^*$ is a regular value, and the reduced space $\mu^{-1}(\xi)/\mathrm{U}(1)$ called the *football orbifold* . It has two orbifold points, where $\Gamma_i \neq \{e\}$, and are exactly the $\mathbb{Z}_s, \mathbb{Z}_t$ as above.

Theorem 4.1.8 ([Sat57])**.** *$B = (|B|, \mathcal{U})$ is an orbifold with effective action if and only if $|M|/G$ for some manifold M and some compact Lie group G with a locally free effective action on M.*

We can work with orbifold just as if they were manifolds: By [Sat57], all usual notions of differential geometry on smooth manifolds are well defined for smooth orbifolds, i.e. (co-)tangent bundles, forms, general bundles, connections, curvature, integration, Riemannian metrics, etc.. We assume this throughout and make implicit orbifold notation when working with locally free actions.

Example 4.1.9. Let $B = (|B|, \mathcal{U})$ be an orbifold. Define $\alpha \in \Omega^l_{\mathrm{orb}}(B, \mathbb{R})$ to be a collection of forms $\tilde{\alpha}_j \in \Omega^l(\tilde{U}_j, \mathbb{R})$ such that $\tilde{\alpha}_j$ are Γ_j-invariant and $\lambda_{jk}^* \tilde{\omega}_j = \tilde{\omega}_k$ for all injections $\lambda_{jk} : (\tilde{U}_k, \Gamma_k, \varphi_k) \to (\tilde{U}_j, \Gamma_j, \varphi_j)$. Let (ψ_j) be a partition of unity subordinate to $\{U_j\}$ and define integration of $\alpha \in \Omega^b_{\mathrm{orb}}(B, \mathbb{R})$ on b-dimensional B by

$$\int_B \alpha := \sum_j \frac{1}{|\Gamma_j|} \int_{\tilde{U}_j} \varphi^*(\psi_j)\tilde{\alpha}_j.$$

Note that exterior differentiation also makes sense on orbifolds.

Definition 4.1.10. A *symplectic orbifold* is a pair (B, ω) where $\omega \in \Omega^2_{\mathrm{orb}}(B; \mathbb{R})$ is closed and non-degenerate on $B = (|B|, \mathcal{U})$.

4.2 Symplectic reduction

In our definition of fibrating null foliation for a pair (Z, σ) where $\sigma \in \Omega^2(Z)$ is closed. We will allow the foliation to be a submersion $\pi : Z \to B$, where $B = (|B|, \mathcal{U})$ is an orbifold. We will take Z to be a manifold.

Theorem 4.2.1 (Reduction on constant rank submanifolds)**.** *If the null foliation of (Z, σ) is fibrating with fibration $\pi : Z \to B$, then there exists a closed non-degenerate form $\omega_B \in \Omega^2_{\mathrm{orb}}(B)$ such that $\pi^*\omega_B = \sigma$.*

Proof. Let $V \in \mathcal{X}_{\mathrm{vert}}(Z)$ relative to the fibration π. By the definition of "fibrating", $\ker \sigma = \ker d\pi$, so since $V \in \ker d\pi$, we have $V \in \ker \sigma$, and so $\iota_V \sigma = 0$. Also since $d\sigma = 0$, we have $\mathcal{L}_V \sigma = d\iota_V \sigma + \iota_V d\sigma = 0$. So overall, $\sigma \in \Omega^2_{\mathrm{basic}}(Z)$. Since $\pi^* : \Omega^2_{\mathrm{orb}}(B) \to \Omega^2_{\mathrm{basic}}(Z)$ is an isomorphism, there exists $\omega_B \in \Omega^2_{\mathrm{orb}}(B)$ such that $\pi^*\omega_B = \sigma$. All we need to check is that ω_B is closed and non-degenerate. Since $d\pi^*\omega_B = \pi^* d\omega_B = d\sigma = 0$, so $d\omega_B = 0$, and finally non-degeneracy follows from the condition that $\ker \sigma = \ker d\pi$ (we leave this as an exercise). □

Definition 4.2.2. Assume (Z, σ) is fibrating. Then the symplectic orbifold (B, ω_B) in the fibration $\pi : Z \to B$ is called the *symplectic reduction* of (Z, σ).

In our applications, $i : Z \hookrightarrow (M, \omega)$ will be a constant rank submanifold, and $\sigma = i^*\omega$ on Z.

Theorem 4.2.3. *Let (M, ω, G, μ) be a Hamiltonian G-space with (proper and effective) group action. Then $\xi \in \mathfrak{g}^*_{\mathrm{reg}}$ if and only if G_p is discrete for all $p \in \mu^{-1}(\xi)$. Also, $\mu^{-1}(\xi) \subseteq M$ is a constant rank submanifold, and the leaf of the null foliation through $p \in \mu^{-1}(\xi)$ is the orbit*

$$\mathcal{O}_{\xi,p} := G_\xi \cdot p = \{g \cdot p \in M \mid g \in G_\xi\}.$$

Proof. Since $\xi \in \mathfrak{g}^*_{\mathrm{reg}}$, by definition $d\mu_p$ is onto. By a previous result, $\operatorname{Im} d\mu_p = \operatorname{ann}(\mathfrak{g}_p) = \mathfrak{g}^*$. Thus $\mathfrak{g}_p = \{0\}$. Thus, $G_p \subseteq G_\xi$ is discrete. Let $Z_\xi := \mu^{-1}(\xi)$ and let $i_\xi : Z_\xi \hookrightarrow M$ be the inclusion. Then since $T_p Z_\xi = \ker d\mu_p$, $\ker d\mu_p = (T_p \mathcal{O}_p)^\omega$, we find

$$\begin{aligned}\ker i_\xi^* \omega|_p &= T_p Z_\xi \cap (T_p Z_\xi)^\omega = T_p Z_\xi \cap T_p \mathcal{O}_p = \ker d\mu_p \cap T_p \mathcal{O}_p \\ &= T_p(\mathcal{O}_{\xi,p}).\end{aligned}$$

In the last equality, we have used that

$$T_p(\mathcal{O}_{\xi,p}) = \{Y_p^\sharp \in T_p M \mid Y \in \operatorname{Lie} G_\xi\}.$$

This proves the theorem. □

Theorem 4.2.4 (Marsden–Weinstein–Mayer)**.** *Let (M, ω, G, μ) be a Hamiltonian G-space, and assume that the foliation of $\mu^{-1}(\xi)$ by G_ξ-orbits is a fibration (e.g. this is guaranteed if G_ξ is compact), and let $\pi_\xi : Z_\xi \to Z_\xi/G_\xi =: M_\xi$ be the quotient map onto the orbit space M_ξ, where Z_ξ is as in the proof of the previous Theorem. Then there exists a unique symplectic form ω_ξ on M_ξ such that $i_\xi^* \omega = \pi_\xi^* \omega_\xi$.*

Proof. Apply Theorem 4.2.1 to the null foliation of $(Z_\xi, i_\xi^* \omega)$. □

Recall that the reason we went through all of this was to see how $i_\xi^* \omega$ varies in $U_\delta \subseteq \mathfrak{g}^*_{\mathrm{reg}}$ for $\delta \in \mathfrak{g}^*_{\mathrm{reg}}$. In order to see this, we would like to observe that our problem can be addressed with $\delta \in \mathfrak{g}^*$ replaced by 0 for a "shifted" moment map, using the coadjoint orbit $\mathcal{O}_\delta := G \cdot \delta \subseteq \mathfrak{g}^*$.

Theorem 4.2.5 (Kirillov–Kostant–Sorian)**.** *Let $\mathcal{O}_\delta \subseteq \mathfrak{g}^*$ be an orbit for the coadjoint action of G on $\mathfrak{g}^*$. There exists a unique symplectic structure on $\mathcal{O}_\delta$, denoted $(\mathcal{O}_\delta, \omega_{\mathrm{KKS}})$ such that $G \circlearrowright \mathcal{O}_\delta$ is Hamiltonian and the moment map $\mu_{\mathrm{KKS}} = i : \mathcal{O}_\delta \hookrightarrow \mathfrak{g}^*$ is the inclusion.*

Recall that

$$\omega_{\mathrm{KKS}}|_\xi(\eta_1^\sharp, \eta_2^\sharp) = -\langle \xi, [\eta_1, \eta_2] \rangle,$$

where $\eta_1, \eta_2 \in \mathfrak{g}$ generate $\eta_1^\sharp, \eta_2^\sharp$ at ξ via the coadjoint action of G on $\mathfrak{g}^*$.

12th lecture, October 3rd 2011

The reduced space at 0 is denoted $M_0 =: M/\!\!/G := \mu^{-1}(0)/G$ for a Hamiltonian G-space (M, ω, G, μ). Recall the following Lemma

Lemma 4.2.6 (Product Hamiltonian structure)**.** *Let $(M_j, \omega_j, G, \mu_j)$, $j = 1, 2$ be two Hamiltonian G-spaces. Let G act on $M_1 \times M_2$ diagonally and let $\tilde{\omega} = \mathrm{pr}_1^* \omega_1 + \mathrm{pr}_2^* \omega^* 2$ be a symplectic form on $M_1 \times M_2$, where $\mathrm{pr}_j : M_1 \times M_2 \to M_j$ be the projections. Define $\tilde{\mu} : M_1 \times M_2 \to \mathfrak{g}^*$ by*

$$\tilde{\mu}(p_1, p_2) := \mu_1(p_1) + \mu_2(p_2).$$

Then $(M_1 \times M_2, \tilde{\omega}, G, \tilde{\mu})$ is a Hamiltonian G-space.

The proof is left as an exercise.

Let $\mathcal{O}_\delta^-$ denote $(\mathcal{O}_\delta, -\omega_{\mathrm{KKS}}, G, -\mu_{\mathrm{KKS}})$. Then the above lemma tells us that $M \times \mathcal{O}_\delta^-$ is a Hamiltonian G-space (for (M, ω, G, μ) a Hamiltonian G-space) with moment map $\tilde{\mu} = \tilde{\mu}_\delta : M \times \mathcal{O}_\delta^- \to \mathfrak{g}^*$ defined by

$$\tilde{\mu}(p, \xi) = \mu(p) - \xi.$$

The reason for considering $\mathcal{O}_\delta^-$ is to obtain the minus sign in this equation, as we are going to consider the pre-image of 0 under this moment map. Let $\tilde{\omega} = \tilde{\omega}_\delta$ denote the symplectic form on $M \times \mathcal{O}_\delta^-$.

Theorem 4.2.7 (Shift trick)**.** *Let (M, ω, G, μ) be a Hamiltonian G-space with proper group action. Let $\delta \in \mathfrak{g}^*_{\mathrm{reg}}$ for μ, and $\mathcal{O} = \mathcal{O}_\delta$ the coadjoint orbit through δ, let $\mathcal{O}^-$ be the corresponding KKS-space, and let $\tilde{\mu}$ be as above. Let $j : \mu^{-1}(\delta) \to \tilde{\mu}^{-1}(0)$ be the map $j(p) = (p, \delta)$, and let*

$$J : M_\delta := \mu^{-1}(\delta)/G_\delta \to (M \times \mathcal{O}^-)/\!\!/G := \tilde{\mu}^{-1}(0)/G$$

be the map induced by the following commutative diagram:

$$\begin{array}{ccc} \mu^{-1}(\delta) & \xrightarrow{\ j\ } & \tilde{\mu}^{-1}(0) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ M_\delta & \xrightarrow{\ J\ } & (M \times \mathcal{O}^-)/\!\!/G \end{array}$$

Then J is a symplectomorphism, so $M_\delta \cong (M \times \mathcal{O}^-)/\!\!/G$. Also the G_δ-action on $\mu^{-1}(\delta)$ is (locally) free if and only if the G-action on $\tilde{\mu}^{-1}(0)$ is locally free.

Proof. As a first technical observation, note that smoothness of j implies smoothness of J by a standard argument.

The map J is injective: Let $[p_1], [p_2] \in M_\delta$. Then by definition, $J([p_1]) = \tilde{\pi}(j(p_1))$ for any $p_1 \in \pi^{-1}([p_1])$. Now $\tilde{\pi}(j(p_1)) = \tilde{\pi}(p_1, \delta) = [p_1, \delta]$. Now if $J([p_1]) = [p_1, \delta] = [p_2, \delta] = J([p_2])$ then there exists $g \in G$ such that $(gp_1, \mathrm{Ad}_g\, \delta) = (p_2, \delta)$ (where here as always Ad_g denotes the coadjoint action). This implies that $g \cdot p_1 = p_2$ for $g \in G_\delta$ so $[p_1] = [p_2] \in M_\delta$.

The map J is surjective: Let $[p_1, \xi] \in \tilde{\mu}^{-1}(0)/G$. Then there exists an element $h \in G$ such that $\mathrm{Ad}_h\, \xi = \delta$ and $[p, \xi] = [h \cdot p, \delta]$. let $\tilde{p} = h \cdot p$ so

$$\mu(\tilde{p}) = \mu(h \cdot p) = \mathrm{Ad}_h\, \mu(p) = \mathrm{Ad}_h\, \xi = \delta.$$

Thus $\tilde{p}\mu^{-1}(\delta)$ and $J([p]) = [p, \xi]$.

The map J is symplectic: We want to show that $J^*\tilde{\omega} = \omega_\delta$. Since $\pi^* : \Omega^*(M_\delta) \to \Omega(\mu^{-1}(\delta))$ is an isomorphism, it suffices to show that $\pi^* J^* \tilde{\omega} = \pi^* \omega_\delta$. Since $J \circ \pi = \tilde{\pi} \circ j$, we find $\pi^* J^* \tilde{\omega} = j^* \tilde{\pi}^* \tilde{\omega}$. If $i : \tilde{\mu}^{-1}(0) \hookrightarrow M \times \mathcal{O}^-$ is the inclusion map, then by definition of the symplectic form from the Marsden–Weinstein–Mayer theorem, we have

$$\tilde{\pi}^* \tilde{\omega} = i^*(\mathrm{pr}_1^*\, \omega - \mathrm{pr}_2^*\, \omega_{\mathrm{KKS}}).$$

Let $I := i \circ j : \mu^{-1}(\delta) \hookrightarrow M \times \mathcal{O}^-$ be the map $I(p) = (p, \delta)$. We get

$$\pi^* J^* \tilde{\omega} = j^* i^* (\mathrm{pr}_1^*\, \omega - \mathrm{pr}_2^*\, \omega_{\mathrm{KKS}}) = I^* \,\mathrm{pr}_1^*\, \omega - I^* \,\mathrm{pr}_2^*\, \omega_{\mathrm{KKS}},$$

where $\mathrm{pr}_1 \circ I = \mathbf{1}_{\mu^{-1}(\delta)}$, and $\mathrm{pr}_2 \circ I = \delta$ is constant. So

$$\begin{aligned} I^* \,\mathrm{pr}_1^*\, \omega &= \omega|_{\mu^{-1}(\delta)} = i_\delta^* \omega, \\ I^* \,\mathrm{pr}_2^*\, \omega_{\mathrm{KKS}} &= 0. \end{aligned}$$

Hence $\pi^* J^* \tilde{\omega} = i_\delta^* = \pi^* \omega_\delta$.

Also, G_δ is (locally) free on $\mu^{-1}(\delta)$ if and only if G is (locally) free on $\tilde{\mu}^{-1}(0)$ since j is G_δ-equivariant and J is surjective. □

We consider now a slight generalization of Lemma 4.0.46:

Lemma 4.2.8. *Let (V,ω) be a symplectic vector space and $Q \subseteq V$ any subspace (not necessarily isotropic). Then ω induces canonical forms $\tilde{\omega}$ on both $Q^\omega/(Q^\omega \cap Q)$ and on $Q/(Q^\omega \cap Q)$. In both cases, the forms are defined by $\tilde{\omega}([u],[v]) = \omega(u,v)$ for $[u],[v] \in Q^\omega/Q^\omega \cap Q$ or $[u],[v] \in Q/Q^\omega \cap Q$.*

Proof. The proof is exactly analogous to that of Lemma 4.0.46. □

A vector bundle is called *symplectic* if every fibre is a symplectic vector space.

Definition 4.2.9. For any constant submanifold $i : Z \hookrightarrow M$ for (M,ω) symplectic, the *symplectic normal bundle* is the symplectic vector bundle

$$N(Z) := TZ^\omega/(TZ \cap TZ^\omega).$$

For a coisotropic submanifold Z (i.e. one with $TZ^\omega \subseteq TZ$), the symplectic normal bundle is 0. For an isotropic submanifold (i.e. one with $TZ \subseteq TZ^*\omega$) of dimension l, the symplectic normal bundle has rank $2(n-l)$, where $\dim M = m = 2n$.

The following theorem is due to Marle and extends earlier results of Weinstein (who studied the case Z Lagrangian or symplectic) and Gotay (who studied Z coisotropic).

Theorem 4.2.10 (Constant Rank Embedding Theorem)**.** *Let (M_j,ω_j), $j = 1,2$, be symplectic manifolds. Suppose $I_j : Z \hookrightarrow M_j$, $j = 1,2$, are constant rank embeddings with isomorphic symplectic normal bundles*

$$N_1(Z) = TZ^{\omega_1}/(TZ^{\omega_1} \cap TZ) \cong N_2(Z) = TZ^{\omega_2}/(TZ^{\omega_2} \cap TZ),$$

such that $I_1^*\omega_1 = I_2^*\omega_2$. Then there exist open sets $U_1 \supseteq I_1(Z)$, $U_2 \supseteq I_2(Z)$, and a diffeomorphism $\psi : U_1 \to U_2$ such that $\psi \circ I_1 = I_2$ and $\psi \circ \omega_2 = \omega_1$.

If G acts properly on Z and M_j, preserves ω_j, and I_j are G-equivariant, then U_j can be chosen G-invariant and ψ can be chosen G-equivariant.

Furthermore, if (M_j,ω_j,G_j,μ_j), $j = 1,2$, are Hamiltonian and $\mu_1 \circ I_1 = \mu_2 \circ I_2$, then $\mu_2 \circ \psi = \mu_1$.

Proof. First, let $\nu := TZ \cap TZ^\omega$, $E := TZ/\nu$, $F := TZ^\omega/\nu$, $G = \nu \oplus \nu^*$. We claim that as a symplectic vector bundle,

$$TM|_Z \cong E \oplus F \oplus G.$$

Here, the symplectic form on $G = \nu \oplus \nu^*$ is the one given by

$$\omega_{\nu\oplus\nu^*}(u + l^*, \tilde{u} + \tilde{l}^*) := l^*(\tilde{u}) - \tilde{l}^*(u),$$

where $u, \tilde{u} \in \nu_z$ and $l^*, \tilde{l}^* \in \nu_z^*$ for $z \in Z$. This claim establishes the first part of the theorem by the Darboux–Weinstein theorem: Indeed, writing ν_j, E_j, F_j, G_j in the notation above, if $I_2 : Z \hookrightarrow (M_2,\omega_2)$ is another constant rank embedding with $I_2^*\omega_2 = I_1^*\omega_1$ and $F_1 \cong F_2$. Observe that $\nu_1 = \nu_2 \subseteq TZ$ since $I_2^*\omega_2 = I_1^*\omega_1$ and thus $G_1 = G_2$ and $E_1 = E_2$. From this it follows that

$$TM_1|_Z \cong E_1 \oplus F_1 \oplus G_2 \cong E_2 \oplus F_2 \oplus G_2 \cong TM_2|_Z.$$

Now the Darboux–Weinstein theorem says that a neighbourhood of a submanifold Z in a symplectic manifold (M,ω) is symplectically determined by the symplectic vector bundle $TM|_Z$. Thus $TM_1|_Z \cong TM_2|_Z$ implies our theorem.

To prove the symplectic bundle isomorphism $TM|_Z \cong E \oplus F \oplus G$, let $J : TM \to TM$ be an almost complex structure, i.e. $J^2 = -\mathbf{1}$, adapted to the symplectic form ω, i.e. $\omega(J\cdot, J\cdot) = \omega(\cdot,\cdot)$, and let $g(\cdot,\cdot) :\equiv \omega(\cdot, J\cdot)$ be a positive definite metric. Such almost complex structures always exist.

We start by looking at TZ/ν inside TZ, i.e. $TZ \cap \nu^\perp$ is a symplectic subspace of TZ, and

$$\nu^\perp \cap TZ \xrightarrow{\cong} TZ/\nu, \quad a \mapsto [a]$$

is a symplectomorphism. Here $^\perp$ denotes orthogonal complement with respect to g, i.e. $TZ \cap \nu^\perp := \{a \in TZ \mid g(a,b) = 0 \, \forall b \in \nu\}$. Clearly, also $TZ = (\nu^\perp \cap TZ) \oplus \nu$, and $\omega(a^\perp, b) = 0$ for any $a^\perp \in \nu^\perp TZ, b \in \nu$.

We want to extend TZ by a subspace W so that $\nu \oplus W$ is symplectic and also $W \subseteq (\nu^\perp \cap TZ)^\omega$. Observe that $\omega(a^\perp, Jb) = g(a^\perp, b) = 0$ for any $a^\perp \in \nu^\perp \cap TZ$ and $b \in \nu$. Thus $J\nu \subseteq (\nu^\perp \cap TZ)^\omega$. Also $J\nu \cap \nu = \{0\}$ since

$$g(a, Jb) = \omega(a, J(Jb)) = \omega(a, -b) = 0$$

since $a, b \in \nu = TZ \cap TZ^\omega$. This implies that $J\nu \subseteq TZ^\perp$ and that $TZ^\perp \cap \nu = \{0\}$. Finally, $\nu \oplus J\nu$ is a symplectic subspace since $\omega(a, Jb) = g(a, b)$ is non-degenerate.

Let $\nu \cong \nu^*$ via $a \mapsto g(a, \cdot) =: a^* \in \nu^*$. Notice that

$$\omega(a, Jb) = g(a, b) = a^*(b) = b^*(a) = \omega_{\nu \oplus \nu^*}(b^*, a).$$

If $\varphi : J\nu \to \nu^*$ is the map $Jb \mapsto b^*$ then $\mathbf{1} \times \varphi : \nu \oplus J\nu \to \nu \oplus \nu^*$ is a symplectomorphism. Hence $TZ \oplus J\nu \cong E \oplus \nu \oplus \nu^*$.

Since $TZ \oplus J^\nu$ is a symplectic subbundle of $TM|_Z$, then also its complement $(TZ \oplus J\nu)^\omega$ is a symplectic subbundle, and by properties of symplectic subbundles,

$$TM|_Z = (TM \oplus J\nu) \oplus (TZ \oplus J\nu)^\omega.$$

Since $\nu \subseteq TZ$, then $\nu \cap (TZ \oplus J\nu)^\omega = \{0\}$. Also, that $\nu \subseteq TZ^\omega$ implies that $(TZ \oplus J\nu)^\omega \oplus \nu \subseteq TZ^\omega$. We claim that $(TZ \oplus J\nu)^\omega \oplus \nu = TZ^\omega$. We give a proof based on dimensions: We have

$$\begin{aligned} \dim(TZ \oplus J\nu)^\omega &= \dim M - \dim(TZ \oplus J\nu) = \dim M - \dim Z - \dim \nu \\ &= \dim TZ^\omega - \dim \nu. \end{aligned}$$

Hence, $\dim(TZ \oplus J\nu)^\omega + \dim \nu = \dim TZ^\omega$. Taking quotients by ν, we get an identification

$$(TZ \oplus J\nu)^\omega \cong TZ^\omega / \nu =: F.$$

Notice that all isomorphisms above have been symplectomorphisms, and overall we have $TM|_Z \cong E \oplus F \oplus G$, symplectically.

If (the group) G acts properly on M, we can the isomorphism $TM_Z \cong E \oplus F \oplus G$ equivariant by choosing a G-invariant almost complex structure J. The rest of the claim follows from the G-equivariant part of the Darboux–Weinstein theorem.

Finally, if $(M_j, \omega_j, G, \mu_j)$, $j = 1, 2$ are Hamiltonian and $\mu_1 \circ I_1 = \mu_2 \circ I_2$, where $I_j : Z \hookrightarrow M_j$ are constant rank embeddings, then $\psi^* \mu_2 = \mu_1$. This follows from the next lemma. □

13th lecture, October 7th 2011

Lemma 4.2.11 (Uniqueness of moment maps)**.** *If $\mu_1, \mu_2 : M \to \mathfrak{g}^*$ are two moment maps for the same group action $G \circlearrowright M$, and assume that M is connected, then $\mu_1 = \mu_2 + \xi$ for some $\xi \in (\mathfrak{g}^*)^G$, where as always G acts on $\mathfrak{g}^*$ by the coadjoint action.*

Proof. Since $d\mu_1^Y = \iota_{Y\sharp}\omega = d\mu_2^Y$, we have $d(\mu_1^Y - \mu_2^Y) = 0$. Since M is connected, this implies that $\mu_1^Y - \mu_2^Y = c_Y$ for a constant c_Y for each $Y \in \mathfrak{g}$. But $\mu_1^Y - \mu_2^Y = \langle \mu_1 - \mu_2, Y \rangle$, so c_Y is linear in Y. Thus there exists a $\xi \in \mathfrak{g}^*$ such that $c_Y = \langle \xi, Y \rangle$. Since μ_1, μ_2 are G-equivariant, we have $\xi \in (\mathfrak{g}^*)^G$. □

Proof of the last part of the Constant Rank Embedding Theorem. By the Lemma, $\psi^* \mu_2 = \mu_1 + \xi$ for some $\xi \in (\mathfrak{g}^*)^G$. Using that $\mu_1 \circ I_1 = \mu_2 \circ I_2$ and that $\psi \circ I_1 = I_2$, we conclude that $\xi = 0$, so $\psi^* \mu_2 = \mu_1$. □

Using the shift trick, we may always assume that $0 \in \mathfrak{g}^*_{\text{reg}}$ for a given Hamiltonian G-space (M, ω, G, μ). We want to see how M_ξ for $\xi \in U_0 \subseteq \mathfrak{g}^*_{\text{reg}}$, $0 \in U_0$ are related to M_0.

Assume that G is compact, let $Z = \mu^{-1}(0)$, let $i : Z \hookrightarrow M$ be the inclusion, and let $\pi : Z \to Z/G =: M_0$ be the quotient map. By a previous theorem (since G acts (locally) freely and is compact), $\pi : Z \to Z/G$ is a principal bundle over Z/G.

So there exists a connection 1-form $\alpha \in \Omega^1(Z, \mathfrak{g})$, on Z, i.e. α satisfies

$$g^*\alpha = \mathrm{Ad}_{g^{-1}}\alpha, \quad i_{\xi^\sharp}\alpha = \xi,$$

for all $g \in G$, $\xi \in \mathfrak{g}$. Note here that $\xi^\sharp \in \Gamma(TZ)$ is the vector field generated by $\xi \in \mathfrak{g}$,

$$\xi^\sharp_p = \frac{d}{dt}|_{t=0}(p \cdot \exp(t\xi)),$$

and we assume that $G \circlearrowright M$ is a right action to be in line with what is usually done when working with principal bundles. Also, the adjoint action is

$$\mathrm{Ad}_g = d(L_g \circ R_{g^{-1}}) : \mathfrak{g} \to \mathfrak{g}$$

is a left action, which is compatible with the pullback acting as a left action.

Recall that in the orbifold case, we can construct $\alpha \in \Omega^1(Z, \mathfrak{g})$ as follows: Let $I_g : Z \times \mathfrak{g} \to TZ$ be the embedding $I_g(p, Y) = Y^\sharp_p$. Let $V = I_g(Z \times \mathfrak{g}) \subset TZ$ be what will turn out to be the vertical subbundle of TZ. Choose a G-invariatn Riemannian metric on Z, and let $\pi^\perp : TZ \to V$ be the orthogonal projection with respect to this metric, and let $\widetilde{\mathrm{pr}}_2 : Z \times \mathfrak{g} \to \mathfrak{g}$ be the standard projection. For $v_p \in T_pZ$, define $\alpha_p(v_p)$ by

$$\alpha_p(v_p) := \widetilde{\mathrm{pr}}_2 \circ I_g^{-1} \circ \pi^\perp(v_p).$$

Then $H := \ker \alpha$ defines a G-invariant subbundle of TZ that is complimentary to V so $TZ \cong V \oplus H$.

We are working towards expressing the symplectic space near $\mu^{-1}(0) \subseteq M$ in terms of something easier to understand. This behaviour will be modellen in by the space $Z \times \mathfrak{g}^*$. Let pr_1 and pr_2 be projections from $Z \times \mathfrak{g}^*$ onto the first and second factors respectively. Define $\sigma \in \Omega^2(Z \times \mathfrak{g}^*, \mathbb{R})$ by

$$\sigma := \mathrm{pr}_1^*\, \pi^*\omega_0 - d\langle \mathrm{pr}_2 \wedge \mathrm{pr}_1^*\, \alpha\rangle,$$

where $\omega \in \Omega^2_{\text{orb}}(M_0, \mathbb{R})$ is the MWM reduced symplectic form, so $\pi^*\omega_0 = i^*\omega$, and $\mathrm{pr}_2 \wedge \mathrm{pr}_1^*\, \alpha \in \Omega^1(Z \times \mathfrak{g}^*, \mathfrak{g}^* \otimes \mathfrak{g})$ is composed with the dual pairing $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \otimes \mathfrak{g} \to \mathbb{R}$ to give $\langle \mathrm{pr}_2 \wedge \mathrm{pr}_1^*\, \alpha\rangle \in \Omega^1(Z \times \mathfrak{g}^*, \mathbb{R})$.

Let $G \circlearrowright Z \times \mathfrak{g}^*$ be the diagonal action using the coadjoint action on $\mathfrak{g}^*$, so

$$(p, \xi) \cdot g := (p\dot{g}, \mathrm{Ad}^*_{g^{-1}}\, \xi).$$

Theorem 4.2.12 (Local normal form)**.** *The form $\sigma \in \Omega^2(Z \times \mathfrak{g}^*, \mathbb{R})$ is non-degenerate (hence symplectic) on some neighbourhood of $Z \times \{0\}$ in $Z \times \mathfrak{g}^*$ and satisfies the moment map condition*

$$\iota_{Y^\sharp}\sigma = d\langle \mathrm{pr}_2, Y\rangle$$

for all $Y \in \mathfrak{g}$, i.e. pr_2 is the moment map, and there exists an equivariant symplectomorphism ψ between open neighbourhoods $U, \tilde{U}_0$ so that $Z \subseteq U \subseteq M$, and $Z \times \{0\} \subseteq \tilde{U}_0 \subseteq Z \times \mathfrak{g}^*$ such that $\psi^* \mathrm{pr}_2 = \mu$.

Proof. Recall that $\psi := \mathrm{pr}_1^*\, \pi^*\omega_0 - d\langle \mathrm{pr}_2 \wedge \mathrm{pr}_1^*\, \alpha\rangle$. Clearly $d\sigma = 0$ since $d\omega_0 = 0$ and $d^2 = 0$. Let

$$\tilde{\sigma} := \mathrm{pr}_1^*\, \pi^*\omega_0 - \langle d\,\mathrm{pr}_2 \wedge \mathrm{pr}_1^*\, \alpha\rangle$$

so that $\sigma = \tilde{\sigma} - \langle \mathrm{pr}_2 \wedge d\,\mathrm{pr}_1^*\, \alpha\rangle$. Notice that $\langle \mathrm{pr} \wedge d\,\mathrm{pr}_1^*\, \alpha$ vanishes on $Z \times \{0\}$, since $\mathrm{pr}_2\,|_{Z\times\{0\}} = 0$. If we can show that $\tilde{\sigma}$ is non-degenerate on $Z \times \{0\}$, since non-degeneracy is an open condition, it follows that σ is non-degenerate on a neighbourhood of $Z \times \{0\}$ in $Z \times \mathfrak{g}^*$. Given $(p, 0) \in Z \times \mathfrak{g}^*$, we

can identify $T_{(p,0)}(Z \times \mathfrak{g}^*) \cong V_p \oplus H_p \oplus \mathfrak{g}^*$ where V_p, H_p are the vertical and horizontal subspaces for α respectively.

Let $w := Y_p^\sharp + h + \eta \in V_p \oplus H_p \oplus \mathfrak{g}$, such that $\iota_w \tilde{\sigma} = 0$ and let us show that $w = 0$. We have

$$\begin{aligned}\iota_w \tilde{\sigma}(\cdot) &= \iota_w(\operatorname{pr}_1^* \pi^* \omega_0 - \langle d \operatorname{pr}_2 \wedge \operatorname{pr}_1^* \alpha \rangle)(\cdot) \\ &= \pi^* \omega_0(h, \cdot) - \langle \eta, \operatorname{pr}_1^* \alpha(\cdot) \rangle + \langle d \operatorname{pr}_2(\cdot), Y \rangle.\end{aligned}$$

Since each term depends on h, η and Y independently, then $i_w \tilde{\sigma} = 0$ implies that all three terms vanish individually. But $\pi^* \omega_0$ is non-degenerate on H_p, so $\pi^* \omega_0(h,) = 0$, so $h = 0$. Also $\langle \eta, \operatorname{pr}_1^* \alpha(\cdot) = 0 \rangle$ means that

$$0 = \langle \eta, \operatorname{pr}_1^* \alpha(\tilde{Y}_p^\sharp) \rangle = \langle \eta, \tilde{Y} \rangle$$

for all $\tilde{Y} \in \mathfrak{g}$, so $\eta = 0$. Finally, $\langle d \operatorname{pr}_2(\cdot), Y \rangle = 0$ implies that

$$\langle d \operatorname{pr}_2(\tilde{\eta}), Y \rangle = \langle \tilde{\eta}, Y \rangle = 0$$

for all $\tilde{\eta} \in \mathfrak{g}^*$, so $Y = 0$. We conclude that $w = 0$. Hence, $\tilde{\sigma}$ is non-degenerate on $Z \times \{0\}$, and so σ is non-degenerate on an open set in $Z \times \mathfrak{g}^*$ containing $Z \times \{0\}$.

Next, we show that $\iota_{Y^\sharp} \sigma) d\langle \operatorname{pr}_2, Y \rangle$. Plugging in the definitions, and using that $Y^\sharp \in \ker d\pi$, we find that

$$\begin{aligned}\iota_{Y_p^\sharp} \sigma &= \iota_{Y_p^\sharp}(\operatorname{pr}_1^* \pi^* \omega_0 - d\langle \operatorname{pr}_2 \wedge \operatorname{pr}_1^* \alpha \rangle) \\ &= \iota_{Y_p^\sharp} \operatorname{pr}_1^* \pi^* \omega_0 - \iota_{Y_p^\sharp} d\langle \operatorname{pr}_2 \wedge \operatorname{pr}_1^* \alpha \rangle \\ &= -\iota_{Y_p^\sharp} d\langle \operatorname{pr}_2 \wedge \operatorname{pr}_1^* \alpha \rangle.\end{aligned}$$

Observe that $\langle \operatorname{pr}_2 \wedge \operatorname{pr}_1^* \alpha \rangle$ is invariant under the natural induced G-action: To see this, let $Y_p^\sharp + h + \eta_\xi \in T_{(p,\xi)}(Z \times \mathfrak{g}^*) \cong V_p \oplus H_p \oplus T_\xi \mathfrak{g}^*$. Then since $g^* \alpha = \operatorname{Ad}_{g^{-1}} \alpha$ and since $\alpha_p(Y_p^\sharp) = Y$, we can write

$$\begin{aligned}g^* \langle \operatorname{pr}_2 \wedge \operatorname{pr}_1^* \alpha \rangle_{(p,\xi)}(Y_p^\sharp + h + \eta_\xi) &= \langle g^* \operatorname{pr}_2 |_{(p,\xi)}, g^* \operatorname{pr}_1^* \alpha|_{(p,\xi)}(Y_p^\sharp + h + \eta_\xi) \rangle \\ &= \langle \operatorname{Ad}_{g^{-1}}^* \xi, \operatorname{Ad}_{g^{-1}} \alpha_p(Y_p^\sharp) \rangle \\ &= \langle \operatorname{Ad}_{g^{-1}}^* \xi, \operatorname{Ad}_{g^{-1}} Y \rangle \\ &= \langle \xi, Y \rangle = \langle \operatorname{pr}_2 \wedge \operatorname{pr}_1^* \alpha \rangle_{(p,\xi)}(Y_p^\sharp + h + \eta_\xi),\end{aligned}$$

which proves that $\langle \operatorname{pr}_2 \wedge \operatorname{pr}_1^* \alpha \rangle$ is G-invariant. Therefore,

$$0 = \mathcal{L}_{Y^\sharp} \langle \operatorname{pr}_2 \wedge \operatorname{pr}_1^* \alpha \rangle = (\iota_{Y^\sharp} d + d \iota_{Y^\sharp})(\langle \operatorname{pr}_2 \wedge \operatorname{pr}_1^{\alpha} \rangle),$$

and we find that

$$\begin{aligned}\iota_{Y_p^\sharp} \sigma &= -\iota_{Y_p^\sharp} d\langle \operatorname{pr}_2 \wedge \operatorname{pr}_1^* \alpha \rangle \\ &= d \iota_{Y_p^\sharp} \langle \operatorname{pr}_2 \wedge \operatorname{pr}_1^* \alpha \rangle = d\langle \operatorname{pr}_2, Y \rangle.\end{aligned}$$

Hence, pr_2 satisfies the moment map condition for σ near $Z \times \{0\}$ in $Z \times \mathfrak{g}^*$, i.e. on an open set U_0. To be a moment map, it also has to satisfy the Hamiltonian condition. Since σ is G-invariant, we may choose U_0 to be G-invariant. Thus, $(U_0, \sigma, G, \operatorname{pr}_2)$ is a Hamiltonian G-space.

Let $i_0 : Z \hookrightarrow Z \times \mathfrak{g}^*$ be the embedding $p \mapsto (p, 0)$ which is constant rank since $i_0(Z) = Z \times \{0\} = \operatorname{pr}_2^{-1}(0)$, and $0 \in \mathfrak{g}^*_{\text{reg}}$ for pr_2. Also $(T_p Z)^\sigma = (\ker d \operatorname{pr}_2)^\sigma = T_p \mathcal{O}_p \subseteq T_p Z$ since $\mathcal{O}_p \subseteq Z$. Thus, Z is a coisotropic subspace of $U_0 \subseteq Z \times \mathfrak{g}^*$, and $N_\sigma(Z) = TZ^\sigma / (TZ^\sigma \cap TZ) \cong 0$. Also, $i : Z \hookrightarrow M$ is a coisotropic constant rank embedding for μ for similar reasons, where (M, ω, G, μ) is the original Hamiltonian G-space. The orbit $\mathcal{O}_p$ is contained in Z in this case as well. Thus,

$N_\omega(Z) = TZ^\omega/(TZ^\omega \cap TZ) \cong 0 \cong N_\sigma(Z)$. Since G acts (properly) on Z, M, U_0, since it preserves ω, σ, and since i, i_0 are G-equivariant, the constant rank embedding theorem implies that there exists G-invariant open sets $U, \tilde{U}_0$, such that $i(Z) \subseteq U \subseteq M$, $i_0(Z) \subseteq \tilde{U}_0 \subseteq U_0 \subseteq Z \times \mathfrak{g}^*$, and a G-equivariant diffeomorphism $\psi : U \to \tilde{U}_0$, such that $\psi \circ i = i_0$ and $\psi^*\sigma = \omega$. Finally, since $\mu \circ i = \mathrm{pr}_2 \circ i_0$, we have $\psi^* \mathrm{pr}_2 = \mu$. □

This result is important and will give a nice local description of M_ξ near $0 \in \mathfrak{g}^*_{\mathrm{reg}}$, but we need some more background before we can use it. This background will not be necessary for abelian groups, but we try to be as general as possible.

14th lecture, October 28th 2011

Recall Theorem 4.2.12 which gave us a Hamiltonian G-space $(\tilde{U}_0, \sigma, G, \mathrm{pr}_2)$ with an equivariant symplectomorphism from $U \subseteq M$ to $\tilde{U}_0 \subseteq Z \times \mathfrak{g}^*$, $\psi : U \to \tilde{U}_0$ such that $\psi^* \mathrm{pr}_2 = \mu$.

We want to use this to get a nice local description of $\mu^{-1}(\xi)/G =: M_\xi$ for $\xi \in \mathfrak{g}^*_{\mathrm{reg}}$ close to 0 (the last thing we can always assume by the shift trick).

4.3 Cotangent bundles

Definition 4.3.1. Let Q be a manifold, $M = T^*Q$. Let $\pi_Q : M = T^*Q \to Q$ be the bundle projection and $d\pi_Q : TM \to TQ$ the tangent map. Define the *canonical* 1*-form* on T^*Q, denoted $\theta \in \Omega^1(M, \mathbb{R})$, by

$$\theta_p(W_p) := \langle p, d\pi_Q|_p(W_p) \rangle,$$

where $W_p \in T_pM$, $p \in T^*_{\pi_Q(p)}Q$, and $\langle\ ,\ \rangle$ is the usual pairing between TQ and T^*Q.

Proposition 4.3.2. *The canonical* 1*-form* θ *is the unique* 1*-form such that for any* $\tau \in \Omega^1(Q)$*, we have* $\tau = \tau^*\theta$*, where on the right hand side, we view* τ *as a section* $\tau : Q \to T^*Q$*.*

Proof. Clearly, $\pi_Q \circ \tau = \mathbf{1}_Q : Q \to Q$ and thus, for any $v \in T_qQ$,

$$d\pi_Q|_{\tau(q)} \circ d\tau_q(v) = v.$$

Consider,

$$\begin{aligned}(\tau^*\theta)_q(v) &= \theta_{\tau(q)}(d\tau_q(v)) = \langle \tau(q), d\pi_Q|_{\tau(q)} \circ d\tau_q(v) \rangle \\ &= \langle \tau(q), v \rangle = \tau_q(v),\end{aligned}$$

which establishes the first claim. Uniqueness follows by observing that any non-vertical vector in $w \in T_pM = T_pT^*Q$ (i.e. $w \notin \ker d\pi_Q$), we can write $w = d\tau_q(v)$ for some $\tau \in \Omega^1(Q)$, and $v \in T_qQ$ where $\tau(q) = p$. Assume $\tilde{\theta}$ also satisfies the property of the Proposition. Then

$$\tilde{\theta}_p(w) = \tilde{\theta}_{\tau(q)}(d\tau_q(v)) = \tau_q(v) = \theta_p(w).$$

Since non-vertical vectors span T_pM, we are done. □

It is also useful to work in coordinates. Recall that on $T^*Q|_U = T^*U$ there exist local coordinates, called *local cotangent coordinates* $q_1, \ldots, q_n, p_1, \ldots, p_n$, where $T^*Q|_U \cong U \times \mathbb{R}^n$, and q_i are coordinates on U, p_i on $\mathbb{R}^n$.

Lemma 4.3.3. *In local cotangent coordinates* $q_1, \ldots, q_n, p_1, \ldots, p_n$ *on* T^*U*, the canonical* 1*-form* θ *is*

$$\theta|_{T^*U} = \sum_j p_j dq_j.$$

Proof. Let $\tau = \sum_j \tau_j \tau_j dq_j$ be a 1-form on U. Then $\tau^* p_j = \tau_j$ and $\tau^* q_j = q_j$ (viewing p_j, q_j as functions $T^*Q|_U \to \mathbb{R}$). Then, $\tau^*(\sum_j p_j dq_j) = \sum_j \tau_j dq_j = \tau$, i.e. $\tau^*\theta = \tau$ on T^*U so by the uniqueness part of Proposition 4.3.2, $\theta|_{T^*U} = \sum_j p_j dq_j$. □

Theorem 4.3.4. *Let $M = T^*Q$ with canonical 1-form $\theta \in \Omega^1(T^*Q, \mathbb{R})$, then $\omega := -d\theta$ is a symplectic form.*

Proof. In local coordinates,

$$\omega = \sum_j dq_j \wedge dp_j.$$

□

Let Q_1, Q_2 be manifolds, and consider $f : Q_1 \to Q_2$. We call the induced map $df : TQ_1 \to TQ_2$ the *tangent lift* of f and write $df = Tf$. We do this in order to introduce the *cotangent lift* of f, $T^*f : T^*Q_2 \to T^*Q_1$, given by $(T^*f\alpha)(v) \equiv \alpha(Tfv)$ for $\alpha \in T_p^*Q_2$, $v \in T_qQ_1$.

Proposition 4.3.5. *Let $T^*f : T^*Q_2 \to T^*Q_1$ be the cotangent lift of a diffeomorphism $f : Q_1 \to Q_2$. Then $(T^*f)^*\theta_1 = \theta_2$ for the canonical 1-forms θ_1, θ_2, and hence T^*f is a symplectomorphism, $(T^*f)^*\omega_1 = \omega_2$.*

We leave the proof as an easy exercise.

Proposition 4.3.6. *If (M, ω) is a symplectic manifold with an exact symplectic form, $\omega = -d\theta$, then any G-action on M preserving θ is a Hamiltonian with moment map $\mu : M \to \mathfrak{g}^*$ given by*

$$\mu^Y = \iota_{Y^\sharp}\theta.$$

Proof. Since θ is G-invariant, $\mathcal{L}_{Y^\sharp}\theta = 0$. We calculate and find

$$d\mu^Y = d\iota_{Y^\sharp}\theta = -\iota_{Y^\sharp}d\theta + \mathcal{L}_{Y^\sharp}\theta = \iota_{Y^\sharp}\omega.$$

To see G-equivariance, compute

$$\begin{aligned} g^*\langle \mu, Y\rangle_p &= g^*\iota_{Y^\sharp}\theta_p = g^*[\theta(Y^\sharp)]_p \\ &= \theta_{p\cdot g}(Y^\sharp_{p\cdot g}). \end{aligned}$$

Observe that $Y^\sharp_{p\cdot g} = g_*(\mathrm{Ad}_g Y)^\sharp_p$. Then

$$\begin{aligned} \theta_{p\cdot g}(Y^\sharp_{p\cdot g}) = \theta_{p\cdot g}(g_*(\mathrm{Ad}_g Y)^\sharp_p) = (g^*\theta)_p((\mathrm{Ad}_g Y)^\sharp_p) \\ = \theta_p((\mathrm{Ad}_g Y)^\sharp_p) = \iota_{\mathrm{Ad}_g Y^\sharp}\theta_p = \langle \mu, \mathrm{Ad}_g Y\rangle_p. \end{aligned}$$

Thus $g^*\langle \mu, Y\rangle = \langle \mu, \mathrm{Ad}_g Y\rangle$. □

We move now to the case of the cotangent bundle of a Lie group. Let G be a (compact) Lie group. Then we consider $(T^*G, \mu, G, -d\theta)$ Hamiltonian structures coming from natural left/right group actions on G.

Definition 4.3.7. The *left/right invariant vector fields* on G are vector fields $Y^L, Y^R \in \Gamma(TG)$ invariant under tangent lifts of the left/right multiplication maps $L_g, R_g : G \to G$, i.e.

$$TL_g Y^L = dL_g Y^L = Y^L, \quad TR_g Y^R = dR_g Y^R = Y^R,$$

for $Y \in \mathfrak{g}$ (identifying $Y^L_e = Y^R_e = Y$).

Similarly, *left/right invariant 1-forms* on G are $\xi^L, \xi^R \in \Gamma(T^*G)$ invariant under cotangent lifts of L_g, R_g, i.e.

$$(T^*L_g)\xi^L = \xi^L, \quad (T^*R_g)\xi^R = \xi^R,$$

viewing as before ξ as an element of $\mathfrak{g}^*$.

Note that we write $Y_g^L = dL_g Y$, $Y_g^R = dR_g Y$ for $Y \in \mathfrak{g}$. Also, we have $\xi_g^L(Y_g^L) = \xi(Y)$, $\xi_g^R(Y_g^R) = \xi(Y)$ for $\xi \in \mathfrak{g}^*$. These are the main properties that we need.

Definition 4.3.8. The maps $\psi_L, \psi_R : G \times \mathfrak{g} \to TG$ mapping $(g, Y) \mapsto Y_g^L, Y_g^R$ respectively, and $\tilde{\psi}_L, \tilde{\psi}_R : G \times G^* \to T^*G$ by $(g, \xi) \mapsto \xi_g^L, \xi_g^R$ are called the *left/right trivializations* of the tangent and cotangent bundles.

We will mainly work in the left trivialization of the tangent and cotangent bundles. Note that the maps $\psi_L, \psi_R, \tilde{\psi}_L, \tilde{\psi}_R$ are vector bundle isomorphisms.

Claim 4.3.9. *The invariant vector fields and* 1*-forms satisfy*

$$Y_g^L = (\mathrm{Ad}_g Y)_g^R, \quad Y_g^R = (\mathrm{Ad}_{g^{-1}} Y)_g^L$$
$$\xi_g^L = (\mathrm{Ad}_g^* \xi)_g^R, \quad \xi_g^R = (\mathrm{Ad}_{g^{-1}}^* \xi)_g^L.$$

Consider for example the first one. We want to write $Y_g^L = dR_g \tilde{Y} = \tilde{Y}_g^R$ for some $\tilde{Y} \in \mathfrak{g}$ and show that $\tilde{Y} = \mathrm{Ad}_g Y$. Observe that

$$Y_g^L = dL_g Y = dR_g(dR_{g^{-1}} dL_g Y) = dR_g(\mathrm{Ad}_g Y) = (\mathrm{Ad}_g Y)^R g.$$

Exercise 4.3.10. Prove the three other identities.

Proposition 4.3.11. *If $TL_g, TR_g : G \times \mathfrak{g} \to G \times \mathfrak{g}$ are tangent lifts of L_g, R_g with respect to the left trivialization of TG, then*

$$TL_g(h, Y) = (gh, Y), \quad TR_g(h, Y) = (hg, \mathrm{Ad}_{g^{-1}} Y).$$

Also, if $T^*L_g, T^*R_g : G \times \mathfrak{g}^* \to G \times \mathfrak{g}^*$ are the induced cotangent lifts with respect to the left trivialization, then

$$T^*L_g(h, \xi) = (gh, \xi), \quad TR_g(h, \xi) = (hg, \mathrm{Ad}_{g^{-1}}^* \xi).$$

Proof. We prove that $T^*R_g(h, \xi) = (hg, \mathrm{Ad}_{g^{-1}}^* \xi)$. To see this, we want to write $T^*R_g(h, \xi) = \tilde{\xi}_{h \cdot g}^L$ for some $\tilde{\xi} \in \mathfrak{g}^*$ (i.e. $\tilde{\xi} = \mathrm{Ad}_{g^{-1}}^* \xi$). As before, $\tilde{\xi}_g^L(Y_g^L) = \xi(Y)$. We consider

$$\begin{aligned}(T^*R_g)(\xi^L)_h(Y_h^L) &\equiv \xi_{h \cdot g}^L(dR_g Y_h^L) = \xi_{h \cdot g}^L(dL_h(dR_g Y)) \\ &= \xi_{h \cdot g}^L(dL_h(\mathrm{Ad}_g Y)_g^L) = \xi_g^L((\mathrm{Ad}_g Y)_g^L) = \xi(\mathrm{Ad}_g Y) = (\mathrm{Ad}_{g^{-1}}^* \xi)(Y) \\ &= (\mathrm{Ad}_{g^{-1}}^* \xi)_{h \cdot g}^L(Y_{h \cdot g}^L).\end{aligned}$$

Overall this implies that $T^*R_g(h, \xi) = (hg, \mathrm{Ad}_{g^{-1}}^* \xi)$. We leave the others as an exercise. □

15th lecture, October 31st 2011

We take for the left action on G the vector field generated by the flow $\exp(-tY) \cdot g$. For the right action on G the on generated by the flow $g \cdot \exp(tY)$.

Since the generating vector field for the left action on G is *right* invariant and also has value $-Y$ at e, we conclude it must be $-Y^R \in \Gamma(TG)$. Also, the generating vector field for the right action is $Y^L \in \Gamma(TG)$.

For the cotangent lift of the left/right actions on T^*G, use the fact that there is a unique vector field $\tilde{V} \in \Gamma(TT^*G)$ for each $V \in \Gamma(TG)$ such that the flow of $\tilde{V}$ is the cotangent lift of the flow of V. Locally, if $V = \sum V_j \frac{\partial}{\partial q_j}$, using cotangent coordinates, then

$$\tilde{V} = \sum V_j \frac{\partial}{\partial q_j} - \sum_{j,k} p_j \frac{\partial V_j}{\partial q^k} \frac{\partial}{\partial p_k}.$$

Call the map $\Gamma(TG) \to \Gamma(TT^*G)$ given by $V \mapsto \tilde{V}$ the *cotangent lift of a vector field* . Note that if $\pi_G : T^*G \to G$ is the standard projection map, then $\tilde{V}$ projects to V under π_G, i.e. $d\pi_G(\tilde{V}) = V$. Recall that $\theta \in \Omega^1(T^*G)$ denotes the canonical 1-form, and recall that $(T^*G, -d\theta = \omega_c)$ is symplectic.

We would like to observe that the cotangent lifts of the left/right actions on G to T^*G are Hamiltonian with respect to some moment maps $\mu_L, \mu_R : T^*G \to \mathfrak{g}^*$ respectively.

Lemma 4.3.12. *Given any $V \in \Gamma(TG)$, the cotangent lift $\tilde{V} \in \Gamma(TT^*G)$ is a Hamiltonian vector field for ω_c with Hamiltonian function $H \equiv \iota_{\tilde{V}}\theta$.*

Proof. Recall that cotangent lifts of functions $G \to G$ automatically preserve θ. In particular, the flow of $\tilde{V}$ preserves θ, i.e. $\mathcal{L}_{\tilde{V}}\theta = 0$. Thus, to check the Hamiltonian condition, we compute

$$dH = d\iota_{\tilde{V}}\theta = -\iota_{\tilde{V}}d\theta + \mathcal{L}_{\tilde{V}}\theta = -\iota_{\tilde{V}}d\theta = \iota_{\tilde{V}}\omega_c.$$

□

Recall that since (T^*G, ω_c) is an exact symplectic manifold , i.e. that ω_c is exact, any G-action preserving θ on T^*G is Hamiltonian with moment map $\mu^Y := \iota_{Y^\sharp}\theta$ for $Y \in \mathfrak{g}$.

For the group actions given by the cotangent lifts of the left/right group actions, this is the Hamiltonian structure we consider. Let $\mu^L, \mu^R : T^*G \to \mathfrak{g}^*$ be moment maps for the left/right actions respectively. Recall that the vector field generated by the left action is $-Y^R$, and the one by the right action is Y^L, on G. Let $-\tilde{Y}^R, \tilde{Y}^L \in \Gamma(TT^*G)$ denote the cotangent lifts.

Then, the moments map are $\mu_L^Y := \iota_{-\tilde{Y}^R}\theta$, $\mu_R^Y := \iota_{\tilde{Y}^L}\theta$. We can actually compute these moment maps explicitly: Let $(g, \xi) \in G \times \mathfrak{g}^* \cong T^*G$ in the left trivializations, i.e. $(g, \xi) \mapsto \xi_g^L$. Then, since as we saw last time, $Y_g^R = (\mathrm{Ad}_{g^{-1}} Y)_g^L = (g, \mathrm{Ad}_{g^{-1}} Y)$, we find that

$$\begin{aligned}
\mu_L^Y &= \iota_{-\tilde{Y}^R}\theta|_{(g,\xi)} = \theta_{(g,\xi)}(-\tilde{Y}^R_{(g,\xi)}) \\
&= \langle (g,\xi), d\pi_G|_{(g,\xi)}(-\tilde{Y}^R_{(g,\xi)})\rangle = \langle (g,\xi), (g, -\mathrm{Ad}_{g^{-1}} Y)\rangle \\
&= \langle \xi, -\mathrm{Ad}_{g^{-1}} Y\rangle = \langle -\mathrm{Ad}_g^* \xi, Y\rangle = \langle \mu_L(g,\xi), Y\rangle.
\end{aligned}$$

Thus,

$$\mu_L(g, \xi) = -\mathrm{Ad}_g^* \xi$$

(at least up to a constant, but we will assume it vanishes at 0). Similarly,

$$\mu_R(g, \xi) = \xi.$$

We will now consider what happens when we symplectically reduce with respect to these moment maps. Note first that every $\xi \in \mathfrak{g}^*$ is a regular value for μ_L, μ_R since the cotangent lifts of the left/right actions are free. Thus we can symplectically reduce at any value, and we find the following theorem.

Theorem 4.3.13. *The symplectic reduction of T^*G at $\xi \in \mathfrak{g}^*$ by the right action, which we write $T^*G/\!\!/^\xi G := \mu_R^{-1}(\xi)/G_\xi$, is a Hamiltonian G-space with G-action induced from the left G-action on T^*G via the quotient $\mu_R^{-1}(\xi) \to T^*G/\!\!/^\xi G$. In fact, $T^*G/\!\!/^\xi G$ is symplectomorphic to the co-adjoint orbit $G \cdot (-\xi)$ with the KKS Hamiltonian structure.*

Proof. Clearly, $\mu_R^{-1}(\xi) = G \times \{\xi\}$ by definition of μ_R. Then

$$\mu_R^{-1}(\xi)/G_\xi = (G/G_\xi) \times \{\xi\}.$$

Clearly, μ_L descends to a map

$$\overline{\mu}_L : (G/G_\xi) \times \{\xi\} \to G \cdot (-\xi) \subseteq \mathfrak{g}^*$$

and is a moment map for the reduced symplectic form ω_ξ on $T^*G/\!\!/^\xi G$ with left G-action, since it was a moment map on T^*G.

Define $\tilde{I} : G \cdot (-\xi) \to (G/G_\xi) \times \{\xi\}$ as $\tilde{I} = \overline{\mu}_L^{-1}$. Let $\tilde{\omega}_\xi := \tilde{I}^*\omega_\xi$. Then clearly $\tilde{I}$ is a G-equivariant symplectomorphism from $(G \cdot (-\xi), \tilde{\omega}_\xi)$ to $(T^*G/\!\!/^\xi G, \omega_\xi)$. If $i : G \cdot (-\xi) \hookrightarrow \mathfrak{g}^*$ is inclusion, then i is a moment map for $\tilde{\omega}_\xi$ with respect to coadjoint action.

Thus, $\tilde{\omega}$ must be the KKS symplectic form, and $(G \cdot (-\xi), \tilde{\omega}_\xi, G, i)$ is the KKS space. This follows from the Proposition that ω_{KKS} is the unique symplectic structure on $G \cdot (-\xi)$ such that the co-adjoint action is Hamiltonian for moment map the inclusion. □

Recall, that for (M, ω, G, μ) a Hamiltonian G-space we saw that M looks like a neighbourhood of $Z \times \{0\}$ in $Z \times \mathfrak{g}^*$ with Hamiltonian structure $(U_0 \subseteq Z \times \mathfrak{g}^*, \sigma, G, \text{pr}_2)$.

Observe that we may also view $Z \times \mathfrak{g}^*$ as a quotient $(Z \times T^*G)/G$ using the left trivialization $T^* \cong G \times \mathfrak{g}^*$, where G acts diagonally on the left. That is, $\tilde{\psi} : Z \times T^*G \to Z \times T^*G$ defined by $(p, h, \xi) \mapsto (p \cdot h, h, \xi)$ induces a diffeomorphism

$$\psi : (Z \times T^*G)/G \xrightarrow{\cong} Z \times \mathfrak{g}^*$$

since $\tilde{\psi}$ takes the diagonal action of the left to the left action on the factor T^*G in $Z \times T^*G$. Also, the right action on T^*G in $Z \times T^*G$ goes to the diagonal action on the right in $Z \times \mathfrak{g}^*$. Hence, ψ is a G-equivariant diffeomorphism, and let us turn it into a symplectomorphism. Let $\tilde{\sigma} := \psi^*\sigma$, and define $\tilde{\mu}_R : (Z \times T^*G)/G \to \mathfrak{g}^*$ by $[p, h, \xi] \mapsto \xi$. Then by construction, if $U_0 \subseteq Z \times \mathfrak{g}^*$ denotes the neighbourhood of $Z \times \{0\}$ from before such that $(U_0, \sigma, G, \text{pr}_2)$ is Hamiltonian, then ψ is a symplectomorphism to $(\tilde{U}_0, \tilde{\sigma}, G, \tilde{\mu}_R)$, where $\tilde{U}_0 := \psi^{-1}(U_0)$ that also preserves the moment map pr_2, i.e. $\text{pr}_2 \circ \psi = \tilde{\mu}_R$ (we leave this as an exercise).

After some checking, we can see that the reduced space M_ξ for $\xi \in \mathfrak{g}^*_{\text{reg}}$ close to 0 must be symplectomorphic to

$$[Z \times G \cdot (-\xi)]/G.$$

This is the main theorem in the non-abelian case.

Theorem 4.3.14. *The reduced space M_ξ for ξ close to 0 fibers over M_0 with fibers coadjoint orbits:*

$$M_\xi \cong [Z \times G \cdot (-\xi)]/G.$$

4.4 Proof of the Duistermaat–Heckman theorem

The previous theorem was what we wanted to prove in the general non-abelian case. In the case that $G = T$, a compact torus, we can say more. For $G = T$, the coadjoint action is trivial, and we may identify the reduced spaces at ξ near 0 as

$$M_\xi \cong Z/T \times \{-\xi\} \cong M_0.$$

Recall that $\alpha \in \Omega^1(Z, \mathfrak{t})$, $\mathfrak{t} = \text{Lie}(T)$, is a connection on a torus bundle T —

and so $d\alpha$ is the curvature of this connection. Let $F_\alpha \in \Omega^2(M_0, \mathfrak{t})$ denote the curvature on M_0, so $\pi^*F_\alpha = d\alpha$. It is straightforward to identify the symplectic form ω_ξ on $M_\xi \cong Z/G \times \{-\xi\} \cong M_0$ as

$$\omega_\xi = \omega_0 + \langle \xi, F_\alpha \rangle,$$

where $\omega_\xi \in \Omega^2(M_0, \mathbb{R})$ under the identification $M_\xi \cong M_0$. One can see this using that

$$\sigma = \text{pr}_1^*\, \pi^*\omega_0 - d\langle \text{pr}_2 \wedge \text{pr}_1^*\, \alpha \rangle.$$

Recall that our identification M_ξ with M_0 depends upon a choice of diffeomorphism $\Phi_\alpha : U_0 \subseteq Z \times \mathfrak{g}^* \to U \subseteq M$, which in turn depended upon the choice of connection $\alpha \in \Omega^1(Z, \mathfrak{g})$. So, $\omega_\xi = \omega_0 + \langle \xi, F_\alpha \rangle$ is not canonical, e.g. compare the cases α flat to non-flat.

It turns out that any two such identifications Φ_α, Φ_β for choices of α, β induce identifications $M_\xi \cong M_0$ that are isotopic. Thus, although $\Omega^2(M_\xi, \mathfrak{t}) \cong \Omega^2(M_0, \mathfrak{t})$ is not canonical, the induced isomorphism on cohomology $H^2(M_\xi, \mathfrak{t}) \cong H^2(M_0, \mathfrak{t})$ is canonical by isotopy invariance of induced maps.

Hence we have the following

Theorem 4.4.1 (Duistermaat–Heckman, 1982)**.** *There exists a convex set $U \subseteq \mathfrak{g}^*_{\text{reg}}$ containing 0 such that for any $\xi \in U$,*

$$[\omega_\xi] = [\omega_0] + \langle \xi, [c] \rangle,$$

where $[c] \in H^2(M_0, \mathfrak{t})$ is the first Chern class of the torus bundle $\mu^{-1}(0) = Z \xrightarrow{\pi} M_0$, where we have identified $H^2(M_\xi, \mathfrak{t}) \cong H^2(M_0, \mathfrak{t})$ canonically as above.

Theorem 4.4.2 (Duistermaat–Heckman, 1982; 3.3)**.** *Let (M, ω, T, μ) be a Hamiltonian T-space with effective action, $\dim M = 2n$, $\dim T = d$. The Radon–Nikodym derivative of m_{DH} with respect to the Lebesgue measure $d\xi$ on $\mathfrak{t}^*$, which we write $f = \frac{dm_{\text{DH}}}{d\xi}$, is a polynomial of degree less than or equal to $n - d$ on each connected component of $\mathfrak{t}^*_{\text{reg}}$ of μ.*

Proof. Assume without loss of generality that $U \subseteq \mathfrak{t}^*_{\text{reg}}$ is a neighbourhood of 0. Recall that f is defined by the relation

$$\int_{\mu^{-1}(U)} \frac{\omega^n}{n!} = \int_U f(\xi) d\xi.$$

We will express

$$\int_{\mu^{-1}(U)} \frac{\omega^n}{n!} = \int_U (\int_Z F(\xi)) d\xi$$

for some form $F(\xi)$ using the isomorphism of Hamiltonian spaces $\mu^{-1}(U) \cong Z \times U$, where $Z \times U$ has symplectic form

$$\sigma \equiv \text{pr}_1^* \, \pi^* \omega_0 - d\langle \text{pr}_2 \wedge \text{pr}_1^* \, \alpha \rangle.$$

At this point, to simplify notation, and to get rid of the notational dependence on the projection maps, we let ξ_j be coordinates on $U \subseteq \mathfrak{g}^*$ and write

$$\langle \text{pr}_2 \wedge \text{pr}_1^* \, \alpha \rangle \equiv \langle \xi, \alpha \rangle = \sum_{j=1}^{d} \xi_j \alpha_j,$$

for $\alpha_j \in \Omega^1(Z, \mathbb{R})$. Rewrite σ as

$$\sigma = \pi^* \omega_0 - \langle \xi, d\alpha \rangle + \sum \alpha_j d\xi_j,$$

dropping the notation pr_1. Thus

$$\begin{aligned} \int_{\mu^{-1}(U)} \frac{\omega^n}{n!} &= \int_{Z \times U} \frac{\sigma^n}{n!} = \int_{Z \times U} \frac{1}{n!} (\pi^* \omega_0 - \langle \xi, d\alpha \rangle + \sum_j \alpha_j d\xi_j)^n \\ &= \int_{Z \times U} \frac{1}{(n-d)!} (\pi^* \omega_0 - \langle \xi, d\alpha \rangle)^{n-d} \wedge \alpha_1 \wedge \cdots \wedge \alpha_d \wedge d\xi. \end{aligned}$$

By Fubini's theorem, we can write this as

$$\int_{\mu^{-1}(U)} \frac{\omega^n}{n!} = \int_U [\int_Z \frac{1}{(n-d)!}(\pi^*\omega_0 - \langle \xi, d\alpha\rangle)^{n-d}\alpha_1 \wedge \cdots \wedge \alpha_d]d\xi =: \int_U f(\xi)d\xi.$$

If we use orbifold integration, we can write

$$f(\xi) = \int_{M_0} \frac{1}{(n-d)!}[\omega_0 - \langle \xi, F_\alpha\rangle]^{n-d} = \mathrm{Vol}(M_\xi).$$

□

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Theorem 4.4.3 (Duistermaat–Heckman, [DH82] Thm. 4.1)**.** *Let (M, ω, T, μ) be a Hamiltonian T-space of dimension $\dim M = 2n$, let $\beta = \omega^n/n!$, and let $Y \in \mathrm{Lie}(T)$ such that M^Y is isolated. Then*

$$I_{\mu^Y,\beta}(\hbar) = \int_M e^{i\mu^Y/\hbar}\beta = (2\pi\hbar)^n \sum_{p_l \in M^Y} \frac{i^n e^{i\mu^Y(p_l)/\hbar}}{\prod_{j=1}^n \alpha_{p_l,j}(Y)}.$$

Proof. First make a change of variables $\hbar \to 1/t$. Let

$$F_t^Y(p) := \frac{i^n e^{i\mu^Y(p)t}}{\prod_{j=1}^n \alpha_{p,j}(Y)}.$$

Recall that we made the following simplifying observations: We can reduce to the case where $Y \in \mathrm{Lie}(T) = \mathfrak{t}$ with $T^Y = \langle \exp(\mathbb{R}Y)\rangle \cong \mathrm{U}(1)$ and $\alpha_{p_l,j}(Y) \neq 0$ for all $j = 1, \ldots, n$, $p_l \in M^Y$. Here, recall that $\alpha_{p_l,j}(Y)$ are the weights of the isotropy representation on $T_{p_l}M$. Under the identification $T^Y \cong \mathrm{U}(1)$, set $Y = 1$, so we can write $\mu = \mu^Y$. We saw that by stationary phase we could write

$$I_{\mu,\beta}(t) = \int_M e^{it\mu}\beta = \left(\frac{2\pi}{t}\right)^n \sum_{p_l \in \mathrm{Crit}\,\mu} F_t(p_l) + s(t),$$

where $s(t) = O(t^\infty)$ (meaning that it falls off faster than any polynomial) is Schwartz on $\mathbb{R} \setminus \{0\}$. Recall that $s(t)$ came from contributions associated to the lemma of non-stationary phase. We want to show that $s(t) = 0$.

Multiplication by t^n gives us

$$t^n s(t) = t^n \int_M e^{i\mu t}\beta - (2\pi)^n \sum_{p_l \in \mathrm{Crit}\,\mu} F_t(p_l),$$

so $t^n s(t)$ must be a smooth Schwartz function of t on all of $\mathbb{R}$. We will show that this is impossible unless $s(t) = 0$, using the fact the Duistermaat–Heckman polynomial $f(\xi)$ is a piecewise polynomial of degree less than or equal to $n-1$. Here, recall that for a given function h on $\mathfrak{g}^*$ that

$$\int_M (h \circ \mu)\beta := \int_{\mathfrak{g}^*} h(\xi)dm_{\mathrm{DH}} = \int_{\mathfrak{g}^*} h(\xi)f(\xi)d\xi.$$

In our case, $h = e^{it\xi}$ and $\mathfrak{g}^* \cong \mathbb{R}$, and we get

$$\int_M e^{it\mu}\beta = \int_{\mathbb{R}} f(\xi)e^{it\xi}d\xi,$$

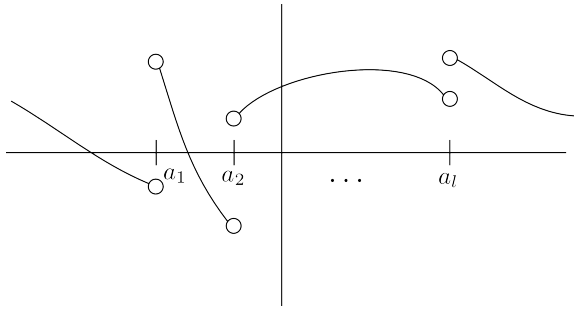

Figure 5: One way of thinking of the Duistermaat–Heckman polynomial f.

where $f(\xi)$ is the Duistermaat–Heckman polynomial. Using properties of the Fourier transform, we find

$$t^n \int_{\mathbb{R}} f(\xi) e^{it\xi} d\xi = t^n \mathcal{F}^{-1} f(t) = \left(\frac{1}{i}\right)^n \mathcal{F}^{-1}\left(\frac{d^n f}{d\xi^n}\right)(t).$$

Now, as observed, f is a piecewise degree less than or equal to $n-1$; see Figure 5 where $a_l = \mu(p_l) \in \mathbb{R}$. On the intervals away from a_l we have $\frac{d^n f}{d\xi^n} = 0$ since $\deg f \leq n-1$. At the points a_l, $\frac{d^n f}{d\xi^n}(a_l)$ is identified as a distribution with a q_l'th derivative (for $q_l \leq n-1$) of the Dirac delta distribution

$$\frac{d^n f}{d\xi^n}(a_l) = \frac{d^{q_l}\delta_{a_l}}{d\xi^{q_l}}.$$

For example, if $\mu(p) = a \in \mathbb{R}$ and f is a Heavyside function at a, say

$$f = \begin{cases} -1, & \xi < a \\ 1, & \xi > a \end{cases}.$$

Then $f' = \delta(a) = \delta_a$ and $\mathcal{F}^{-1} f'(t) = e^{ita}$. Recall that

$$\frac{d^q \delta_a}{d\xi^q}[\varphi] := (-1)^q \frac{d^q \varphi}{d\xi^q}(a)$$

for φ some test function. In general, $\mathcal{F}^{-1}(\frac{d^n f}{d\xi^n})(t)$ is a linear combination of terms $t^{q_l} e^{ia_l t}$. So,

$$t^n s(t) = \sum_l c_l t^{q_l} e^{ia_l t}$$

for some constants $c_l \in \mathbb{C}$. Clearly, the right hand side is not Schwarz unless it vanishes, so $s(t) = 0$ (since $\mathbb{C}$ is an integral domain $\ddot{\smile}$). □

Note that we may extend to $(\operatorname{Lie} T) \otimes \mathbb{C} = \mathfrak{t} \otimes \mathbb{C} = \mathfrak{t}^{\mathbb{C}}$ and write

$$\int_M e^{t\mu^Y} \beta = \left(\frac{2\pi}{t}\right)^n \sum_{p \in M^Y} \frac{e^{t\mu^Y(p)}}{\prod_{j=1}^n \alpha_{p,j}(Y)},$$

where $Y \in \mathfrak{t}^{\mathbb{C}}$.

4.5 Applications

One often applies the Duistermaat–Heckman theorem to compute Liouville volumes of Hamiltonian spaces (M, ω, T, μ).

Example 4.5.1. Let $(M, \omega, \mathrm{U}(1), H)$ be a Hamiltonian U(1)-space with Hamiltonian H. Identify $\mathrm{Lie}(\mathrm{U}(1)) \cong \mathbb{R}$ such that the integral lattice its dual are $\Lambda \cong \mathbb{Z}$, $\Lambda^* \cong \mathbb{Z}$. Then the Duistermaat–Heckman theorem says that

$$\int_M e^{tH} \frac{\omega^n}{n!} = \left(\frac{2\pi}{t}\right)^n \sum_{p \in \mathrm{Crit}\, H} \frac{e^{tH(p)}}{\prod_{j=1}^n \alpha_{j,p}} \tag{12}$$

for isolated fixed points. In this example, since the left hand side is smooth in t, expanding $e^{tH(p)}$ in t, the terms of order less than n must vanish, and we get a description of the symplectic volume in terms of the weights.

More precisely, expanding

$$e^{tH(p)} = \sum_{k=0}^{\infty} \frac{(H(p))^k}{k!} t^k,$$

we see, by observing that the left hand side of (12) is smooth in t, and noting that the right hand side

$$(2\pi)^n \sum_{p \in \mathrm{Crit}\, H} \frac{\sum_{k=0}^{\infty} \frac{(H(p))^k}{k!} t^{k-n}}{\prod_{j=1}^n \alpha_{j,p}}$$

has singular contributions for $k \leq n$, that

$$\sum_{p \in \mathrm{Crit}\, H} \frac{(H(p))^k}{\prod_{j=1}^n \alpha_{j,p}} = 0$$

for $k < n$. Also,

$$\mathrm{Vol}(M) = \int_M \frac{\omega^n}{n!} = \frac{(2\pi)^n}{n!} \sum_{p \in \mathrm{Crit}\, H} \frac{(H(p))^n}{\prod_{j=1}^n \alpha_{j,p}}.$$

An infinite dimensional analogue of this can be used to give formulas for volumes of moduli spaces, such as the moduli space of flat connections in Chern–Simons theory.

5 The Berline–Vergne theorem

Next, following [BGV04], we generalize the Duistermaat–Heckman theorem to a more general class of integrands. So far, we have looked at

$$\int_M e^{\mu^Y} \beta,$$

where (M, ω, T, μ) is a Hamiltonian T-space, where $\beta = \omega^n/n!, \dim M = 2n, Y \in \mathfrak{t}^{\mathbb{C}}, \mu : M \to \mathfrak{t}^*$. We will try to generalize to manifolds M with just a G-action and drop the Hamiltonian condition as much as possible. We will try to find special integrands that might replace $e^{\mu^Y} \beta$.

5.1 Equivariant cohomology

In other words, what other forms like $e^{\mu^Y} \beta \in \Omega^{2n}(M, \mathbb{C})$ satisfy a localization principle?

Write $e^{\mu^Y} \beta = e^{\mu^Y} \frac{\omega^n}{n!}$ and consider

$$e^{\mu^Y + \omega} = e^{\mu^Y} \left(1 + \omega + \frac{\omega^2}{2!} + \cdots \frac{\omega^n}{n!}\right),$$

and we are led to understand $\mu^Y + \omega$. We have

$$\int_M e^{\mu^Y} \beta = \int_M e^{\mu^Y + \omega},$$

since $\int_M \alpha = 0$ if $\deg \alpha \neq 2n$. The idea is to view the form as a map $\mu + \omega : \mathfrak{g} \to \Omega^\bullet(M)$, where $\Omega^\bullet(M) := \Omega^\bullet(M, \mathbb{C})$ is the algebra of smooth complex valued differential forms on M. This motivates us to consider general forms $\alpha : \mathfrak{g} \to \Omega^M$. We look at properties of $e^{\mu^Y + \omega}$ to mimick – in fact we look at $\mu^Y + \omega$, where we have the moment map condition relating the derivatives of the two forms. Here, we have switched to the notation $\Omega^\bullet := \Omega^*$ to avoid confusion later.

If we assume that $G \circlearrowright M$ is a *right* action, then the formula for the moment map condition is

$$g^* \mu = \mathrm{Ad}^*_{g^{-1}} \mu.$$

For $(\mu + \omega)(Y) = \mu^Y + \omega \in \Omega^0(M) \oplus \Omega^2(M)$, the moment map condition says that

$$g^*(\mu + \omega)(Y) = (\mu + \omega)(\mathrm{Ad}_g Y).$$

That is, $\mu + \omega : \mathfrak{g} \to \Omega^\bullet(M)$ is G-equivariant. This is the first property we generalize.

Definition 5.1.1. A *C^∞-G-equivariant differential form* on (M, G) is a C^∞-map $\alpha : \mathfrak{g} \to \Omega^\bullet(M)$ which is G-equivariant with respect to the adjoint on $\mathfrak{g}$ and the pullback action on $\Omega^\bullet(M)$, i.e. $g^* \alpha(Y) = \alpha(\mathrm{Ad}_g Y)$ for all $g \in G$, $Y \in \mathfrak{g}$.

Denote the space of C^∞-G-equivariant differential forms on (M, G) as $\Omega^{\infty,\bullet}_G(M)$.

Define a wedge product of $\alpha, \beta : \mathfrak{g} \to \Omega^\bullet(M)$ by

$$(\alpha \wedge \beta)(Y) := \alpha(Y) \wedge \beta(Y).$$

If α, β are G-equivariant then so is $\alpha \wedge \beta$, since

$$\begin{aligned} g^*(\alpha \wedge \beta)(Y) &= (g^* \alpha) \wedge (g^* \beta)(Y) = (g^* \alpha(Y)) \wedge (g^* \beta(Y)) \\ &= \alpha(\mathrm{Ad}_g Y) \wedge \beta(\mathrm{Ad}_g Y) = (\alpha \wedge \beta)(\mathrm{Ad}_g Y). \end{aligned}$$

So, $\Omega^{\infty,\bullet}_G(M)$ is an algebra under the wedge product.

Note that if G is abelian, then the adjoint action is trivial, and G-equivariance reduces to G-invariance $g^* \alpha(Y) = \alpha(Y)$ for $Y \in \mathfrak{g}$. In general, we define an action of G on the space $\mathrm{Map}^\infty(\mathfrak{g}, \Omega^\bullet(M))$ of smooth maps from $\mathfrak{g}$ to $\Omega^\bullet(M)$ by

$$(g \cdot \alpha)(Y) \equiv g^* \alpha(\mathrm{Ad}_{g^{-1}} Y).$$

Then for $\alpha \in \mathrm{Map}^\infty(\mathfrak{g}, \Omega^\bullet(M))$, we have $\alpha \in \Omega^{\infty,\bullet}_G(M)$ if and only if $g \cdot \alpha = \alpha$, i.e. if and only if α is invariant under this G-action.

The other moment map condition says that $d\mu^Y = \iota_{Y^\sharp} \omega$. Since $d\omega = 0$ for symplectic forms, consider $d(\mu^Y + \omega) = d\mu^Y = \iota_{Y^\sharp} \omega$, which we write as

$$d(\mu^Y + \omega) - \iota_{Y^\sharp} \omega = 0.$$

The idea is to generalize this to equivariant forms.

Definition 5.1.2. Given $\alpha \in \Omega^{\infty,\bullet}_G(M)$, the *equivariant deRham derivative* , denoted $d_\mathfrak{g} : \Omega^{\infty,\bullet}_G(M) \to \mathrm{Map}^\infty(\mathfrak{g}, \Omega^\bullet(M))$, is defined as

$$d_\mathfrak{g} \alpha(Y) := d\alpha(Y) - \iota_{Y^\sharp} \alpha(Y)$$

where $d : \Omega^\bullet(M) \to \Omega^\bullet(M)$ is the ordinary deRham derivative.

Lemma 5.1.3. *We have the following:*

1. *The equivariant deRham derivative $d_{\mathfrak{g}}$ is well-defined as a map $\Omega_G^{\infty,\bullet}(M) \to \Omega_G^{\infty,\bullet}(M)$.*

2. *It satisfies $d_{\mathfrak{g}}^2 = 0$.*

Remark 5.1.4. There is another definition of equivariant cohomology which is purely algebraic topological which matches with this one under certain conditions; e.g. when M and G are compact. We discuss this next time.

Proof. Consider

$$g^*(d_{\mathfrak{g}}\alpha)(Y) = g^* d\alpha(Y) - g^* \iota_{Y^\sharp}\alpha(Y) = dg^*\alpha(Y) - g^*\iota_{Y^\sharp}\alpha(Y),$$

where $g^*\alpha(Y) = \alpha(\mathrm{Ad}_g Y)$ since $\alpha \in \Omega_G^{\infty,\bullet}(M)$.

We now at $g^*\iota_{Y^\sharp}\alpha(Y)$. Recall that $g^*\iota_{Y^\sharp} = \iota_{(\mathrm{Ad}_g Y)^\sharp} g^*$ (or prove this as an exercise). Thus

$$g^*\iota_{Y^\sharp}\alpha(Y) = \iota_{(\mathrm{Ad}_g Y)^\sharp} g^*\alpha(Y) = \iota_{(\mathrm{Ad}_g Y)^\sharp}\alpha(\mathrm{Ad}_g Y).$$

Overall,

$$g^*(d_{\mathfrak{g}}\alpha(Y)) = d\alpha(\mathrm{Ad}_g(Y)) - \iota_{(\mathrm{Ad}_g Y)^\sharp}\alpha(\mathrm{Ad}_g(Y)) = d_{\mathfrak{g}}\alpha(\mathrm{Ad}_g Y),$$

which proves the first part. To see $d_{\mathfrak{g}}^2 = 0$ observe that $d^2 = 0$ and $\iota_{Y^\sharp}^2 = 0$ so that

$$\begin{aligned} d_{\mathfrak{g}}^2\alpha(Y) &= d_{\mathfrak{g}}(d_{\mathfrak{g}}\alpha)(Y) = d(d_g\alpha)(Y) - \iota_{Y^\sharp}(d_{\mathfrak{g}}\alpha)(Y) \\ &= d(d\alpha(Y) - \iota_{Y^\sharp}\alpha(Y)) - \iota_{Y^\sharp}(d\alpha(Y) - \iota_{Y^\sharp}\alpha(Y)) \\ &= -(d\iota_{Y^\sharp} + \iota_{Y^\sharp} d)\alpha(Y) = -\mathcal{L}_{Y^\sharp}\alpha(Y). \end{aligned}$$

Observe that $\alpha(Y)$ is invariant under the action of the 1-parameter subgroup $\langle \exp(tY)\rangle$, i.e.

$$(\exp(tY))^*\alpha(Y) = \alpha(\mathrm{Ad}_{\exp(tY)} Y) = \alpha(Y), \tag{13}$$

where the last equality follows since from

$$\mathrm{Ad}_{\exp(tY)} Y = \frac{d}{ds}|_{s=0}[\exp(tY)\exp(sY)\exp(-tY)] = \frac{d}{ds}|_{s=0}[\exp(tY + sY - tY)] = Y,$$

where the second to last equality follows from the fact that $[Y, Y] = 0$. Equation (13) implies that $\mathcal{L}_{Y^\sharp}\alpha(Y) = 0$ and $d_{\mathfrak{g}}^2 = 0$. □

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Recall that

$$\Omega_G^{\infty,\bullet}(M) = \{\alpha : \mathfrak{g} \to \Omega^\bullet(M) \mid g^*\alpha(Y) = \alpha(\mathrm{Ad}_g Y), \forall g \in G, Y \in \mathfrak{g}\}$$

is the set of G-equivariant differential forms on (M, G), where M is a manifold with right group action $G \circlearrowleft M$.

We restrict to polynomial maps in $\mathfrak{g}$,

$$\alpha : \mathfrak{g} \to \Omega^\bullet(M),$$

i.e. consider a subalgebra $\Omega_G^{\cdot}(M) := (S(\mathfrak{g}^*) \otimes \Omega^\bullet(M))^G \subseteq \Omega_G^{\infty,\bullet}(M)$ of polynomial G-invariant forms. Here, $S(\mathfrak{g}^*)$ denotes the *symmetric algebra* on the vector space $\mathfrak{g}^*$, which for our purposes can be viewed as the polynomial ring over $\mathbb{C}$ in $\dim G = d$ variables. Namely, if $\{\xi_1, \ldots, \xi_d\}$ is a basis for $\mathfrak{g}^*$, then viewing ξ_j as formal variables, we have the identification

$$S(\mathfrak{g}^*) \cong \mathbb{C}[\xi_1, \ldots, \xi_d].$$

Take the usual grading by $\mathbb{N}_0 = \{0, 1, 2, \dots\} = \mathbb{Z}^{\geq 0}$ on

$$S(\mathfrak{g}^*) \cong \bigoplus_{i \in \mathbb{N}_0} S^i(\mathfrak{g}^*),$$

where $S^i(\mathfrak{g}^*)$ are the homogeneous polynomials of degree i. Define an $\mathbb{N}_0$-grading on $\Omega^\bullet_G(M)$ by

$$\Omega^\bullet_G(M) = \bigoplus_{l \in \mathbb{N}_0} \Omega^l_G(M),$$

where

$$\Omega^l_G(M) := \bigoplus_{2i+j=l} [S^i(\mathfrak{g}^*) \otimes \Omega^j(M)]^G.$$

A typical element of $\Omega^l_G(M)$, called an *elementary form* , looks like

$$\alpha = p^i(\xi) \otimes \alpha_{[j]},$$

where p^i is a degree i homogeneous polynomial, and α is a degree j differential form, and $2i+j = l$. The reason we define the grading in this way is so that the equivariant deRham derivative $d_\mathfrak{g}$ increases degree by one.

Lemma 5.1.5. *If $\alpha \in \Omega^l_G(M)$, then $d_\mathfrak{g}\alpha \in \Omega^{l+1}_G(M)$.*

Proof. It suffices to consider consider an elementary form $\alpha = p^i(\xi) \otimes \alpha_{[j]}$, where $2i + j = l$. Let $\{Y_1, \dots, Y_d\}$ be a basis for $\mathfrak{g}$ and $\{\xi_1, \dots, \xi_d\}$ the corresponding dual basis for $\mathfrak{g}^*$, i.e. $\xi_p(Y_q) = \delta_{pq}$.

The main observation is that

$$d_\mathfrak{g}\alpha = p^i(\xi) \otimes d\alpha_{[j]} - \sum_{q=1}^d \xi_q p^i(\xi) \otimes \iota_{Y_q^\sharp}\alpha_{[j]}. \tag{14}$$

To see this, evaluate both sides at $Y \in \mathfrak{g}$, where $Y = \sum_{q=1}^d a_q \cdot Y_q$ and note that $a_q = \xi_q(Y)$. Recall that by definition,

$$d_\mathfrak{g}\alpha(Y) = d\alpha(Y) - \iota_{Y^\sharp}\alpha(Y),$$

and clearly, $d\alpha(Y) = p^i(\xi)(Y) \otimes d\alpha_{[j]}$, and lastly

$$\begin{aligned}
\left(\sum_{q=1}^d \xi_q p^i(\xi) \otimes \iota_{Y_q^\sharp}\alpha_{[j]}\right)(Y) &= \sum_{q=1}^d \xi_q(Y) p^i(\xi)(Y) \otimes \iota_{Y_q^\sharp}\alpha_{[j]} \\
&= p^i(\xi)(Y) \otimes \left[\sum_{q=1}^d \xi_q(Y)\iota_{Y_q^\sharp}\alpha_{[j]}\right] \\
&= p^i(\xi)(Y) \otimes \iota_{\left[\sum_{q=1}^d \xi_q(Y) Y_q^\sharp\right]}\alpha_{[j]} \\
&= p^i(\xi)(Y) \otimes \iota_{Y^\sharp}\alpha_{[j]} = \iota_{Y^\sharp}\alpha(Y),
\end{aligned}$$

where the third to last equation follows from

$$\sum_{q=1}^d \xi_q(Y) Y_q^\sharp = \left(\sum_{q=1}^d (a_q Y_q)^\sharp\right) = Y^\sharp.$$

Thus we find that

$$\left[\sum_{q=1}^d \xi_q p^i(\xi) \otimes \iota_{Y_q^\sharp}\alpha_{[j]}\right](Y) = \iota_{Y^\sharp}\alpha(Y),$$

which implies (14). From (14) we can read of degrees. Namely,

$$\deg[p^i(\xi) \otimes d\alpha_{[j]}] = 2i + (j+1) = l+1,$$

and similarly

$$\deg\left[\sum_{q=1}^{d} \xi_q p^i(\xi) \otimes \iota_{Y_q^\sharp} \alpha_{[j]}\right] = 2(i+1) + (j-1) = l+1,$$

and overall, $\deg[d_{\mathfrak{g}}\alpha] = \deg\alpha + 1$. □

Definition 5.1.6. Let (M, G) be a manifold with a (right) Lie group action $G \circlearrowleft M$. The *equivariant deRham cohomology* $H_G^\bullet(M)$ of (M, G) is the cohomology of the *equivariant deRham complex*

$$0 \xrightarrow{d_{\mathfrak{g}}} \Omega_G^0(M) \xrightarrow{d_{\mathfrak{g}}} \Omega_G^1(M) \xrightarrow{d_{\mathfrak{g}}} \cdots \xrightarrow{d_{\mathfrak{g}}} \Omega_G^l(M) \xrightarrow{d_{\mathfrak{g}}} \cdots,$$

i.e.

$$H_G^l(M) := \frac{\mathrm{Ker}(d_{\mathfrak{g}} : \Omega_G^l(M) \to \Omega_G^{l+1}(M))}{\mathrm{Im}(d_{\mathfrak{g}} : \Omega_G^{l-1} \to \Omega_G^l(M))}.$$

We briefly outline the algebraic topology approach to (known as the Borel construction of) equivariant deRham cohomology. References for the following theorems are [MS74], [May99].

Theorem 5.1.7. *For any Lie group G, there exists a contractible topological space, denoted EG, on which G acts freely.*

The following theorem tells us that EG is unique up to homotopy equivalence.

Theorem 5.1.8. *If E_1, E_2 are contractible, and G acts freely on them, then there exist G-equivariant maps $\varphi_1 : E_1 \to E_2$, $\varphi_2 : E_2 \to E_1$, and G-equivariant homotopies $\varphi_1 \circ \varphi_2 \cong \mathbf{1}_{E_2}$, $\varphi_2 \circ \varphi_1 \cong \mathbf{1}_{E_1}$.*

Definition 5.1.9. The *classifying space* of a Lie group is the quotient space

$$BG := EG/G.$$

Note again that BG is unique up to homotopy equivalence. Also, since the action of G on $EG$is free, each fiber of the quotient $\pi : EG \to G$ is an orbit $G \cdot \{Q\}$ homeomorphic to G.

Example 5.1.10. If $S^\infty = \varinjlim S^{2k+1}$, the direct limit of odd dimensional spheres $S^{2k+1} \subseteq \mathbb{C}^{k+1}$, with respect to natural inclusions, then one can show that S^∞ is contractible, and S^1 acts freely on it (acting on every S^{2k+1}). Thus, $ES^1 = S^\infty$, and

$$BS^1 = ES^1/S^1 = \varinjlim \mathbb{CP}^k =: \mathbb{CP}^\infty.$$

If the action $G \circlearrowleft M$ is not free, the quotient space M/G can be poorly behaved, but we would like a way to talk about the (co)homology of M/G. Borel's idea was to replace M by $M \times EG$, where $G \circlearrowleft M \times EG$ through the diagonal action $g \cdot (p, Q) \equiv (p \cdot g, g^{-1} \cdot Q)$. Then

$$M \times_G EG := (M \times EG)/G$$

is well behaved since the action of G on $M \times EG$ is free, and $M \cong M \times EG$ are homotopy equivalent.

Definition 5.1.11. The *equivariant cohomology* of M is defined as $\tilde{H}_G(M) := H^\bullet(M \times_G EG)$.

Note that $\tilde{H}_G^\bullet(M)$ is well-defined with respect to any coefficient system R, i.e. we can define $\tilde{H}_G^\bullet(M,R) := H^\bullet(M \times_G EG, R)$ for any ring R. Equivariant deRham cohomology was defined with respect to $\mathbb{R}$ and $\mathbb{C}$ (we really only defined it for the complex numbers, but restricting to real differential forms, we can define it over the real numbers as well), and in this case, it turns out that the two notions of equivariant cohomology are equivalent, as the following theorem tells us. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, and let $H_G^\bullet(M,\mathbb{F})$ denote the equivariant deRham cohomology over $\mathbb{F}$.

Theorem 5.1.12 (Cartan, 1950)**.** *Let (M,G) be a compact smooth manifold with a Lie group action $G \circlearrowright M$ by a (compact) group. Then,*

$$\tilde{H}_G^\bullet(M,\mathbb{F}) \cong H_G^\bullet(M,\mathbb{F}).$$

Let $\alpha : \mathfrak{g} \to \Omega^\bullet(M)$ be a C^∞-equivariant differential form, let $Y \in \mathfrak{g}$, and let $\alpha(Y)_{[j]}$ denote the part of $\alpha(Y)$ in $\Omega^j(M)$. If $d_\mathfrak{g}\alpha(Y) = \sum_j \beta(Y)_{[j]}$, we have

$$\begin{aligned}
\beta(Y)_{[0]} &= -\iota_{Y^\sharp}\alpha(Y)_{[1]},\\
\beta(Y)_{[1]} &= d\alpha(Y)_{[0]} - \iota_{Y^\sharp}\alpha(Y)_{[2]},\\
&\vdots\\
\beta(Y)_{[j]} &= d\alpha(Y)_{[j]} - \iota_{Y^\sharp}\alpha(Y)_{[j+1]},\\
&\vdots\\
\beta(Y)_{[m]} &= d\alpha(Y)_{[m-1]},
\end{aligned}$$

where $m = \dim M$.

Definition 5.1.13. Define *equivariant integration* of $\alpha \in \Omega_G^{\infty,\bullet}(M)$ as the map $\int_M : \Omega_G^{\infty,\bullet} \to S(\mathfrak{g}^*)^G$, $\alpha \mapsto \int_M \alpha$, where $\left(\int_M \alpha\right)(Y) := \int_M \alpha(Y)_{[m]}$, and where $m = \dim M$.

Note that we have abused notation and that $\int_M \alpha(Y)_{[m]}$ denotes ordinary integration. Some authors use $\oint$ for equivariant integration, but we reserve this notation for later.

Observe that $\int_M d_\mathfrak{g}\alpha = 0$ since

$$\left(\int_M d_\mathfrak{g}\alpha\right)(Y) := \int_M (d_\mathfrak{g}\alpha(Y))_{[m]} = \int_M d(\alpha(Y)_{[m-1]}) = 0,$$

where the last equality follows from Stokes' theorem and the previous observation that $(d_\mathfrak{g}\alpha(Y))_{[m]} = \beta(Y)_{[m]} = d(\alpha(Y)_{[m-1]})$. This observation tells us that equivariant integration descends to cohomology, and we get a well-defined map

$$\int_M : H_G^\bullet(M) \to S(\mathfrak{g}^*)^G.$$

Example 5.1.14 (Equivariant cohomology of a point)**.** This example demonstrates a major difference between $H_G^\bullet(M)$ and $H^\bullet(M)$. Let $M = \{\text{pt}\}$. Then for $\mathbb{F}$ either $\mathbb{R}$ or $\mathbb{C}$, we have

$$\Omega^j(M) = \begin{cases} \mathbb{F}, & \text{for } j = 0,\\ 0, & \text{for } j > 0. \end{cases}$$

We have

$$\Omega_G^l(M) = \bigoplus_{2i=l} (S^i(\mathfrak{g}^*))^G = \begin{cases} (S^{l/2}(\mathfrak{g}^*))^G, & l \text{ even},\\ 0, & l \text{ odd}. \end{cases}$$

The equivariant deRham complex is

$$0 \xrightarrow{d_\mathfrak{g}} S^0(\mathfrak{g}^*)^G \xrightarrow{d_\mathfrak{g}} 0 \xrightarrow{d_\mathfrak{g}} (S^1(\mathfrak{g}^*))^G \xrightarrow{d_\mathfrak{g}} 0 \xrightarrow{d_\mathfrak{g}} \cdots,$$

and we get

$$H^l_G(\mathrm{pt}) = \begin{cases} (S^{l/2}(\mathfrak{g}^*))^G, & l \text{ even}, \\ 0, & l \text{ odd}. \end{cases}$$

People therefore write $H^\bullet_G(\mathrm{pt}) \cong (S^\bullet(\mathfrak{g}^*))^G$. Note that this of course depends heavily on G. One recovers ordinary cohomology when G is the trivial group.

Example 5.1.15. Consider the action of $G = S^1$ on itself, $M = S^1$, by rotations. We compute $H^\bullet_{S^1}(S^1)$ using the equivariant deRham model. One would expect that since the action is already free, $H^\bullet_{S^1}(S^1) \cong H^\bullet(S^1/S^1)$ – we return to this point later, and instead compute the equivariant deRham cohomology explicitly.

First, observe that

$$\Omega^l_G(S^1) = \begin{cases} \mathbb{F}^{l/2}[\xi] \otimes \Omega^0(S^1), & l \text{ even}, \\ \mathbb{F}^{(l-1)/2}[\xi] \otimes \Omega^1(S^1), & l \text{ odd}, \end{cases}$$

where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. Consider first the case where $l \geq 1$ is odd. Then $\alpha_l \in \Omega^l_{S^1}(S^1)$ is closed if and only if

$$d_{\mathfrak{g}}\alpha_l = -\xi p^{(l-1)/2}(\xi) \otimes \iota_{\theta^\sharp}\alpha_{[1]} = 0,$$

where $\xi \in (\mathrm{Lie}\, S^1)^*$ and $\theta \in (\mathrm{Lie}\, S^1)$ are standard bases. Thus $d_{\mathfrak{g}}\alpha = 0$ if and only if $\iota_{\theta^\sharp}\alpha_{[1]} = 0$ for $\alpha_{[1]} \in \Omega^1(S^1)$, where $\alpha_{[1]} = f(\theta)\, d\theta$ implies that $\iota_{\theta^\sharp}(f(\theta)d\theta) = f(\theta) = 0$, and $\alpha_{[1]} = 0$. Thus, if α_l is closed, we have $\alpha_l = 0$, and so

$$H^l_{S^1}(S^1) = 0,$$

when $l \geq 1$ is odd.

Consider the case $l \geq 2$ even. Then $\alpha_l \in \Omega^l_{S^1}(S^1)$ is closed if and only if $d_{\mathfrak{g}}\alpha_l = p^{l/2}(\xi) \otimes d\alpha_{[0]} = 0$, which is equivalent to $\alpha_{[0]} \in \mathbb{F}$ being a constant or, in other words, that $\alpha_l \in \mathbb{F}^{l/2}[\xi]$. On the other hand, $\alpha_l \in \Omega^l_{S^1}(S^1)$ is exact if and only if

$$\alpha_l = d_{\mathfrak{g}}\alpha_{l-1} = -\xi p^{(l-2)/2}(\xi) \otimes \iota_{\theta^\sharp}\alpha_{[1]},$$

where $d\iota_{\theta^\sharp}\alpha_{[1]} = 0$ since $d^2_{\mathfrak{g}} = 0$, and this implies that also $\iota_{\theta^\sharp}\alpha_{[1]} \in \mathbb{F}$ is a constant. Thus, α_l is exact if and only if $\alpha_l \in \mathbb{F}^{l/2}[\xi]$. Hence,

$$H^l_{S^1}(S^1) = 0$$

for all $l \geq 1$.

Finally, for $l = 0$, $\alpha_0 \in \Omega^0_{S^1}(S^1)$ is closed if and only if $d_{\mathfrak{g}}\alpha_0 = d\alpha_{[0]} = 0$, which holds if and only if $\alpha_0 \in \mathbb{F}$ is constant. Since exact forms in $\Omega^0_{S^1}(S^1)$ are $0 \in \mathbb{F}$, we get that

$$H^0_{S^1}(S^1) = \mathbb{F}.$$

Overall, we observe that $H^\bullet_{S^1}(S^1) \cong H^\bullet(S^1/S^1) = H^\bullet(\mathrm{pt})$.

18th lecture, November 11th 2011

Recall, last time we saw $H^\bullet_{S^1}(S^1) \cong H^\bullet(S^1/S^1) \cong H^\bullet(\mathrm{pt})$. Our basic observation is that this action is free (and transitive). Generalizing this observation, we arrive at the following Proposition.

Proposition 5.1.16. *If $G \circlearrowright M$ locally freely, then the projection map $\pi : M \times_G EG \to M/G$ induces an isomorphism*

$$\tilde{H}^\bullet_G(M, \mathbb{F}) \cong H^\bullet(M/G, \mathbb{F})$$

for $\mathbb{F}$ either $\mathbb{R}$ or $\mathbb{C}$.

If G acts freely, then $\tilde{H}^\bullet_G(M, R) \cong H^\bullet(M/G, R)$ for any coefficient ring R.

For a proof of this result, see [GS99] or [Car51b] and [Car51a].
Note that if G acts locally freely but not freely, then the map induced by projection

$$\pi^* : H^\bullet(M/G, \mathbb{Z}) \to \tilde{H}^\bullet_G(M, \mathbb{Z})$$

may fail to be an isomorphism. One example of this is the action $S^1 \circlearrowright S^3 \subseteq \mathbb{C}^2$ given by $\eta \cdot (z_1, z_2) \equiv (\eta z_1, \eta^2 z_2)$ (check).

Theorem 5.1.17 (Mayer–Vietoris)**.** *If $U, V \subseteq (M, G)$ are G-invariant subsets such that $M = U^\circ \cup V^\circ$, then there is a long exact sequence*

$$\cdots \to \tilde{H}^l_G(M) \xrightarrow{r} \tilde{H}^l_G(U) \oplus \tilde{H}^l_G(V) \xrightarrow{s} \tilde{H}^l_G(U \cap V) \xrightarrow{\delta} \cdots,$$

where r, s, δ are the usual Mayer–Vietoris maps.

The proof follows from the non-equivariant version; see e.g. [Mas80, p. 58].

Example 5.1.18. Let $G = S^1$, $M = S^2$, where $S^1 \circlearrowright S^2$ by rotations about the z-axis. We compute $H^\bullet_{S^1}(S^2)$ using Mayer–Vietoris. Let $S, N \in S^2$ be the south and north poles respectively, and let $U_S := S^2 \setminus \{N\}$ and $U_N := S^2 \setminus \{S\}$. These are open S^1-invariant subsets of S^2. Note that $U_S \cap U_N$ equivariantly retracts onto the equator S_E at $z = 0$, and the action $S^1 \circlearrowright S_E$ is free. Hence

$$H^\bullet_{S^1}(U_S \cap U_N) \cong H^\bullet_{S^1}(S^1_E) \cong H^\bullet(\mathrm{pt}),$$

and $H^\bullet_{S^1}(U_S) \cong H^\bullet_{S^1}(\{S\}) = \mathbb{F}[\xi_S]$, $H^\bullet_{S^1}(\{N\}) \cong \mathbb{F}[\xi_N]$. Now Mayer–Vietoris implies that

$$0 \to \tilde{H}^l_{S^1}(S^2) \to \tilde{H}^l_{S^1}(U_S) \oplus \tilde{H}^l_{S^1}(U_N) \to 0.$$

This implies that $\tilde{H}^l_{S^1}(S^2) \cong \mathbb{F}^l[\xi_S] \oplus \mathbb{F}^l[\xi_N]$. We observe that $\xi_S \cdot \xi_N \equiv 0$ in the ring structure to get this identification. Thus as a ring,

$$\tilde{H}^\bullet_{S^1}(S^2) \cong \mathbb{F}[\xi_S, \xi_N]/\langle \xi_S \cdot \xi_N \rangle,$$

where $\langle \xi_S \cdot \xi_N \rangle$ is the ideal generated by $\xi_S \cdot \xi_N$.

We note the following properties of equivariant cohomology:

1. G-equivariant maps $\psi : M_1 \to M_2$ induce pullbacks on equivariant differential forms and cohomology, $\psi^* : \Omega^\bullet_G(M_2) \to \Omega^\bullet_G(M_1)$ and $\psi^* : H^\bullet_G(M_2) \to H^\bullet_G(M_1)$.
2. Let $R_G := H^\bullet(BG) \cong H^\bullet_G(\mathrm{pt}) \cong (S^\bullet(\mathfrak{g}^*))^G$. Then the map $\pi_{\mathrm{pt}} : M \to \mathrm{pt}$ induces an R_G-module structure on $\Omega^\bullet_G(M)$ and $H^\bullet_G(M)$, so $\bigoplus_q \Omega^q_G(M)$ and $\bigoplus_q H^q_G(M)$ are graded R_G-modules. Recall that the module structure is defined by $(\cdot, \cdot) : R_G \times \Omega^\bullet_G(M) \to \Omega^\bullet_G(M)$ by $(\varphi, \alpha) \mapsto \pi^*_{\mathrm{pt}}(\varphi) \wedge \alpha$.
3. For $\psi_t : M_1 \to M_2$, $t \in [0, 1]$, a G-equivariant homotopy, the induced maps $\psi_t^* : H^\bullet_G(M_2) \to H^\bullet_G(M_1)$ are equivalent for all $t \in [0, 1]$.

5.2 The Atiyah–Bott–Berline–Vergne theorem

Lemma 5.2.1. *Let $\alpha \in \Omega^l_G(M)$, $\beta \in \Omega^k_G(M)$, and let $|\alpha| := l$, $|\beta| := k$ denote degrees. Then,*

(i) $d_{\mathfrak{g}}(\alpha \wedge \beta) = d_{\mathfrak{g}}(\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d_{\mathfrak{g}}\beta$, *and*

(ii) $\alpha \wedge \beta = (-1)^{|\alpha| \cdot |\beta|} \beta \wedge \alpha$.

Proof. For $a \in \Omega^{|}a|(M)$, $b \in \Omega^{|}b|(M)$, we know that

$$d(a \wedge b) = da \wedge b + (-1)^{|a|} a \wedge db,$$
$$\iota_v(a \wedge b) = \iota_v a \wedge b + (-1)^{|}a| a \wedge \iota_v b,$$

for $v \in \Gamma(TM)$, and $a \wedge b = (-1)^{|a||b|} b \wedge a$, for ordinary forms. Combining this with the fact that if $\alpha = p^i(\xi) \otimes \alpha_{[j]} \in \Omega^l_G(M)$, then $(-1)^l = (-1)^j$ since $l = 2i + j$, we get what we want. □

Proposition 5.2.2 ([BGV04], Prop. 7.10)**.** *Let G be a compact Lie group and let $\alpha \in \mathrm{Map}^\infty(\mathfrak{g}, \Omega^\bullet(M))$ be equivariantly closed , i.e. $d_{\mathfrak{g}}\alpha = 0$ (and note that we do not require α to be in $\Omega^{\infty,\bullet}_G(M)$). Let $Y \in \mathfrak{g}$ and $M^Y = \{p \in M \mid Y^\sharp(p) = 0\}$ (which we do not necessarily assume to be discrete). Then $\alpha(Y)_{[m]} \in \Omega^m(M)$, $m = \dim M$, is exact on $M \setminus M^Y$.*

Proof. Since G is compact, we can pick a G-invariant Riemannian metric on M, $\langle \cdot, \cdot \rangle \in \mathrm{Sym}^2 T^*M$. Define $\theta \in \mathrm{Map}^\infty(\mathfrak{g}, \Omega^1(M))$ by $\theta(Y)(\cdot) \equiv Y^\sharp, \cdot\rangle$. It is easy to see that $\theta \in \Omega^3_G(M)$ since it is linear in Y and is G-equivariant, i.e.

$$\begin{aligned}(g^*\theta)(Y)_p(v) \equiv \theta(Y)_{p\cdot g}(g_* v) \equiv \langle Y^\sharp_{p\cdot g}, g_*(v_p)\rangle = \langle g_*(\mathrm{Ad}_g\, Y)^\sharp_p, g_* v_p\rangle \\ = \langle (\mathrm{Ad}_g\, Y)^\sharp_p, v_p\rangle,\end{aligned}$$

thus $g^*\theta(Y) = \theta(\mathrm{Ad}_g\, Y)$. Observe that $\theta_Y := \theta(Y) \in \Omega^1(M)$ is non-zero on $M \setminus M^Y$ for all $Y \in \mathfrak{g}$ since $\iota_{Y^\sharp}\theta_Y = \|Y^\sharp\|^2 \neq 0$ on $M \setminus M^Y$. Also, $d_{\mathfrak{g}}\theta(Y) = d\theta(Y) - \|Y^\sharp\|^2 \in \Omega^\bullet(M \setminus M^Y)$ is invertible. Recall that in general, if $\tilde{C}$ is an algebra, $n \in \tilde{C}$ is nilpotent and $c \in \tilde{C}$ any invertible element commuting with n, then

$$(-c + n)^{-1} = -c^{-1}(1 - n/c)^{-1} = -c^{-1} - c^{-2}n - c^{-3}n^2 - \cdots$$

is a finite sum and defines the inverse operation in $\tilde{C}$. Thus in our case, we have the finite sum

$$(d_{\mathfrak{g}}\theta(Y))^{-1} = -\|Y^\sharp\|^{-2} - \|Y^\sharp\|^{-4}(d\theta(Y)) - \cdots - \|Y^\sharp\|^{-2(l+1)}(d\theta(Y))^l - \cdots \in \Omega^\bullet(M \setminus M^Y)$$

and satisfies $(d_{\mathfrak{g}}\theta(Y)) \wedge (d_{\mathfrak{g}}\theta(Y))^{-1} = 1$. Let $d_Y := d - \iota_{Y^\sharp} : \Omega^\bullet(M) \to \Omega^\bullet(M)$. Observe that for any $\beta \in \Omega^\bullet_G(M)$ we have $(d_{\mathfrak{g}}\beta)(Y) \equiv d_y(\beta(Y))$, and

$$d_Y^2(\beta(Y)) = (d^2_{\mathfrak{g}}\beta)(Y) = 0.$$

In particular, $d_Y(\alpha(Y)) = (d_{\mathfrak{g}}\alpha)(Y) = 0$ by assumption, so $d_Y\theta_Y = d_{\mathfrak{g}}\theta(Y)$ and $d^2_Y\theta_Y = d^2_{\mathfrak{g}}\theta(Y) = 0$.

Observe that $d_Y[d_{\mathfrak{g}}\theta(Y)]^{-1} = 0$. This follows from

$$d_Y[d_{\mathfrak{g}}\theta(Y) \wedge (d_{\mathfrak{g}}\theta(Y))^{-1}] = d_Y 1 = 0$$

but on the other hand

$$\begin{aligned}d_Y[d_{\mathfrak{g}}\theta(Y) \wedge (d_{\mathfrak{g}}\theta(Y))^{-1}] = d_Y d_{\mathfrak{g}}\theta(Y) \wedge (d_{\mathfrak{g}}\theta(Y))^{-1} \pm d_{\mathfrak{g}}\theta(Y) \wedge d_Y[d_{\mathfrak{g}}\theta(Y)]^{-1} \\ = \pm d_{\mathfrak{g}}\theta(Y) \wedge d_Y[d_{\mathfrak{g}}\theta(Y)]^{-1},\end{aligned}$$

and multiplying by $[d_{\mathfrak{g}}\theta(Y)]^{-1}$ yields the claim. Let $\nu_Y := \theta_Y \wedge [d_{\mathfrak{g}}\theta(Y)]^{-1} \in \Omega^\bullet(M \setminus M^Y)$. Then

$$d_Y\nu_Y = d_Y\theta Y \wedge [d_{\mathfrak{g}}\theta(Y)]^{-1} = (d_{\mathfrak{g}}\theta(Y)) \wedge [d_{\mathfrak{g}}\theta(Y)]^{-1} = 1.$$

Thus

$$d_Y(\nu_Y \wedge \alpha(Y)) = d_Y\nu_Y \wedge \alpha(Y) \pm \nu_Y \wedge d_Y\alpha(Y) = d_Y\nu_Y \wedge \alpha(Y) = \alpha(Y).$$

Taking the top degree component of this equation, implies that

$$\alpha(Y)_{[m]} = d[(\nu_Y \wedge \alpha(Y))_{[m-1]}].$$

□

Note that the assumption that G is compact is necessary, as the following example from [BGV04] shows.

Example 5.2.3. Let $M = S^1 \times S^1$, $G = \mathbb{R}$, and let $x, y \in \mathbb{R}/2\pi\mathbb{Z}$ be coordinates on M. Let $\mathbb{R}$ act on M via the flow of the non-vanishing vector field $Y = (2 + \sin x)\frac{\partial}{\partial y}$. Let

$$\alpha(Y) = -(7\cos x + \sin(2x)) + (1 - 4\sin x)dx \wedge dy.$$

One verifies that $d_Y\alpha(Y) = 0$, but

$$\int_{S^1\times S^1} \alpha(Y) = \int_{S^1\times S^1} \alpha(Y)_{[2]} = (2\pi)^2,$$

and so $\alpha(Y)_{[2]}$ is *not* exact.

This last proposition suggest that $\int_M \alpha(Y)$, for $d_{\mathfrak{g}}\alpha = 0$, depends only on the restriction of $\alpha(Y)$ to the fixed point set M^Y.

We will study the case for M^Y finite. First, recall the definition of the isotropy representation (Definition 3.2.7).

Lemma 5.2.4. *Let $G \circlearrowright M$, $v \in T_pM$, $g \in G$ such that $p \cdot g = p$. Let $\exp_p : T_p \to M$ be the Riemannian exponential map for a G-invariant metric $h_G \in \mathrm{Sym}^2 T^*M$. Then*

$$\exp_p(v) \cdot g = \exp_p(g_* v).$$

Proof. This follows by uniqueness of the map $\exp_p$ as follows: Let

$$\gamma_g^v(t) := \exp_p(tv) \cdot g.$$

Then clearly $\gamma_g^v(0) = \exp_p(0) \cdot g = p \cdot g = p$. Also

$$\frac{d}{dt}|_{t=0}\gamma_g^v(t) = g_* \frac{d}{dt}|_{t=0} \exp_p(tv) = g_* v.$$

Since $\exp_p(tg_*v)$ is the unique flow satisfying these two conditions, we see that $\gamma_g(1) = \exp_p(v)\cdot g = \exp(g_*v)$. □

Recall Proposition 3.2.8:

Proposition 5.2.5. *Let $h_G \in \mathrm{Sym}^2 T^*M$ be a G-invariant Riemannian metric on (M, G). Let $\exp_p : T_pM \to M$ defined by h_G. Then for any $p \in M^Y$, $Y \in \mathfrak{g}$, we have that $v \in \ker(L_p(Y))$ if and only if $\exp_p(v) \in M^Y$.*

Proof. Let $q = \exp_p(v)$ for $v \in T_pM$, $p \in M^Y$. Let $\varphi_t : M \to M$ be the flow

$$\varphi_t(q) \equiv q \cdot \exp(tY) = \exp_p(v) \cdot \exp(tY) = \exp_p((\exp tY)_* v).$$

Then

$$\begin{aligned} Y_q^\sharp &= \frac{d}{dt}|_{t=0}(\varphi_t(q)) = \frac{d}{dt}|_{t=0}[\exp(\exp(tY)_* v)] \\ &= \frac{d}{dt}|_{t=0}[\exp(tY)_* v] \end{aligned}$$

since $d\exp_p|_v = \mathbf{1}_{T_pM}$ under the identification $T_vT_pM \cong T_pM$. Let $\tilde{v} \in \Gamma(TM)$ be a vector field with $\tilde{v}_p = v$. Then $\tilde{v}_{\varphi_{-t}(p)} = \tilde{v}_p = v$ since $\varphi_t(p) = p \cdot \exp(tY) = p$ for all $t \in (-\varepsilon, \varepsilon)$, $p \in M^Y$. So,

$$\begin{aligned} Y_q^\sharp &= \frac{d}{dt}|_{t=0}[\exp(tY)_* v] = \frac{d}{dt}|_{t=0}[(\varphi_t)_* \tilde{v}_{\varphi_{-t}(p)}] \\ &= (\mathcal{L}_{Y^\sharp}\tilde{v})_p = L_p(Y)(v). \end{aligned}$$

Thus, for $p \in M^Y$, we have $v \in \ker L_p(Y)$ if and only if $Y_q^\sharp = L_p(Y)(v) = 0$ which holds if and only if $q = \exp_p(v)$ is in M^Y. □

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A corollary to Proposition 5.2.5 is the following:

Corollary 5.2.6. *A fixed point $p \in M^Y$ is isolated if and only if $L_p(Y)$ is invertible.*

Proof. Assume that $L_p(Y)$ is not invertible. Then there exists $0 \neq v \in \ker L_p(Y)$, and in fact, $0 \neq tv \in \ker L_p(Y)$ for $t \in (-\varepsilon, \varepsilon)$. By Proposition 5.2.5, $\exp_p(tv) \in M^Y$ for $t \in (-\varepsilon, \varepsilon)$. For any open neighbourhood U_p of $p = \exp_p(0)$, there exists $t_0 \in (-\varepsilon, \varepsilon)$ such that $\exp_p(t_0 v) \in U_p$. This means that p can not be isolated.

If $p \in M^Y$ is not isolated, there exists a sequence $q_j \in M^Y$, $j \in \mathbb{N}$, such that $\lim_{j\to\infty} q_j = p$. Choosing a small enough open neighbourhood V_p containing p, we know $\exp_p : W_0 \to V_p$, $W_0 \subseteq T_pM$, is a local diffeomorphism. Then for $q \in V_p \cap M^Y \neq \emptyset$, there exists $0 \neq v_q \in T_pM$ such that $q = \exp_p(v_q)$. By Proposition 5.2.5, $v_q \in \ker L_p(Y)$, and $L_p(Y)$ is not invertible. □

Recall that $L_p(Y)$ is skew-symmetric with respect to a G-invariant metric h_G, i.e.

$$h_G(L_p(Y)v, w) = -h_G(v, L_p(Y)w)$$

for all $v, w \in T_pM$. If $p \in M^Y$, then since $L_p(Y)$ is invertible and skew-symmetric, $\dim M = m = 2n$ must be even. Let $\{e_1, \dots, e_n\}$ be an oriented h_G-orthogonal basis of T_pM such that

$$L_p(Y)e_{2i-1} = -\lambda_i e_{2i}, \quad L_p(Y)e_{2i} = \lambda_i e_{2i-1}, \tag{15}$$

for $\lambda_i \in \mathbb{R}^{\neq 0}$, $1 \leq i \leq n$. That is, in these coordinates,

$$L_p(Y) = \begin{pmatrix} \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix} \end{pmatrix}.$$

Let $x = (x_1, \dots, x_m) = x_1 e_1 + \cdots + x_m e_m$ be a coordinate system on T_pM. Consider $Y^\sharp(x)$ with respect to this coordinate system, i.e.

$$Y^\sharp(x) = (\exp_p^{-1})_* Y^\sharp_q,$$

where $q = \exp_p(x)$. Observe that

$$\begin{aligned} Y^\sharp(x) &= (\exp_p^{-1})_* \frac{d}{dt}|_{t=0}[\varphi_t(q)] = (\exp_p^{-1})_* \frac{d}{dt}|_{t=0}[q \cdot \exp_p(tY)] \\ &= (\exp_p^{-1})_* \frac{d}{dt}|_{t=0}[\exp_p((\exp tY)_* x)] = (\exp_p^{-1})_* (\exp_p)_* \frac{d}{dt}|_{t=0}[\exp(tY)_* x], \end{aligned}$$

where the third step follows from a previous lemma. In particular $Y^\sharp(x)$ is linear in $x \in T_pM$. Write

$$Y^\sharp(x) = \sum_{i=1}^m l_i(x) \frac{\partial}{\partial x_i}$$

for some linear functions $l_i : T_pM \to \mathbb{R}$. We want to compute $l_i(x)$. To do this, we use that $(\mathcal{L}_{Y^\sharp}\tilde{v})_0 = [Y^\sharp, \tilde{v}]|_0 = L_0(Y)(v)$ for $\tilde{v} \in \Gamma(T_pM)$ such that $\tilde{v}|_0 = v$. Consider $\tilde{v}_i := \frac{\partial}{\partial x_i}$ and identify $\tilde{v}|_0 = e_i$.

By (15), and since $Y_j^\sharp(0) = 0$, the formula for the Lie bracket gives

$$\begin{aligned} [Y^\sharp, \tilde{v}_{2i-1}]|_0 &= \sum_{j,k} \left[Y_j^\sharp(y) \frac{\partial (v_{2i-1})_k}{\partial x_j}(0) - (v_{2i-1})_j(0) \frac{\partial Y_k^\sharp}{\partial x_j}(0) \right] e_k \\ &= \sum_k \left[-\frac{\partial Y_k^\sharp}{\partial x_{2i-1}}(0) \right] e_k = L_0(Y)e_{2i-1} = -\lambda_i e_{2i}. \end{aligned}$$

Thus

$$\frac{\partial Y_k^\sharp}{\partial x_{2i-1}}(0) = 0, \quad k \neq 2i,$$
$$\frac{\partial Y_{2i}^\sharp}{\partial x_{2i-1}}(0) = \lambda_i.$$

Similarly, one finds that

$$\frac{\partial Y_k^\sharp}{\partial x_{2i}}(0) = 0, \quad k \neq 2i-1,$$
$$\frac{\partial Y_{2i-1}^\sharp}{\partial x_{2i}}(0) = -\lambda_i.$$

Observe that these relations imply that

$$\frac{\partial Y_{2i}^\sharp}{\partial x_j} = 0, \quad j \neq 2i-1,$$
$$\frac{\partial Y_{2i}^\sharp}{\partial x_{2i-1}}(0) = \lambda_i,$$
$$\frac{\partial Y_{2i-1}^\sharp}{\partial x_j}(0) = 0, \quad j \neq 2i,$$
$$\frac{\partial Y_{2i-1}^\sharp}{\partial x_{2i}}(0) = -\lambda_i.$$

Since $Y_k^\sharp(x) = l_k(x)$ is linear in x, these relations imply that

$$l_{2i}(x) = \lambda_i x_{2i-1}, \quad l_{2i-1}(x) = -\lambda_i x_{2i},$$

i.e.

$$Y^\sharp(x) = \lambda_i \left[x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right] + \cdots + \lambda_n \left[x_{m-1} \frac{\partial}{\partial x_m} - x_m \frac{\partial}{\partial x_{m-1}} \right].$$

Definition 5.2.7. Let $\mathcal{U}_G = \{U_j\}_{j\in\mathcal{O}}$ be a G-invariant open cover of (M, G). A *G-invariant partition of unity subordinate to $\mathcal{U}_G$* is an ordinary partition of unity subordinate to $\mathcal{U}_G$, $\{\varphi_j\}_{j\in\mathcal{O}}$, where the functions $\varphi_j : U_j \to [0, 1]$ are G-invariant, i.e. $\varphi_j(p \cdot g) = \varphi_j(p)$ for all $p \in U_j$, $g \in G$, $j \in \mathcal{O}$.

The proof of the following Lemma is a non-trivial argument. See [GGK02, Cor. B.3.3] and note that it is a corollary of the slice theorem.

Lemma 5.2.8. *Let $G \circlearrowright M$ be a proper group action of a Lie group G on a smooth manifold M. Then for any G-invariant open cover $\mathcal{U}_G$, there exists a G-invariant partition of unity subordinate to that cover.*

One of the key parts of the proof of the Berline–Vergne theorem is the following: For $Y \in \mathfrak{g}$, we construct a 1-form, $\Omega_Y \in \Omega^1(M)$, with the following properties:

$$d_Y^2 \Omega_Y = 0, \quad \iota_{Y^\sharp} \Omega_Y \neq 0. \tag{16}$$

Recall here that $d_Y := d - \iota_{Y^\sharp} : \Omega^\bullet \to \Omega^\bullet(M)$ and for $\beta \in \Omega^\bullet(M)$, we have $d_Y^2 \beta = \mathcal{L}_{Y^\sharp} \beta$. In particular, the latter will vanish for invariant forms β. Also, recall from the proof of a previous proposition that $\iota_{Y^\sharp} \beta \neq 0$ implied that $d_Y \beta$ is invertible in $\Omega^\bullet(M \setminus M^Y)$ in the sense that

$$(d_Y\beta)^{-1} = -(\iota_{Y^\sharp}\beta)^{-1} - (\iota_{Y^\sharp}\beta)^{-2}(d_Y\beta) - \cdots - (\iota_{Y^\sharp}\beta)^{-(l+1)}(d_Y\beta)^l - \cdots \in \Omega^\bullet(M \setminus M^Y)$$

is well-defined and $(d_Y\beta) \wedge (d_Y\beta)^{-1} = 1$.

Proposition 5.2.9. *Let $\alpha \in \mathrm{Map}^\infty(\mathfrak{g}, \Omega^\bullet(M))$ be equivariantly closed, i.e. $d_{\mathfrak{g}}\alpha = 0$ (but not necessarily in $\Omega_G^{\infty,\bullet}(M)$). For any $\beta \in \Omega^1(M)$ satisfying (16), i.e. $d_Y^2\beta = 0$, $\iota_{Y^\sharp}\beta \not\equiv 0$, let $\nu_Y := \beta \wedge (d_Y\beta)^{-1} \in \Omega^\bullet(M \setminus M^Y)$. Then $\alpha(Y)_{[m]} = d[(\nu_Y \wedge \alpha(Y))_{[m-1]}]$ on $M \setminus M^Y$.*

The proof of this is exactly the same as that of Propostion 5.2.2. Before we construct Ω_Y, we construct forms $\Omega_Y^p \in \Omega^1(U_p)$ for each $p \in M^Y$, where U_p are G-invariant opens sets containing p. We view U_p as subsets of T_pM via $\exp_p : T_pM \to M$. The forms are defined explicitly by

$$\Omega_Y^p := \lambda_1^{-1}[x_1dx_2 - x_2dx_1] + \cdots + \lambda_n^{-1}[x_{m-1}dx_m - x_mdx_{m-1}].$$

Claim 5.2.10. *These forms satisfy (16), i.e. (a) $d_Y^2\Omega_Y^p = 0$ and (b) $\iota_{Y^\sharp}\Omega_Y^p \neq 0$.*

Proof. (a) Since $d_Y^2\Omega_Y^p = -\mathcal{L}_{Y^\sharp}\Omega_Y^p$, we can calculate

$$\mathcal{L}_{Y^\sharp}\Omega_Y^p = \iota_{Y^\sharp}d\Omega_Y^p + d\iota_{Y^\sharp}\Omega_Y^p$$

directly. Observe that it suffices to consider the restriction of Ω_Y^p to, say, the x_1-x_2-plane. Then

$$d\iota_{Y^\sharp}\Omega_Y^p = d[\iota_{(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1})}(x_1dx_2 - x_2dx_1)] = d[x_1^2 + x_2^2] = 2x_1dx_1 + 2x_2dx_2,$$

and on the other hand

$$\iota_{Y^\sharp}d\Omega^p = \iota_{(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1})}d[x_1dx_2 - x_2dx_1] = \iota_{(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1})}[2dx_1 \wedge dx_2] = -2x_1dx_1 + 2x_2dx_2,$$

so overall, $\mathcal{L}_{Y^\sharp}\Omega_Y^p = 0$.

(b) We saw above that, restricted to the x_1-x_2-plane, $\iota_{Y^\sharp}\Omega_Y^p = x_1^2 + x_2^2$. In general, $\iota_{Y^\sharp}\Omega_Y^p = \sum_{i=1}^m x_i^2 \neq 0$ for $x \neq 0$. □

For fixed $Y \in \mathfrak{g}$, let $\theta_Y(\cdot) := \langle Y^\sharp, \cdot\rangle \in \Omega^1(M)$ for any G-invariant $\langle\ ,\ \rangle \in \mathrm{Sym}^2 T^*M$. Then as we have seen, $\iota_{Y^\sharp}\theta_Y > 0$ on $M \setminus M^Y$ and $d_Y^2\theta_Y$, i.e. θ_Y satisfies (16). Let $\mathcal{U}_G = \{U_p, M \setminus M^Y\}_{p\in M^Y}$ be a finite open G-invariant cover of M, where U_p are the G-invariant open sets associated to the forms Ω_Y^p above. Let $\{\varphi_p, \varphi\}_{p\in M^Y}$ be a G-invariant partition of unity, subordinate to $\mathcal{U}_G$. Define the global form

$$\Omega_Y = \varphi \cdot \theta_Y + \sum_{p\in M^Y} \varphi_p \cdot \Omega_Y^p \in \Omega^1(M).$$

Claim 5.2.11. *The form Ω_Y satisfies (16).*

This follows from the facts that θ_Y and Ω_Y^p satisfy (16) and that φ, φ_p are G-invariant. Also, one uses that $\mathcal{L}_{Y^\sharp}(a \wedge b) = (\mathcal{L}_{Y^\sharp}(a)) \wedge b + a \wedge (\mathcal{L}_{Y^\sharp}(b))$.

By definition of partitions of unity, we know that $\operatorname{supp}\varphi \subseteq M \setminus M^Y$. Since M^Y is assumed to be finite, we can find $\varepsilon > 0$ such that $B_\varepsilon(p) \cap \operatorname{supp}\varphi = \emptyset$ for all $p \in M^Y$, where $B_\varepsilon(p) = \{x \in U_p \mid \|x\|^2 \leq \varepsilon\}$ and $\|x\|^2 = \sum x_i^2$.

Thus, $\Omega_Y|_{B_\varepsilon(p)} = \Omega_Y^p|_{B_\varepsilon(p)}$, and we have a nice explicit description of Ω_Y near $p \in M^Y$.

Theorem 5.2.12 (Atiyah–Bott–Berline–Vergne)**.** *Let G be a compact Lie group acting on a smooth oriented compact manifold M, and let $\alpha \in \mathrm{Map}^\infty(\mathfrak{g}, \Omega^\bullet(M))$ be any equivariantly closed differential form on M, i.e. $d_{\mathfrak{g}}\alpha = 0$ (but not necessarily in $\Omega_G^{\infty,\bullet}(M)$). Let $Y \in \mathfrak{g}$ such that M^Y is isolated. Then*

$$\int_M \alpha(Y) = (-2\pi)^n \sum_{p\in M^Y} \frac{\alpha(Y)_{[0]}(p)}{\mathrm{Pf}(L_p(Y))},$$

where $\dim M = m = 2n$, and $\alpha(Y)_{[0]}(p)$ denotes the value of the function $\alpha(Y)_{[0]}$ at $p \in M$.

Proof. First, observe that since $\Omega_Y \in \Omega^1(M)$ satisfies (16), if $\nu_Y := \Omega_Y \wedge (d_Y\Omega_Y)^{-1} \in \Omega^\bullet(M\backslash M^Y)$, we may write

$$\alpha(Y)_{[m]} = d[(\nu_Y \wedge \alpha(Y))_{[m-1]}] = d\left[\left(\frac{\Omega_Y \wedge \alpha(Y)}{d_Y\Omega_Y}\right)_{[m-1]}\right].$$

Write $S_\varepsilon(p) = \{x \in U_p \mid \|x\|^2 = \varepsilon = -\partial B_\varepsilon(p)$, where the sign comes from exchanging the interior for exterior orientation of $S_\varepsilon(p)$. Then by our previous results and by Stokes' theorem, we have

$$\begin{aligned}\int_M \alpha(Y) :&= \int_M \alpha(Y)_{[m]} = \lim_{\varepsilon\to 0^+} \int_{M\backslash\bigcup_{p\in M^Y} B_\varepsilon(p)} \alpha(Y)_{[m]} = \lim_{\varepsilon\to 0^+} \int_{M\backslash\bigcup_{p\in M^Y} B_\varepsilon(p)} d\left[\left(\frac{\Omega_Y \wedge \alpha(Y)}{d_Y\Omega_Y}\right)_{[m-1]}\right] \\ &= -\sum_{p\in M^Y} \lim_{\varepsilon\to 0^+} \int_{S_\varepsilon(p)} \left[\frac{\Omega_Y^p \wedge \alpha(Y)}{d_Y\Omega_Y^p}\right]_{[m-1]}\end{aligned}$$

Recall that

$$(d_Y\Omega_Y^p)^{-1}|_{S^\varepsilon(p)} = -\varepsilon^{-1} - \varepsilon^{-2}(d\Omega_Y^p) - \varepsilon^3(d\Omega_Y^p)^2 - \cdots - \varepsilon^{-(n+1)}(d\Omega_Y^p)^n.$$

This means that

$$\left[\frac{\Omega_Y^p \wedge \alpha(Y)}{d_Y\Omega_Y^p}\right]_{[m-1]}|_{S_\varepsilon(p)} = -\varepsilon^{-n}\alpha(Y)_{[0]}\Omega_Y^p \wedge (d\Omega_Y^p)^{n-1} - \sum_{j=1}^{n-1} \varepsilon^{j-n}\Omega_Y^p \wedge \alpha(Y)_{[2j]} \wedge (d\Omega_Y^p)^{n-j-1}.$$

Observe that, when we integrate and take $\varepsilon \to 0$, the only term to survive is the first one. Make a change of variables $\tilde{x} := \varepsilon^{-1/2}x$ so that $\|\tilde{x}\| = 1$ on $S_\varepsilon(p)$. Then since

$$\Omega_Y^p(x) = \sum_{j=1}^n \lambda_j^{-1}[x_{2j-1}dx_{2j} - x_{2j}dx_{2j-1}],$$

we see that $\Omega_Y^p(x) = \varepsilon\Omega_Y^p(\tilde{x})$ and so

$$\Omega_Y^p(x) \wedge (d\Omega_Y^p(x))^n = \varepsilon^n\Omega_Y^p(\tilde{x}) \wedge (d\Omega_Y^p(\tilde{x}))^{n-1}.$$

Let $\alpha(Y)^\varepsilon_{[0]}(\tilde{x}) := \alpha(Y)_{[0]}(x)$. Then

$$-\varepsilon^{-n}\alpha(Y)_{[0]}(x)\Omega_Y^p(x) \wedge (d\Omega_Y^p(x))^{n-1} = -\alpha(Y)^\varepsilon_{[0]}(\tilde{x})\Omega_Y^p(\tilde{x}) \wedge (d\Omega_Y^p(\tilde{x}))^{n-1}.$$

Changing variables in the integral yields

$$\begin{aligned}\int_M \alpha(Y) &= \sum_{p\in M^Y} \lim_{\varepsilon\to 0^+} \int_{S_1(p)} \alpha(Y)^\varepsilon_{[0]}(\tilde{x})[\Omega_Y^p(\tilde{x}) \wedge (d\Omega_Y^p(\tilde{x}))^{n-1}] \\ &= \sum_{p\in M^Y} \alpha(Y)_{[0]}(p) \int_{S_1(p)} \Omega_Y^p \wedge (d\Omega_Y^p)^{n-1},\end{aligned}$$

since $\lim_{\varepsilon\to 0^+} \alpha(Y)^\varepsilon_{[0]}(\tilde{x}) = \alpha(Y)_{[0]}(p)$. Now by Stokes' theorem,

$$\int_{S_1(p)} \Omega_Y^p \wedge (d\Omega_Y^p)^{n-1} = \int_{B_1(p)} d[\Omega_Y^p \wedge (d\Omega_Y^p)^{n-1}] = \int_{B_1(p)} (d\Omega_Y^p)^n,$$

where $(d\Omega_p^Y)^n = (-2)^n n!(\lambda_1\lambda_2\cdots\lambda_n)^{-1}dx_1 \wedge \cdots \wedge dx_n$. We find that

$$\int_{B_1(p)} (d\Omega_Y^p)^n = \frac{(-2)^n n!}{\left(\prod_{j=1}^n \lambda_j\right)} \int_{B_1(p)} dx_1 \wedge \cdots \wedge dx_n = \frac{(-2)^n n!}{\left(\prod_{j=1}^n \lambda_j\right)} \mathrm{Vol}_{2n}(B_1(p)),$$

where in general, $\mathrm{Vol}2n(B_1(p)) = \frac{\pi^n}{n!}$. Since $\mathrm{Pf}(L_p(Y)) = \prod_{j=1}^n \lambda_j$, we get

$$\int_M \alpha(Y) = (-2\pi)^n \sum_{p\in M^Y} \frac{\alpha(Y)_{[0]}(p)}{\mathrm{Pf}(L_p(Y))}.$$

□

20th lecture, November 18th 2011

This Atiyah–Bott–Berline–Vergne localization theorem is quite more general than the Duistermaat–Heckmann theorem. We will now see how the Duistermaat–Heckmann theorem as a corollary.

Theorem 5.2.13 ([DH82], Theorem 4.1)**.** *Let (M, ω, G, μ) by a compact Hamiltonian G-space with compact Lie group G, and let* $\dim M = m = 2n$*. Assume that $Y \in \mathfrak{g}$ such that M^Y consists of isolated points. Then*

$$\int e^{it\mu^Y}\beta = \left(\frac{2\pi i}{t}\right)^n \sum_{p\in M^Y} \frac{e^{it\mu^Y(p)}}{\mathrm{Pf}(L_p(Y))},$$

where $\beta = \omega^n/n!$.

Proof. Apply the Atiyah–Bott–Berline–Vergne localitzation theorem to

$$\alpha(Y) = e^{it(\mu^Y+\omega)} \in \mathrm{Map}^\infty(\mathfrak{g}, \Omega^\bullet(M)).$$

All we need to check is that $d_{\mathfrak{g}}(e^{it(\mu^Y+\omega)}) = 0$. It is easy to see that

$$d_{\mathfrak{g}}(e^{it(\mu^Y+\omega)}) = itd_{\mathfrak{g}}(\mu^Y + \omega)e^{it(\mu^Y+\omega)},$$

using that

$$e^{it(\mu^Y+\omega)} = e^{it\mu^Y}(1 + it\omega + \cdots + \frac{(it\omega)^n}{n!}).$$

Recall that we our original motivation for defining $d_{\mathfrak{g}}$ as we did, was to ensure that $d_{\mathfrak{g}}(\mu^Y + \omega) = 0$. Working back, we see that this holds by the definition of $d_{\mathfrak{g}}$, using moment map condition, $d\mu^Y = \iota_{Y^\sharp}\omega$, and that $d\omega = 0$. Observe that

$$\int_M e^{it\mu^Y}\beta = (it)^{-n}\int_M e^{it(\mu^Y+\omega)},$$

since $e^{it(\mu^Y+\omega)}_{[m]} = (it)^n e^{it\mu^Y}\beta$. By the Atiyah–Bott–Berline–Vergne theorem, we have

$$\begin{aligned}\int_M e^{it\mu^Y}\beta = (it)^{-n}\int_M e^{it(\mu^Y+\omega)} &= (it)^{-n}(-2\pi)^n \sum_{p\in M^Y} \frac{\alpha(Y)_{[0]}(p)}{\mathrm{Pf}(L_p(Y))} \\ &= \left(\frac{2\pi i}{t}\right)^n \sum_{p\in M^Y} \frac{e^{it\mu^Y(p)}}{\mathrm{Pf}(L_p(Y))},\end{aligned}$$

where $\alpha(Y)_{[0]}(p) = e^{it(\mu^Y+\omega)}_{[0]}(p) = e^{it\mu^Y(p)}$. □

Of course, a lot of more went into the original proof of the Duistermaat–Heckmann theorem, where we applied stationary phase, which is far more general, and it was clear from that proof when the higher order terms vanished, which is not the case for the above proof. The original one is more useful for working with path integrals perturbatively.

5.3 The equivariant Euler class

Next, we study a generalized localization formula, due to Berline and Vergne, that extends to the case of a non-isolated fixed point set. To do so, we need to introduce the equivariant Euler class.

Definition 5.3.1. Let $F \hookrightarrow \mathcal{E} \xrightarrow{\pi} M$ be a fibre bundle over a smooth manifold M, and let G be a Lie group. We say that $(\mathcal{E}, M, F, \pi)$ (or just write $\mathcal{E}$) is a *G-equivariant fibre bundle* if G acts smoothly on both $\mathcal{E}$ and M (on the right), such that $\pi : \mathcal{E} \to \mathcal{M}$ is G-equivariant, i.e.

$$\pi(q \cdot g) = \pi(q) \cdot g,$$

for all $q \in \mathcal{E}, g \in G$.

If $\mathcal{E} = \mathcal{V}$ is a vector bundle, we require that the induced action on fibres is linear and call $\mathcal{E}$ a *G-equivariant vector bundle* . We let $g^{\mathcal{E}} : \mathcal{E}_p \to \mathcal{E}_{p \cdot g}$ denote the fibre map.

Recall that G acts on sections $s \in \Gamma(M, \mathcal{E})$ by

$$(s \cdot g)(q) := s(q \cdot g^{-1}) \cdot g^{\mathcal{E}}.$$

We denote by $\mathcal{L}_Y^{\mathcal{E}}$, for $Y \in \mathfrak{g}$, the corresponding *infinitesimal action* on $\Gamma(M, \mathcal{E})$,

$$\mathcal{E}_Y^{\mathcal{L}} s := \frac{d}{dt}|_{t=0}[s \cdot \exp(tY)].$$

Example 5.3.2. If $G \circlearrowright M$ and $\mathcal{E} = TM$, then for $\tilde{v} \in \Gamma(TM)$,

$$L_Y^{\mathcal{E}}(\tilde{v}) = \mathcal{L}_{Y^\sharp} \tilde{v} = [Y^\sharp, \tilde{v}]$$

is the usual Lie derivative.

Example 5.3.3. If $\mathcal{E} = T^*M$, then for $\alpha \in \Gamma(T^*M)$, $L_Y^{\mathcal{E}}(\alpha) = \mathcal{L}_{Y^\sharp}\alpha$ is the Lie derivative. Recall that the induced action on T^*M from an action on M is the cotangent lift

$$(T^*g)(\alpha)_q(v) := \alpha_p((Tg^{-1})v),$$

where $v \in T_qM$, $\alpha \in T_p^*M$, $q = p \cdot g$, and $Tg^{-1} := (g^{-1})_* : T_qM \to T_qM$ (note that, originally, we defined the cotangent lift using not Tg^{-1} but Tg).

Example 5.3.4. Consider $\mathcal{E} = \wedge^\bullet T^*M \otimes \mathcal{V}$, where $\mathcal{V}$ is a G-equivariant vector bundle over M. The action of G on $\wedge^\bullet T^*M$ is defined as follows: For $\beta \in \wedge_p^l T^*M$ and $v_1 \wedge \cdots \wedge v_l \in \wedge_q^l TM$, $g \in G$, let

$$(\beta \cdot g)_q(v_1 \wedge \cdots \wedge v_l) := \beta_p((Tg^{-1})v_1 \wedge \cdots \wedge (Tg^{-1})v_l),$$

where $q = p \cdot g$. For an element $\beta \otimes w \in \wedge^\bullet T^*M \otimes \mathcal{V}$, then $(\beta \otimes w) \cdot g := (\beta \cdot g) \otimes (w \cdot g)$ gives an action on $\wedge^\bullet T^*M \otimes \mathcal{V}$. As above, we get an induced action on sections, $G \circlearrowright \Gamma(M, \wedge^\bullet T^*M \otimes \mathcal{V})$.

Definition 5.3.5. Given a smooth manifold M and a vector bundle $\mathcal{V}$ over M, the *$\mathcal{V}$-valued* or *twisted differential forms* are defined as

$$\Omega^\bullet(M, \mathcal{V}) := \Gamma(M, \wedge^\bullet T^*M \otimes \mathcal{V}).$$

Definition 5.3.6. A *covariant derivative* or *connection* on a vector bundle $\mathcal{V}$ over a smooth manifold M is a differential operator

$$\nabla : \Omega^0(M, \mathcal{V}) \to \Omega^1(M, \mathcal{V})$$

satisfying the Leibniz rule: For $s \in \Omega^0(M, \mathcal{V})$, $f \in C^\infty(M)$,

$$\nabla(fs) = df \otimes s + f\nabla s.$$

We also use the same symbol ∇ to denote the unique extension of this operator to the map

$$\nabla : \Omega^\bullet(M, \mathcal{V}) \to \Omega^{\bullet+1}(M, \mathcal{V}),$$

satisfying the Leibniz rule: For $\alpha \in \Omega^l(M)$, $\beta \in \Omega^\bullet(M, \mathcal{V})$,

$$\nabla(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^l \alpha \wedge \nabla\beta.$$

Recall that the *curvature* of a covariant derivative ∇ is the $\mathrm{End}(\mathcal{V})$-valued 2-form on M defined by

$$F(w_1, w_2) := [\nabla_{w_1}, \nabla_{w_2}] - \nabla_{[w_1,w_2]}$$

for $w_1, w_2 \in \Gamma(TM)$. Recall that $\nabla^2\alpha = F \wedge \alpha$ for $\alpha \in \Omega^\bullet(M, \mathcal{V})$. We abuse notation and write $\nabla^2 = F$. We have analogous notions in the equivariant setting. From now on, write $\Omega^\bullet_G := \Omega^{\bullet,\infty}_G$.

Definition 5.3.7. If $\mathcal{V}$ is a G-equivariant vector bundle over a smooth manifold M, the space of *equivariant differential forms with values in* $\mathcal{V}$ is the space

$$\Omega^\bullet_G(M, \mathcal{V}) := (S(\mathfrak{g}^*) \otimes \Omega^\bullet(M, \mathcal{V}))^G$$

with $\mathbb{N}_0$-grading defined analagously to that on $\Omega^\bullet_G(M)$.

Recall that for $\alpha \in S(\mathfrak{g}^*) \otimes \Omega^\bullet(M, \mathcal{V})$, the G-action is defined as

$$(\alpha \cdot g)(Y) := [\alpha(\mathrm{Ad}_g Y)] \cdot g.$$

If $\mathcal{V} = M \times \mathbb{C}$, then this action reduces to the usual action of G on $\Omega^\bullet_G(M)$,

$$(\alpha \cdot g)(Y) := (g^{-1})^* \alpha(\mathrm{Ad}_g Y).$$

Also, for $\alpha = p^i(\xi) \otimes \alpha_{[j]} \in S^i(\mathfrak{g}^*) \otimes \Omega^j(M, \mathcal{V})$, let $\deg \alpha := 2i + j$, where $\Omega^j(M, \mathcal{V}) = \Gamma(M, \wedge^j T^*M \otimes \mathcal{V})$.

Definition 5.3.8. A *G-invariant covariant derivative or connection* is an ordinary connection $\nabla : \Omega^\bullet(M, \mathcal{V}) \to \Omega^{\bullet+1}(M, \mathcal{V})$ that commutes with the G-action on $\Omega^\bullet(M, \mathcal{V})$, i.e.

$$\nabla(\alpha \cdot g) = (\nabla\alpha) \cdot g,$$

for all $\alpha \in \Omega^\bullet(M, \mathcal{V})$, $g \in G$.

Note that given a connection ∇ (which always exists), and a compact Lie group G, we can construct a G-invariant connection

$$\tilde{\nabla}\alpha \equiv \int_G \nabla(\alpha \cdot g) \cdot g^{-1} dG,$$

where dG denotes Haar measure on G.

Definition 5.3.9. Given a G-invariant connection ∇, the *equivariant connection* $\nabla_\mathfrak{g}$, defined by ∇, is the operator on $S(\mathfrak{g}^*) \otimes \Omega^\bullet(M, \mathcal{V})$ defined by

$$(\nabla_\mathfrak{g}\alpha)(Y) := (\nabla - \iota_{Y^\sharp})\alpha(Y),$$

for $Y \in \mathfrak{g}$.

Note that $\nabla_\mathfrak{g}$ satisfies the Leibniz rule: For $\alpha \in S(\mathfrak{g}^*) \otimes \Omega^\bullet(M)$, $\beta \in S(\mathfrak{g}^*) \otimes \Omega^\bullet(M, \mathcal{V})$, we have

$$\nabla_\mathfrak{g}(\alpha \wedge \beta) = (d_\mathfrak{g}\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge (\nabla_\mathfrak{g}\beta).$$

Also, one can show that $\nabla_\mathfrak{g} : \Omega^\bullet_G(M, \mathcal{V}) \to \Omega^\bullet_G(M, \mathcal{V})$, so $\nabla_\mathfrak{g}$ preserves G-equivariance.

Definition 5.3.10. The *equivariant curvature* of an equivariant connection $\nabla_\mathfrak{g}$ is defined to be

$$F_\mathfrak{g}(Y) := \nabla_\mathfrak{g}(Y)^2 + \mathcal{L}^\mathcal{V}_Y,$$

for $Y \in \mathfrak{g}$, where $\nabla_\mathfrak{g}(Y) := \nabla - \iota_{Y^\sharp}$.

Proposition 5.3.11. *The equivariant curvature $F_{\mathfrak{g}}$ acts on $\Omega^\bullet_G(M,\mathcal{V})$ as an element of $\Omega^\bullet_G(M, End(\mathcal{V}))$ and satisfies the equivariant Bianchi identity*

$$\nabla_{\mathfrak{g}} F_{\mathfrak{g}} = 0.$$

For a proof of this, see [BGV04, Proposition 7.4]. Note that for $\theta \in \Omega^\bullet_G(M, \mathrm{End}(\mathcal{V}))$, we have $\nabla_{\mathfrak{g}}\theta \in \Omega^\bullet_G(M, \mathrm{End}(\mathcal{V}))$, where $\nabla_{\mathfrak{g}}\theta = [\nabla_{\mathfrak{g}}, \theta]$ and

$$\nabla_{\mathfrak{g}}\theta \wedge \beta := \nabla_{\mathfrak{g}}(\theta \wedge \beta) - \theta \wedge \nabla_{\mathfrak{g}}\beta$$

for $\beta \in \Omega^\bullet_G(M,\mathcal{V})$. The equivariant curvature $F_{\mathfrak{g}}$ acts on $\Omega^\bullet_G(M,\mathcal{V})$ as an element of $\Omega^\bullet_G(M, \mathrm{End}(\mathcal{V}))$ as for any $\alpha \in \Omega^\bullet_G(M,\mathcal{V})$, we have $F_{\mathfrak{g}}\alpha = \tilde{F}_{\mathfrak{g}} \wedge \alpha$ for a particular $\tilde{F}_{\mathfrak{g}} \in \Omega^\bullet_G(M, \mathrm{End}(\mathcal{V}))$. We abuse notation and write $F_{\mathfrak{g}}$ for $\tilde{F}_{\mathfrak{g}}$.

Example 5.3.12. We may justify the formula $F_{\mathfrak{g}}(Y) = \nabla_{\mathfrak{g}}(Y)^2 + \mathcal{L}^{\mathcal{V}}_Y$ by considering the case when $\mathcal{V} = M \times \mathbb{C}$, and $\nabla = d$ is the ordinary deRham operator. Then $\nabla_{\mathfrak{g}} = d_{\mathfrak{g}}$, $\nabla_{\mathfrak{g}}(Y) = d_Y = d - \iota_{Y^\sharp}$, so that $d_Y^2 = -\mathcal{L}_{Y^\sharp}$. Also, $\mathcal{L}^{\mathcal{V}}_Y = \mathcal{L}_{Y^\sharp}$ as we saw before. Hence

$$F_{\mathfrak{g}}(Y) = \nabla_{\mathfrak{g}}(Y)^2 + \mathcal{L}^{\mathcal{V}}_Y = -\mathcal{L}_{Y^\sharp} + \mathcal{L}_{Y^\sharp} = 0.$$

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Continuing from last time, we find

$$\begin{aligned} F_{\mathfrak{g}}(Y) &= (\nabla - \iota_{Y^\sharp})^2 + \mathcal{L}^{\mathcal{V}}_Y = F - (\iota_{Y^\sharp}\nabla + \nabla\iota_{Y^\sharp}) + \mathcal{L}^{\mathcal{V}}_Y \\ &= F - [\iota_{Y^\sharp}, \nabla] + \mathcal{V}^{\mathcal{V}}_Y. \end{aligned}$$

Definition 5.3.13. The *moment* (relative to a G-invariant connection ∇) is the element of $\Omega^2_G(M, \mathrm{End}(\mathcal{V}))$ which at $Y \in \mathfrak{g}$ is given by

$$\mu(Y) := \mathcal{L}^{\mathcal{V}}_Y - [\iota_{Y^\sharp}, \nabla].$$

Note that it makes sense to call this the moment as in special cases, that we will not consider, it reduces to a moment map. Observe also that

$$F_{\mathfrak{g}}(Y) = F + \mu(Y).$$

The reason for going through all of this was to construct the equivariant Euler class. We recall the definition of the Berezin operator.

Definition 5.3.14. Let V be an oriented Euclidean space of dimension m. The *Berezin operator* is the map $\mathrm{Br} : \wedge^* V \to \mathbb{R}$ such that if $\beta \in \wedge^l V$, for $l < m$, then $\mathrm{Br}(\beta) = 0$, and if $\{e_1, \ldots, e_m\}$ is a positively oriented orthonormal basis of V, then

$$\mathrm{Br}(c \cdot e_1 \wedge \cdots \wedge e_n) = c,$$

for any $c \in \mathbb{R}$.

Definition 5.3.15. The *Pfaffian* of an element $\beta \in \wedge^2 V$ is the number

$$\mathrm{Pf}(\beta) := \mathrm{Br}(\exp(\beta)) = \mathrm{Br}(1 + \beta + \frac{\beta^2}{2!} + \cdots) = \begin{cases} \mathrm{Br}\left(\frac{\beta^n}{n!}\right), & \text{if } m = 2n, \\ 0, & \text{otherwise} \end{cases}.$$

Recall that we can define the Pfaffian of an element $A \in \mathfrak{so}(V) = \mathrm{Lie}(\mathrm{SO}(V))$, where we view A as determining an anti-symmetric bilinear form on V via $\beta_A := \sum_{i<j}\langle e_i, Ae_j\rangle e_1 \wedge e_j \in \wedge^2 V$: We then define

$$\mathrm{Pf}(\beta) := \mathrm{Pf}(\beta_A).$$

We can compute $\mathrm{Pf}(A)$ for $A \in \mathfrak{so}(V)$, $\dim V = m = 2n$, by choosing an oriented orthonormal basis $E = \{e_1, \ldots, e_m\}$ of V such that A has the form

$$[A]_E = \begin{pmatrix} \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix} \end{pmatrix},$$

for $\lambda_i \in \mathbb{R}$, $1 \leq i \leq n$. Then

$$\beta_A = \lambda_1 e_1 \wedge e_2 + \cdots + \lambda_n e_{n-1} \wedge e_n,$$

and $\mathrm{Pf}(A) = \prod_{j=1}^n \lambda_j$. Clearly, $\mathrm{Pf}(A)^2 = \det A$.

Recall, a connection ∇ is *compatible* with a metric $\langle\, , \rangle \in \mathrm{Sym}^2 \mathcal{V}^*$, for an oriented vector bundle $\mathcal{V} \to M$, if for any two sections $w_1, w_2 \in \Gamma(\mathcal{V})$, we have

$$d\langle w_1, w_2\rangle = \langle \nabla w_1, w_2\rangle + \langle w_1, \nabla w_2\rangle.$$

For a real G-equivariant oriented vector bundle $\mathcal{V} \to M$, one can use a G-invariant metric $\langle\, , \rangle \in \mathrm{Sym}^2 \mathcal{V}^*$, and a compatible G-invariant connection ∇ (which always exists) to show that the associated $\mu(Y)$ satisfies

$$\langle \mu(Y) w_1, w_2\rangle = -\langle w_1, \mu(Y) w_2\rangle,$$

and the curvature $F = \nabla^2$ satisfies

$$\langle F(v_1, v_2) w_1, w_2\rangle = -\langle w_1, F(v_1, v_2) w_2\rangle,$$

for any $w_1, w_2 \in \Gamma(\mathcal{V})$, $v_1, v_2 \in \Gamma(TM)$. So overall, $\mu(Y) \in \Omega^0_G(M, \mathfrak{so}(\mathcal{V}))$ and $F \in \Omega^2_G(M, \mathfrak{so}(\mathcal{V}))$. Thus $F_{\mathfrak{g}}(Y) = F + \mu(Y) \in \Omega^\bullet_G(M, \mathfrak{so}(\mathcal{V}))$. Viewing the Pfaffian as a map $\mathrm{Pf} : \Omega^\bullet_G(M, \mathfrak{so}(\mathcal{V})) \to \Omega^\bullet_G(M)$.

Definition 5.3.16. The *equivariant Euler form* is defined as

$$\chi^{\mathcal{V}}_{\mathfrak{g}}(Y) \equiv \mathrm{Pf}(F_{\mathfrak{g}}(Y)) \in \Omega^\bullet_G(M).$$

Fact 5.3.17. *The equivariant Euler form $\chi^{\mathcal{V}}_{\mathfrak{g}}$ is equivariantly closed, i.e. $d_{\mathfrak{g}}\chi^{\mathcal{V}}_{\mathfrak{g}} = 0$, and its cohomology class does not depend on the choice of connection ∇ or the metric of $\mathcal{V}$, but only on the orientation of $\mathcal{V}$. See Theorem 7.7 of [BGV04] for an argument that can be extended analogously to this case.*

5.4 Generalized localization

Proposition 5.4.1. *Fix $Y \in \mathfrak{g}$, and let M^Y be the set of zeroes of $Y^\sharp$ in M. Then M^Y is a union of disjoint connected components that are submanifolds of M and of possibly different dimensions. The normal bundle $\mathcal{N}$ over any connected component is an orientable vector bundle (see [BT82] for a definition, one is that its determinant line bundle has a non-zero global section) with even dimensional fibers.*

Proof. Choose a G-invariant Riemannian metric h_G on (M, G), and recall by a previous proposition that, if $\exp_p : T_pM \to M$ denotes the Riemannian exponential map for h_G, then $v \in \ker(L_p(Y))$ if and only if $\exp_p(v) \in M^Y$ for $Y \in \mathfrak{g}$. Since $\exp_p : T_pM \to M$ is a diffeomorphism near p, we see that $\exp_p : \ker(L_p(Y)) \to M^Y$ defines a manifold structure on M^Y.

Using h_G, we may identify $\mathcal{N}_p = (T_pM^Y)^\perp \subseteq T_pM$, so we can write $T_pM = \mathcal{N}_p \oplus T_pM^Y$. Observe that $L_p(Y) : \mathcal{N}_p \to \mathcal{N}_p$ is an isomorphism and it must also be anti-symmetric be previous observations. By basic linear algebra, $\mathcal{N}_p$ must be even dimensional.

It is straightforward to show that the weights $\alpha_{p,j}(Y) = \lambda_j$ are the same for any points p and q in the same connected component of M^Y: Namely, let $p, q \in M^Y$ be in the same connected component $\mathcal{C}$ of M^Y. Assuming that $\exp_p : \mathrm{Ker}(L_p(Y)) \to \mathcal{C}$ is onto (which is probably not necessary), let $v \in \ker(L_p(Y))$, such that $q = \exp_p(v)$. Define $S_v : T_pM \to T_pM$ by $S_v(w) = v + w =: \tilde{w}_v$. Then if $\tilde{S}_v : M^Y \to M^Y$ is defined by $\tilde{S}_v = \exp_q \circ S_v \circ \exp_p^{-1}$, so $\tilde{S}_w(p) = q$ and $\tilde{w} = d\tilde{S}_v(w)$ (since $d\tilde{S}_v = \mathbf{1}$). Then

FiXme Fatal: This can probably be fixed up a bit

$$L_q(Y)(\tilde{w}) \equiv \frac{d}{dt}|_{t=0}(\exp(tY)_*\tilde{w}) = \frac{d}{dt}|_{t=0}(\exp(tY)_*w) = L_p(Y)(w),$$

so under this identification, the isotropy representations are identified, and in particular the weights agree. We may therefore define an orientation on $\mathcal{N}$ by requiring that $\mathrm{Pf}(L_p(Y)|_{\mathcal{N}_p}) > 0$. □

Let $\mathrm{rk}(\mathcal{N}) : M^Y \to \mathbb{N}_0$ be the locally constant function that gives the codimension of each connected component. Let

$$\mathfrak{g}_Y = \{\tilde{Y} \in \mathfrak{g} \mid [\tilde{Y}, Y] = 0\}$$

be the centralizer of $Y \in \mathfrak{g}$, and let G_Y be the connected Lie subgroup of G generated by $\mathfrak{g}_Y$.

Proposition 5.4.2. *The subgroup G_Y preserves M^Y and acts on $\mathcal{N}$.*

Proof. We only need to show that

1. $\exp(\tilde{Y}) : M^Y \to M^Y$ for $\tilde{Y} \in \mathfrak{g}_Y$, and that
2. $\exp(\tilde{Y})_* : \mathcal{N} \to \mathcal{N}$.

For the first claim, let $q = p \cdot \exp(\tilde{Y}) \in M^Y$. Fix $p \in M^Y$ and let $q = \exp_p(v) \in M^Y$ for some $v \in \ker L_p(Y)$. We want to show that $q \cdot \exp(\tilde{Y}) \in M^Y$. By a previous lemma,

$$q \cdot \exp(\tilde{Y}) = \exp_p(v) \cdot \exp(\tilde{Y}) = \exp_p(\exp(\tilde{Y})_* v),$$

and thus by a previous lemma, if we can show that $\exp(\tilde{Y})_* v \in \ker L_p(Y)$, we obtain that $q \cdot \exp(\tilde{Y}) \in M^Y$. Consider

$$L_p(Y)(\exp(\tilde{Y})_* v) = \frac{d}{dt}|_{t=0}[\exp(tY)_* \exp(\tilde{Y})_* v]$$

and notice that, since $[\tilde{Y}, Y] = 0$,

$$\exp(tY)_* \exp(\tilde{Y})_* = (\exp(\tilde{Y}) \exp(tY))_* = (\exp(tY) \cdot \exp(\tilde{Y}))_* = \exp(\tilde{Y})_* \exp(tY)_*.$$

Thus,

$$\begin{aligned} L_p(Y)(\exp(\tilde{Y})_* v) &= \frac{d}{dt}|_{t=0}[\exp(\tilde{Y})_* \exp(tY)_* v] = \exp(\tilde{Y})_* \frac{d}{dt}|_{t=0}[\exp(tY)_* v] \\ &= \exp(\tilde{Y})_*[L_p(Y)(v)] = 0, \end{aligned}$$

since $v \in \ker L_p(Y)$ by assumption.

For the second part, recall that for $q \in M^Y$, we have $w \in \mathcal{N}_q$ if and only if $L_q(Y)w \neq 0$. This follows since $w \in T_qM^Y$ if and only if $w \in \ker L_q(Y)$ and $T_qM^Y \cap \mathcal{N}_q = \{0\}$. Let $q = p \cdot \exp_p(\tilde{Y}) \in M^Y$ and $w = \exp(\tilde{Y})_* v$, for $v \in \mathcal{N}_p$. By the above observation, $v \in \mathcal{N}_p$ if and only if $L_p(Y)v \neq 0$. Using again that $[\tilde{Y}, Y] = 0$, we find that

$$\begin{aligned} L_q(Y)w &= \frac{d}{dt}|_{t=0}[\exp(tY)_* w] = \frac{d}{dt}|_{t=0}[\exp(tY)_* \exp(\tilde{Y})_* v] \\ &= \exp(\tilde{Y})_* \frac{d}{dt}[\exp(tY)_* v] = \exp(\tilde{Y})_* L_p(v). \end{aligned}$$

Since $\exp(\tilde{Y})_*$ is invertible, and $L_p(Y)v \neq 0$, we have $L_q(Y)w \neq 0$, so $w \in \mathcal{N}_q$. □

Choose a G_Y-invariant metric on $\mathcal{N}$, and a compatible G_Y-invariant connection $\nabla^{\mathcal{N}}$ on $\mathcal{N}$. Given $\tilde{Y} \in \mathfrak{g}_Y$, define

$$\nabla^{\mathcal{N}}_{\tilde{Y}} := [\iota_{\tilde{Y}^\sharp}, \nabla^{\mathcal{N}}],$$

and recall that $\mu^{\mathcal{N}} \in \Omega^2_{G_Y}(M, \mathrm{End}(\mathcal{N}))$ is defined by

$$\mu^{\mathcal{N}}(\tilde{Y}) = \mathcal{L}^{\mathcal{N}}_{\tilde{Y}} - \nabla^{\mathcal{N}}_{\tilde{Y}},$$

for $\tilde{Y} \in \mathfrak{g}_Y$. Briefly going back to the case $\tilde{Y} = Y$, we know that $Y^\sharp$ vanishes on M^Y, so that $\mu^{\mathcal{N}}(Y) = \mathcal{L}^N_Y$, and recalling the definition of $\mathcal{L}^{\mathcal{N}}_Y$, for $p \in M^Y$ we have $\mu^N(Y)|_p = L_p(Y)|_{\mathcal{N}_p}$ which is an invertible linear transformation on $\mathcal{N}$. Consider the G_Y-equivariant Euler class $\chi^{\mathcal{N}}_{\mathfrak{g}_Y}(\tilde{Y}) \equiv \mathrm{Pf}(F^{\mathcal{N}}_{\mathfrak{g}_Y}(\tilde{Y})) \in \Omega^\bullet_G(M^Y)$. Recall that we chose an orientation on $\mathcal{N}$ by requiring $\mathrm{Pf}(L_p(Y)|_{\mathcal{N}_p}) > 0$. Observe then that

$$\chi^{\mathcal{N}}_{\mathfrak{g}_Y}(\tilde{Y})|_{[0]} = \mathrm{Pf}(\mu^{\mathcal{N}}(\tilde{Y})) \neq 0$$

for $\tilde{Y}$ sufficiently close to Y, since $\mu^{\mathcal{N}}(Y)$ is invertible.

Theorem 5.4.3 ([BGV04], Thm. 7.13)**.** *Let G be a compact Lie group that acts on a compact oriented smooth manifold M and $\alpha \in \mathrm{Map}^\infty(\mathfrak{g}, \Omega^\bullet(M))$ equivariantly closed, i.e. $d_{\mathfrak{g}}\alpha = 0$ (and as usual, α is not necessarily in $\Omega^\bullet_G(M)$). Let $Y \in \mathfrak{g}$, let M^Y denote the zeroes of $Y^\sharp$ in M, and let $\mathcal{N}$ be the normal bundle of M^Y in M. Choose an orientation $\mathcal{N}$ as above. Then for $\tilde{Y} \in \mathfrak{g}_Y$ sufficiently close to Y, we have*

$$\int_M \alpha(\tilde{Y}) = \int_{M^Y} (-2\pi)^{\mathrm{rk}(\mathcal{N})/2} \frac{\alpha(Y)}{\chi^{\mathcal{N}}_{\mathfrak{g}_Y}(\tilde{Y})},$$

where $\chi^{\mathcal{N}}_{\mathfrak{g}_Y}$ is the equivariant Euler form of the normal bundle. Here, implicitly, $\int_{M^Y}$ denotes the sum of integrals over components of M^Y. In particular,

$$\int_M \alpha(Y) = \int_{M^Y} (-2\pi)^{\mathrm{rk}(\mathcal{N})/2} \frac{\alpha(Y)}{\mathrm{Pf}(\mathcal{L}^{\mathcal{N}}(Y) + F^{\mathcal{N}})},$$

where $F^N \in \Omega^2_G(M^Y, \mathfrak{so}(N))$ is the curvature of $\mathcal{N}$.

We will not prove the theorem in detail but only give some idea about, why a localization should exist.

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In the case that M^Y consists of isolated points, we have $\mathcal{L}^{\mathcal{N}}(Y)|_p = L_p(Y)$, $F^{\mathcal{N}} = 0$, and $(\mathrm{rk}(\mathcal{N}))(p) = m = 2n$ for each $p \in M^Y$. Also, since M^Y is 0-dimensional, we have $\int_p \alpha(Y) = \alpha(Y)_{[0]}(p)$, and overall we recover the formula,

$$\int_M \alpha(Y) = (-2\pi)^n \sum_{p \in M^Y} \frac{\alpha(Y)_{[0]}(p)}{\mathrm{Pf}(L_p(Y))}.$$

Sketch of proof of Theorem 5.4.3. We follow the basic idea of [Bis86], [Wit92] and [AB84]. Choose any $\theta \in \Omega^\bullet_G(M)$ such that $\theta(Y) \in \Omega^1(M)$ and satisfying $\iota_{Y^\sharp}(\theta(Y)) = 0$ if and only if $Y^\sharp = 0$. For example, we can choose $\theta : \mathfrak{g} \to \Omega^\bullet(M)$ linaer in Y, and in fact defined by a G-invariant metric $\langle , \rangle \in \mathrm{Sym}^2 T^*M$, i.e. define $\theta(Y)(\cdot) := \langle Y^\sharp, \cdot \rangle$.

We have already seen that $\theta \in \Omega^3_G(M)$ in Proposition 5.2.2. Observe that if $\alpha \in \mathrm{Map}^\infty(\Omega^\bullet(M))$ is equivariantly closed, $d_{\mathfrak{g}}\alpha = 0$, then

$$\int_M \alpha(Y) = \int_M \alpha(Y) \wedge e^{t d_{\mathfrak{g}} \theta(Y)} \tag{17}$$

for all $t \in \mathbb{R}$. This follows since $\alpha(Y) \wedge [e^{td_{\mathfrak{g}}\theta(Y)} - 1]$ is equivariantly exact. To see this, write

$$e^{td_{\mathfrak{g}}\theta(Y)} = 1 + td_{\mathfrak{g}}\theta(Y) + \frac{t^2(d_{\mathfrak{g}}\theta(Y))^2}{2} + \cdots = 1 + td_{\mathfrak{g}}\beta(Y)$$

for some $\beta \in \Omega^{\bullet}_G(Y)$. Then

$$\alpha(Y) \wedge [e^{td_{\mathfrak{g}}\theta(Y)} - 1] = \alpha(Y) \wedge d_{\mathfrak{g}}\beta(Y) = d_{\mathfrak{g}}(\alpha \wedge \beta)(Y),$$

and hence

$$\int \alpha(Y) \wedge [e^{td_{\mathfrak{g}}\theta(Y)} - 1] = 0,$$

which implies (17). Now, write

$$\int_M \alpha(Y) = \int_M \alpha(Y) \wedge e^{td_{\mathfrak{g}}\theta(Y)} = \int_M \alpha(Y) e^{-t\|Y^\sharp\|^2} \sum_{j \geq 0} \frac{t^j (d\theta(Y))^j}{j!}.$$

Already at this step, we see that integral localizes somehow to a fixed point set. I.e. taking the limit $t \to \infty$, we see that the integral reduces to contributions from $p \in M^y$. That is,

$$\int_M \alpha(Y) = \sum_{\mathcal{C}_l \subseteq M^Y} F_l,$$

for some local contributions F_l, where $\mathcal{C}_l$ are the connected components of M^Y. A careful analysis will show that

$$F_l = \int_{\mathcal{C}_l} (-2\pi)^{\mathrm{rk}(\mathcal{N})/2} \frac{\alpha(Y)}{\mathrm{Pf}(\mathcal{L}^{\mathcal{N}}(Y) + F^{\mathcal{N}})}.$$

□

6 Non-abelian localization

6.1 Motivation from physics – electromagnetism

Let $X = \mathbb{R}^3$, and identify $\mathbb{R}^4 = \mathbb{R} \times X$, so that $(t, x) \in \mathbb{R}^4$ is naturally thought of as t being time and x being spatial position. Let $V \subseteq X$ be a compact 3-manifold with boundary ∂V, say V a ball. Let $D \subseteq X$ be a compact 2-manifold with boundary ∂D, say D a disk. Let $i_t : X \to \mathbb{R} \times X$ be the inclusion map $x \mapsto (t, x)$, and define $i_t^V := i_t|_V$, and $i_t^D := i_t|_D$.

Given $\alpha \in \Omega^{\bullet}(\mathbb{R}^4)$, we view $\int_D \alpha$ and $\int_V \alpha$ as functions of time and define

$$\int_D \alpha := \left(\int_D \alpha \right)(t) := \int_D (\iota_t^D)^* \alpha$$

and similarly for V. Let

$$\overline{E} := E_1\partial_1 + E_2\partial_2 + E_3\partial_3 \in \Gamma(T\mathbb{R}^4),$$
$$\overline{B} := B_1\partial_1 + B_2\partial_2 + B_3\partial_3 \in \Gamma(T\mathbb{R}^4),$$

where $B_i, E_i \in \Omega^0(\mathbb{R}^4)$, $\partial_i = \frac{\partial}{\partial x^i}$, where $\{x^1, x^2, x^3\}$ are standard coordinates on $X = \mathbb{R}^3$, so any $p \in \mathbb{R}^4$ is expressed as $p = (t, x^1, x^2, x^3)$, for some $t, x^i \in \mathbb{R}$, $1 \leq i \leq 3$.

Definition 6.1.1. The vector fields $\overline{E}, \overline{B}$ are called the *electric* and *magnetic fields* respectively

Let $\tilde{g} \equiv (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \in \mathrm{Sym}^2 T^*X$ be the Euclidean metric on X. Recall the Hodge star operator defined with respect to $\tilde{g}$,

$$\tilde{*} : \Omega^l(X, \mathbb{R}) \to \Omega^{3-l}(X, \mathbb{R}),$$

defined as follows: For $\{\partial_1, \partial_2, \partial_3\}$ is a positively oriented orthonormal basis for $\tilde{g}$ at any $x \in X$, $\tilde{g}$ induces the isomorphism $\flat : TX \to T^*X$ via $\partial_i \mapsto g(\partial_i, \cdot) \equiv dx^i$. Let $\flat : \wedge^l TX \to \wedge^l T^*X$, and let $\tilde{\sharp} : \wedge^l T^*M \to \wedge^l TM$ be the induced musical isomorphisms.

Let $\nu = \nu_x := dx^1 \wedge dx^2 \wedge dx^3$ denote the volume element at $x \in X$. Then for $\alpha, \beta \in \Omega^l(X, \mathbb{R})$, we define $\tilde{*}\beta \in \Omega^{3-l}(X)$ by

$$\alpha \wedge \tilde{*}\beta := \langle \alpha, \beta \rangle_{\tilde{g}} \nu,$$

where $\langle \alpha, \beta \rangle_{\tilde{g}} := \beta(\tilde{\sharp}\alpha) = \alpha(\tilde{\sharp}\beta)$. For example, if $\beta = dx^1$, $\alpha = \alpha_i dx^i := \sum_{i=1}^3 \alpha_i dx^i$, then

$$\alpha \wedge \tilde{*} dx^1 = \alpha_1 dx^1 \wedge dx^2 \wedge x^3.$$

This implies in particular that $\tilde{*}dx^1 = dx^2 \wedge dx^3$, $\tilde{*}dx^2 = -dx^3 \wedge dx^1$ and $\tilde{*}dx^3 = dx^1 \wedge dx^2$.

To formula Maxwell's equations, it is useful to identify $\overline{E}, \overline{B} \in \Gamma(T\mathbb{R}^4)$ with differential forms on $\mathbb{R}^4$.

Definition 6.1.2. Define

$$E = \flat(\overline{E}) = E_1 dx^1 + E dx^2 + E_3 dx^3 \in \Omega^1(\mathbb{R}^4, \mathbb{R}),$$
$$B = \tilde{*}(\flat \overline{B}) = B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2 \in \Omega^2(\mathbb{R}^4, \mathbb{R}),$$

which we will still refer to as the *electric* and *magnetic fields* respectively. Define the *electric charge density* by

$$\tilde{\rho} := \rho dx^1 \wedge dx^2 \wedge x^3 \in \Omega^3(\mathbb{R}^4),$$

where $\rho \in \Omega^0(\mathbb{R}^4)$, and define the *electric current density* by

$$j = j_1 dx_2 \wedge dx^3 + j_2 dx^3 \wedge dx^1 + j_3 dx^1 \wedge dx^2 \in \Omega^2(\mathbb{R}^4).$$

Example 6.1.3. We will take our convention to be such that charge flows in the direction of protons. Since protons are taken to have positive charge, we take charge flow to be in direction of positive charge. The right hand rule tells which direction the $\overline{B}$ fields point (see Fig. 6). Here, j has magnitude the average charge per second through a given point.

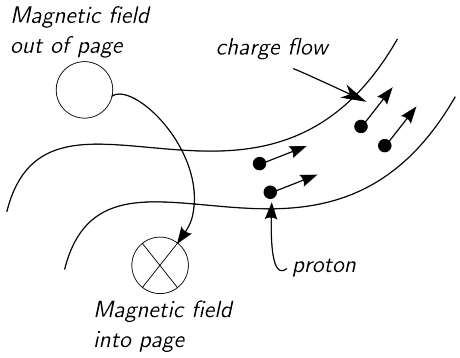

Figure 6: Protons moving in a wire

We write $\tilde{\rho}$ for the average charge per unit volume. Recall that charge can be measured in Coulomb, C. If e^+ denotes protons and e^- electrons, then approximately, the charge of e^+ and e^- is $1.602 \cdot 10^{-19}$C. Maxwell's equations are usually stated with some constants, $\varepsilon_0, \mu_0, c_0$, that is, the electric permittivity, magnetic permeability, and the speed of light (all in free space) respectively. Recall that $c_0 = 1/\sqrt{\varepsilon_0 \mu_0}$. In SI units,

$$\begin{aligned} \varepsilon_0 &\approx \times 8.854 \cdot 10^{-12} \mathrm{C}^2 \cdot \mathrm{N}^{-1} \cdot \mathrm{m}^{-1}, \\ \mu_0 &= 4\pi \cdot 10^{-7} \mathrm{N} \cdot \mathrm{s}^2 \cdot \mathrm{C}^{-2}, \\ c &\approx 2.998 \cdot 10^8 \mathrm{m/s}. \end{aligned}$$

We choose units so that $\varepsilon_0 = \mu_0 = c_0 = 1$. Maxwell's equations can now be stated as follows: The first one is Gauss' magnetic law,

$$\int_{\partial V} B = 0, \tag{18}$$

which can be interpreted as saying that there are no magnetic monopoles. The second one is Faraday's law,

$$\int_{\partial D} E = -\frac{d}{dt} \int_D B, \tag{19}$$

which says that changing the magnetic field induces an electric field. The third one is Gauss' electric law,

$$\int_{\partial V} \tilde{*} E = \int_V \tilde{\rho}, \tag{20}$$

which says that the total electric charge in V is the total electric flux over ∂V. The fourth one is Ampere's law with Maxwell's correction,

$$\int_{\partial D} B = \int_D j + \frac{d}{dt} \int_D \tilde{*} E, \tag{21}$$

which says that changing electric fields will generate magnetic fields. Equations (18) and (19) are called the homogeneous Maxwell equations, and equations (20) and (21) are called the inhomogeneous Maxwell equations. Using Stokes' theorem, we may also write the equations as

$$\begin{aligned} \tilde{d} \tilde{*} E = \tilde{\rho}, \quad \tilde{d} B = 0, \\ \tilde{d} E = -\frac{\partial}{\partial t} B, \quad \tilde{d} B = j + \frac{\partial}{\partial t} (\tilde{*} E), \end{aligned}$$

where $\tilde{d}$ denotes exterior derivative on $X = \mathbb{R}^3$. Observe that the first equation implies that $B = \tilde{d} a$ by the Poincaré lemma on X, for some $a := A_1 dx^1 + A_2 dx^2 + A_3 dx^3 \in \Omega^1(\mathbb{R}^4)$. Also, the second one implies that $E = -\frac{\partial}{\partial t} a + \tilde{d} \varphi$ for some $\varphi \in \Omega^0(\mathbb{R}^4)$. Let $A_0 := \varphi$.

Definition 6.1.4. Define the *electromagnetic gauge potential* $A := A_\mu dx^\mu \in \Omega^1(\mathbb{R}^4)$, $0 \leq \mu \leq 3$. We call $\varphi = A_0$ the *electric potential* and $a = \sum_{i=1}^3 A_i dx^i$ the *magnetic potential* . Also, define *electromagnetic gauge current density* $J := \rho + j \wedge dt \in \Omega^3(\mathbb{R}^4)$ and $F_A := dA = E \wedge dt + B \in \Omega^2(\mathbb{R}^4)$ the *electromagnetic field strength* .

Clearly, $dF_A = 0$ (which we can view as an incarnation of the Bianchi identity) if and only if the first two of Maxwell's equations hold, and $d * F_A = J$ is equivalent to the last two of the equations. Here d is exterior derivative on $\mathbb{R}^4$, and $* : \Omega^k(\mathbb{R}^4) \to \Omega^{4-k}(\mathbb{R}^4)$, relative to the Minkowski metric on $\mathbb{R}^4$.

These equations describe all of *classical* electromagnetism, and we can solve for E, B in $F_A = E \wedge dt + B$ for given $\tilde{\rho}, j$. Indeed, F_A contains all the physics and as a matrix,

$$[F_A]_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}.$$

Recall that there was some ambiguity in the choice of A, but observe that if $A' = A + d\psi$ for some $\psi \in \Omega^0(\mathbb{R}^4)$, then $F_{A'} = F_A$, and A' contains the same physics as A.

23rd lecture, November 28th 2011

We are interested in quantum electromagnetism. Quantum effects in a theory may be concisely encoded in a path integral

$$Z := \fint_{\mathcal{P}} \exp\left[\frac{iS[\varphi]}{\hbar}\right][\mathcal{D}\varphi],$$

where $\mathcal{P}$ is the space of "paths" in the theory, φ is a "path" or a "field", $S[\varphi]$ is the action, $\hbar$ is Planck's constant, and $[\mathcal{D}\varphi]$ is the formal path integral measure.

The first question is: What should $S[\varphi]$ be for electromagnetism? By one of Feynman's postulates, $S[\varphi]$ should be an action for the classical theory,

$$S[\varphi] = \int_R \mathcal{L}[\varphi, \dot{\varphi}], \tag{22}$$

for some region of spacetime, R, and $\mathcal{L}$ a Lagrangian depending on φ and its derivatives $\dot{\varphi} = \{\partial_\nu \varphi\}_{\nu=0,\ldots,n}$ and $\varphi, \partial_\nu \varphi$ are viewed as formal independent variables. Then (22) should yield the classical equations of motion via the Euler–Lagrange equations (or equivalently Hamilton's principle $\delta S[\varphi, \dot{\varphi}] = 0$).

Claim 6.1.5. *The relevant field(s) in Maxwell's theory is $\varphi = A \in \Omega^1(\mathbb{R}^4)$, the electromagnetic gauge potential, and the correct Lagrangian is*

$$\mathcal{L}[A_\mu, \partial_\nu A_\mu] := -\frac{1}{2} F_A \wedge *F_A + A \wedge J.$$

To see that this is the correct Lagrangian, we must check that the Euler–Lagrange equations yield the classical Maxwell's equations, i.e.

$$dF_A = 0, \quad d * F_A = J.$$

Since $dF_A = 0$ is always satisfied (by the Bianchi identity), we must check that we get $d * F_A = J$. Physicists do this by writing everything in components and using the explicit Euler–Lagrange equations. Rather, we use Hamilton's principle and compute stationary points of the action $S[A]$ directly; recall that these are exactly the solutions of the Euler–Lagrange equations. We look at the variational equation

$$\delta S[A] = 0.$$

This raises the question of what we mean by a variation in this infinite-dimensional setting. Recall that in classical mechanics, a variation of a path $x : [a, b] \to M$ in a finite-dimensional manifold M, is a set of parametrized paths

$$\overline{x}(t, \varepsilon) : [a, b] \times (-\varepsilon_0, \varepsilon_0) \to M$$

such that

$$\delta x(t) := \frac{d}{d\varepsilon}\tilde{x}_t(0)$$

satisfying the boundary conditions $\delta x(a) = \delta x(b) = 0$. One approach to defining variations in quantum field theory is to "formally" differentiate functions of field variables.

Example 6.1.6. Say $A \in \Omega^1(X)$, X a compact 3-manifold, and let $\alpha[A] := \int_X A \wedge dA$, and write $A = A_\beta dx^\beta$, so that $dA = (\partial_\nu A_\mu) dx^\nu \wedge dx^\mu$ and

$$A \wedge dA = A_\beta(\partial_\nu A_\mu) dx^\beta \wedge dx^\nu \wedge dx^\mu.$$

Then, allowing ourselves some liberty in mathematical writing, we have

$$\begin{aligned}
\delta\alpha[A] : &= \int_X \left(\frac{\delta\alpha}{\delta A_\beta} \cdot \delta A_\beta + \frac{\delta\alpha}{\delta(\partial_\nu A_\mu)} \cdot \delta(\partial_\nu A_\mu) \right) \\
&= \int_X \left[(\delta A_\beta)(\partial_\nu A_\mu) dx^\beta \wedge dx^\nu \wedge dx^\mu + A_\beta \delta(\partial_\nu A_\mu) dx^\beta \wedge dx^\nu \wedge dx^\mu \right] \\
&= \int_X [(\delta A_\beta)(\partial_\nu A_\mu) + A_\beta(\partial_\nu \delta A_\mu)] dx^\beta \wedge dx^\nu \wedge dx^\mu)
\end{aligned}$$

Here in the last step, we assume that variation commutes with derivatives so that $\delta(\partial_\nu A_\mu) = \partial_\nu(\delta A_\mu)$. If we define $\delta A := (\delta A_\mu) dx^\mu$, then we find

$$\delta\alpha[A] = \int_X [\delta A \wedge dA + A \wedge d\delta A].$$

We take variations δA with "appropriate" boundary conditions.

Consider $A \in \Omega^1_c(\mathbb{R}^4)$ and

$$S[A] = \int_{\mathbb{R}^4} \left[-\frac{1}{2} F_A \wedge *F_A + A \wedge J \right].$$

Then another approach is to formalize the properties we know variations satisfy and *define* a variation.

Definition 6.1.7. A *variation* on $\mathbb{R}^4$ with the Minkowski metric is a map $\delta : \Omega^l(\mathbb{R}^4) \to \Omega^l(\mathbb{R}^4)$ that

(A) commutes with the exterior derivative, i.e. $\delta d = d\delta$, and

(B) commutes with the Hodge star, i.e. $\delta * = *\delta$, and

(C) satisfies $\delta(\alpha \wedge \beta) = \delta\alpha \wedge \beta + \alpha \wedge \delta\beta$.

Using the properties (A)—(C), we compute

$$\begin{aligned}
\delta S[A] : &= \int_{\mathbb{R}^4} \delta\mathcal{L}[A, \dot{A}] = \int_{\mathbb{R}^4} \delta \left[-\frac{1}{2} F_A \wedge *F_A + A \wedge J \right] \\
&= \int_{\mathbb{R}^4} \left[-\frac{1}{2}(d\delta A \wedge *dA + dA \wedge *d(\delta)) + \delta A \wedge J \right].
\end{aligned}$$

Define $\langle \alpha, \beta \rangle := \int_{\mathbb{R}^4} \alpha \wedge \beta$ for $\alpha, \beta \in \Omega^l_c(\mathbb{R}^4)$. Recall the adjoint d^* of $d : \Omega^l(\mathbb{R}^4) \to \Omega^{l+1}(\mathbb{R}^4)$ defined by $\langle d\alpha, \beta \rangle = \langle \alpha, d^*\beta \rangle$, so that $d^* : \Omega^l(\mathbb{R}^4) \to \Omega^{l-1}(\mathbb{R}^4)$ satisfies $d^* = *d*$ on Minkowski $\mathbb{R}^4$. Also, recall that $*^2 = (-1)^{l(4-l)+1}$ on $\Omega^l(\mathbb{R}^4)$. Then

$$\begin{aligned}
\int_{\mathbb{R}^4} d\delta A \wedge *dA =: \langle d\delta A, dA \rangle = \langle \delta A, d^* dA \rangle = \int_{\mathbb{R}^4} \delta A \wedge (*d * dA) \\
= \int_{\mathbb{R}^4} \delta A \wedge (d * dA).
\end{aligned}$$

Also,

$$\int_{\mathbb{R}^4} dA \wedge *d(\delta A) =: \langle dA, d\delta A\rangle = \langle d\delta A, dA\rangle = \langle \delta A, d^* dA\rangle.$$

Hence, overall we get

$$\delta S[A] = \int_{\mathbb{R}^4} [-(\delta A \wedge d * dA) + \delta A \wedge J] = \int_{\mathbb{R}^4} \delta A \wedge [-d * F_A + J].$$

By the Fundamental lemma of calculus of variations we see that $\delta S[A] = 0$ if and only if $d*F_A = J$, and we recover Maxwell's equations.

24th lecture, December 5th 2011

As we saw last time, quantum electro magnetism may be described in terms of a path integral

$$Z := \fint_{A \in \Omega^1(\mathbb{R}^4)} \exp\left[\frac{iS[A]}{\hbar}\right] [\mathcal{D}A],$$

where the action $S[A]$ is given by

$$S[A] := \int_{\mathbb{R}^4} \left(-\frac{1}{2} F_A \wedge *F_A + A \wedge J\right).$$

Example 6.1.8. Consider the Aharonov–Bohm effect: see Fig. **??**, where the shift in the interference pattern, which occurs even if the electron can not interact with the magnetic field. Here, the interference pattern is shifted by $\Phi = \int_D B$, where D is the solonoid and B the magnetic flux through it.

Viewing the path integral as arising from all possible paths from the electron to the detector screen, as previously the magnitude of the pattern may be described by $|Z|^2$.

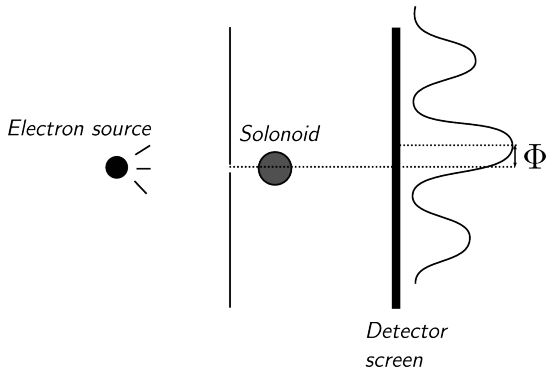

Figure 7: The Aharonov–Bohm effect

6.2 Yang–Mills theory

It turns out to be natural to view $A \in \Omega^1(\mathbb{R}^4)$ as coming from connections on a principal U(1)-bundle. After work of Hilbert, Weyl, Fock, etc., there was a generalization to a formal theory of quantum electro magnetism, due to C. N. Yang and R. Mills, [YM54], giving an attempted explanation for "strong interactions" inside a nucleus of an atom by generalizing from U(1) to more general groups G.

For simplicity, we consider a "toy model" and replace $\mathbb{R}^4$ (or a 4-manifold) by a 2-manifold and we assume that $J = 0$, so there is no external charge in our theory. The theory is known as 2D (quantum) Yang–Mills theory.

Let Σ be a connected closed Riemann surface, G a compact conencted Lie group with Lie algebra $\operatorname{Lie} G = \mathfrak{g}$. Let $P \xrightarrow{\pi} \Sigma$ be a principal G-bundle, and let

$$\Omega^l(P;\mathfrak{g}) = \Gamma(P, \wedge^l T^*P \otimes \mathfrak{g}) = \Omega^l(P) \otimes \mathfrak{g}$$

be the space of Lie algebra valued l-forms on P. Reclal that a typical element $\alpha \in \Omega^l(P;\mathfrak{g})$ may be expressed relative to a basis $\{Y_j\}_{j=1}^{d=\dim \mathfrak{g}}$ of $\mathfrak{g}$ as

$$\alpha = \sum_j \alpha_j \otimes Y_j,$$

for $\alpha_j \in \Omega^l(P)$. The group G acts on $\Omega^l(P;\mathfrak{g})$ as $g \cdot \alpha := \operatorname{Ad}_g(g^*\alpha)$ or $g \cdot \alpha = \sum_j g^*\alpha_j \otimes \operatorname{Ad}_g Y_j$. For each $Y \in \mathfrak{g}$, we get an associated vertical vector field $Y^\sharp \in \Gamma(TP)$ via the natural G-action on P

$$Y^\sharp_p := \frac{d}{dt}|_{t=0}[p \cdot \exp(tY)].$$

Recall that a *connection* is a 1-form $A \in \Omega^1(P,\mathfrak{g})$ that is G-invariant, so $g^*A = \operatorname{Ad}_{g^{-1}} A$ for all $g \in G$, and is also vertical, so that $\iota_{Y^\sharp} A = Y$ for all $Y \in \mathfrak{g}$. Let $\Omega^l(P,\mathfrak{g})^G$ denote the G-invariant Lie algebra valued l-forms on P.

Definition 6.2.1. The *space of connections* on a principal G-bundle P is the set

$$\mathcal{A}_P := \{A \in \Omega^1(P;\mathfrak{g})^G \mid \iota_{Y^\sharp} A = Y, \forall\, Y \in \mathfrak{g}\}.$$

Let $\Omega^l_{\text{hor}}(P;\mathfrak{g}) := \{\beta \in \Omega^l(P;\mathfrak{g}) \mid \iota_{Y^\sharp}\beta = 0, \forall\, Y \in \mathfrak{g}\}$ denote the *space of horizontal $\mathfrak{g}$-valued l-forms* . Let $\operatorname{Ad} P := P \times_G \mathfrak{g} =: \mathfrak{g}(P)$ denote the *associated bundle* for the adjoint representation on $\mathfrak{g}$, $\operatorname{Ad} : G \to \operatorname{Aut}(\mathfrak{g})$, i.e. $P \times_G \mathfrak{g} := (P \times \mathfrak{g})/\sim$, where $(p, Y) \sim (p', Y')$ if and only if there exists $g \in G$ such that $(p', Y') = (p \cdot g, \operatorname{Ad}_{g^{-1}} Y)$, so $\operatorname{Ad}(P)$ is a vector bundle over Σ with fibres isomorphic to $\mathfrak{g}$.

Let $\Omega^l(\Sigma; \operatorname{Ad} P) := \Gamma(\Sigma, \wedge^l T^*\Sigma \otimes \operatorname{Ad} P)$. Recall that we may naturally identify $\Omega^l_{\text{hor}}(P;\mathfrak{g})^G \cong \Omega^l(\Sigma; \operatorname{Ad} P)$. We make this identification throughout. Recall also that $\mathcal{A}_P$ is an affine space modelled on the vector space $\Omega^1_{\text{hor}}(P;\mathfrak{g})^G \cong \Omega^1(\Sigma, \operatorname{Ad} P)$:

Theorem 6.2.2. *If P is a principal bundle over Σ, then $\mathcal{A}_P$ is non-empty, and if $A_0 \in_{\mathcal{P}}$, then we may identify $\mathcal{A}_P = A_0 + \Omega^1_{\text{hor}}(P,\mathfrak{g})^G$.*

Proof. For the first part, see Kobayashi–Nomizu; the idea is to use a partition of unity. For the second part, observe that any connection $A \in \mathcal{A}_P$ that $A - A_0 \in \Omega^1_{\text{hor}}(P;\mathfrak{g})^G$. This is easy to see since

$$\iota_{Y^\sharp}(A - A_0) = \iota_{Y^\sharp} A - \iota_{Y^\sharp} A_0 = Y - Y = 0.$$

Also, if A and A_0 are G-invariant, then so is $A - A_0$. □

For $\alpha = \sum \alpha_j \otimes Y_j \in \Omega^l(P,\mathfrak{g})$, $\beta = \sum \beta_j \otimes Y_j \in \Omega^r(P;\mathfrak{g})$, define

$$\begin{aligned} d\alpha :&= (d \otimes \mathbf{1}_{\mathfrak{g}})(\alpha) = \sum d\alpha_j \otimes Y_j, \\ [\alpha \wedge \beta] :&= (\wedge \otimes [\cdot,\cdot])(\alpha,\beta) = \sum_j \sum_k (\alpha_j \wedge \beta_k) \otimes [Y_j, Y_k] \in \Omega^{l+r}(P;\mathfrak{g}). \end{aligned}$$

We leave the proof of the following as a straight-forward exercise.

Proposition 6.2.3. *If $\alpha \in \Omega^l(P;\mathfrak{g})$, $\beta \in \Omega^r(P;\mathfrak{g})$, then*

(A) $[\alpha \wedge \beta] = (-1)^{lr+1}[\beta \wedge \alpha]$,

(B) $d[\alpha \wedge \beta] = [d\alpha \wedge \beta] + (-1)^l[\alpha \wedge d\beta]$,

(C) if α, β are G-invariant, then so are $d\alpha, d\beta$, and $[\alpha \wedge \beta]$, and

(D) if α, β are horizontal, then so is $[\alpha \wedge \beta]$.

Definition 6.2.4. The *curvature of a connection* $A \in \mathcal{A}_P$ is defined as the 2-form $F_A \in \Omega^2(P;\mathfrak{g})$ given by

$$F_A := dA + \frac{1}{2}[A \wedge A].$$

One can show that $F_A \in \Omega^2_{\text{hor}}(P;\mathfrak{g})^G$. Accordingly, we view $F_A \in \Omega^2(\Sigma, \operatorname{Ad} P)$.

Definition 6.2.5. Given a connection $A \in \mathcal{A}_P$, define the *twisted deRham derivative* $d_A : \Omega^l_{\text{hor}}(P;\mathfrak{g})^G \to \Omega^{l+1}_{\text{hor}}(P;\mathfrak{g})^G$ by

$$d_A\alpha := d\alpha + [A \wedge \alpha].$$

Under the usual identifications, we can view d_A as a map $d_A : \Omega^l(\Sigma, \operatorname{Ad} P) \to \Omega^{l+1}(\Sigma, \operatorname{Ad} P)$. It is easy to show that

$$d_A[\alpha \wedge \beta] = [d_A\alpha \wedge \beta] + (-1)^{|\alpha|}[\alpha \wedge d_A\beta].$$

Definition 6.2.6. If $A \in \mathcal{A}_P$ satisfies $F_A = 0$, then A is called a *flat connection* . We let $\mathcal{A}^{\text{f}}_P$ denote the space of flat connections.

Proposition 6.2.7. *If $\alpha \in \Omega^l_{\text{hor}}(P,\mathfrak{g})^G$, then $d_A^2\alpha = [F_A \wedge \alpha]$.*

Proof. This is an exercise using definitions and properties of $[\cdot \wedge \cdot]$. □

Corollary 6.2.8. *If $A \in \mathcal{A}^{\text{f}}_P$ is a flat connection, then d_A forms a differential on $\Omega^\bullet_{\text{hor}}(P;\mathfrak{g})^G$, and we get a complex*

$$0 \xrightarrow{d_A} \Omega^0_{\text{hor}}(P;\mathfrak{g})^G \xrightarrow{d_A} \Omega^1_{\text{hor}}(P;\mathfrak{g})^G \xrightarrow{d_A} \cdots . \tag{23}$$

Definition 6.2.9. The complex (23) is called the *twisted deRham complex* . We say that this complex is twisted by a flat connection A. We denote the twisted deRham cohojmology as

$$H^l(\Sigma; d_A) := \frac{\ker(d_A : \Omega^l_{\text{hor}}(P;\mathfrak{g})^G \to \Omega^{l+1}_{\text{hor}}(P;\mathfrak{g})^G)}{\operatorname{Im}(d_A : \Omega^{l-1}_{\text{hor}}(P;\mathfrak{g})^G \to \Omega^l_{\text{hor}}(P;\mathfrak{g})^G)}.$$

These cohomology groups are important and have various interpretations, e.g. in terms of the moduli space of flat connections, where H^1 can be understood as the tangent space at a connection A.

Recall, in Maxwell's theory on $\mathbb{R}^4$, that forms $A, A' \in \Omega^1(\mathbb{R}^4)$ were "physically" equivalent, if the differ by an exact form, so $A' = A + d\psi$ for $\psi \in \Omega^0(\mathbb{R}^4)$. We formalize the transformation $A \mapsto A + d\psi$ and consider the analogue for "arbitrary" groups G. If we view $\Omega^1(\mathbb{R}^4)$ as $\Omega^1(\mathbb{R}^4; \operatorname{Ad} P)$, where $P \xrightarrow{\pi} \mathbb{R}^4$ is a principal $\mathrm{U}(1)$-bundle (up to a multiple of i, if we identify $\operatorname{Lie}\mathrm{U}(1) \cong i\mathbb{R}$), then the transformation $A + d\psi$ may be viewed as an action of a group on $\mathcal{A}_P$.

Indeed, define $\mathcal{G}_P := \operatorname{Maps}^\infty(\mathbb{R}^4, \mathrm{U}(1))$ with group structure being pointwise multiplication, and for $u = e^\psi \in \mathcal{G}_P$, $\psi \in i\Omega^0(\mathbb{R}^4)$, define $A \cdot u := A + u^{-1}du$, so that $du = de^\psi = e^\psi \cdot d\psi$, and $u^{-1}du = e^{-\psi}(e^\psi d\psi) = d\psi$.

So we see that $\mathcal{G}_P$ acts on $\mathcal{A}_P$ and the orbits of this action consist of connections that are physically equivalent. Then it is natural to consider $\mathcal{A}_P/\mathcal{G}_P$.

Definition 6.2.10. For a principal G-bundle $P \xrightarrow{\pi} \Sigma$, we define the *gauge group* $\mathcal{G}_P$ to be the set

$$C^\infty(P,G)^G := \{u \in C^\infty(P,G) \mid u(p \cdot g) = g^{-1}u(p) \cdot g\}$$

with group structure given by pointwise multiplication.

Note that sometimes it is useful to identify $\mathcal{G}_P \cong \mathrm{Aut}(P) := \{\Phi \in (\mathrm{Diff}(P,P))^G \mid \pi \circ \Phi = \pi\}$, via the correspondence $\mathcal{G}_P \ni u \mapsto \Phi_u \in \mathrm{Aut}(P)$, where

$$\Phi_u(p) := p \cdot u(p).$$

One can check that this is a group isomorphism, where the group structure on $\mathrm{Aut}(P)$ is given by composition.

Definition 6.2.11. The *gauge action* of $\mathcal{G}_P = C^\infty(P,G)^G$ on $\mathcal{A}_P$ (as a right action) is defined as follows: For $u \in \mathcal{G}_P$, and $A \in \mathcal{A}_P$, define

$$A \cdot u := \mathrm{Ad}_{u^{-1}} A + u^*\theta^L,$$

where $\theta^L \in \Omega^1(G,\mathfrak{g})$ is the *left-invariant Maurer–Cartan form* defined by

$$\theta^L_g(v_g) \equiv (L_{g^{-1}})_* v_g,$$

where $v \in \Gamma(TG)$.

One can show in a matrix representation G, one has $\theta^L_g = g^{-1}dg$ (one can show that every compact Lie group embeds into some $\mathrm{GL}(N,\mathbb{C})$).

Proposition 6.2.12. *Let $A \in \mathcal{A}_P$, $u \in \mathcal{G}_P$, then $F_{A\cdot u} = \mathrm{Ad}_{u^{-1}} F_A$.*

Proof. Recall that $A \cdot u = \mathrm{Ad}_{u^{-1}} A + u^*\theta^L$. One finds that

$$F_{A\cdot u} = d[\mathrm{Ad}_{u^{-1}} A] + \frac{1}{2}[\mathrm{Ad}_{u^{-1}} A \wedge \mathrm{Ad}_{u^{-1}} A] + d[u^*\theta^L] + \frac{1}{2}[u^*\theta^L \wedge u^* \wedge u^*\theta^L] + [u^*\theta^L \wedge \mathrm{Ad}_{u^{-1}} A].$$

This follows from the formula $[\alpha \wedge \beta] = (-1)^{lr+1}[\beta \wedge \alpha]$ for $\alpha \in \Omega^l(P;\mathfrak{g}), \beta \in \Omega^r(P,g)$, applied to $u^*\theta^L, \mathrm{Ad}_{u^{-1}} A \in \Omega^1(P;\mathfrak{g})$. Recall the Maurer–Cartan structure equations which says that

$$d\theta^L + \frac{1}{2}[\theta^L, \theta^L] = 0,$$

and so,

$$d[u^*\theta^L] + \frac{1}{2}[u^*\theta^L, u^*\theta^L] = u^*[d\theta^L + \frac{1}{2}[\theta^L, \theta^L]] = 0.$$

Now, write

$$d[\mathrm{Ad}_{u^{-1}} A] = d(\mathrm{Ad}_{u^{-1}}) \wedge A + \mathrm{Ad}_{u^{-1}} dA.$$

We claim that $d(\mathrm{Ad}_{u^{-1}})A = -[u^*\theta^L \wedge \mathrm{Ad}_{u^{-1}} A]$. To see this, let $\gamma : [-\varepsilon, \varepsilon] \to P$ be a curve with $\gamma(0) = p$, $\dot\gamma(0) = v$, and write $u^{-1}(\gamma(t)) = u^{-1}(p) \cdot g(t)$, where $g(t) := L_{u(p)} := L_{u(p)} \cdot u^{-1}(\gamma(t))$ for $L : G \to G$ the left multiplication map. Then clearly $g(0) = e$, and $\mathrm{Ad}_{u^{-1}(\gamma(t))} = \mathrm{Ad}_{u^{-1}(p)} \cdot \mathrm{Ad}_{g(t)}$. Also

$$\dot g(0) = \frac{d}{dt}|_{t=0} = ((u^{-1})^*\theta^L)(\dot\gamma(0)),$$

by definition of θ^L. Hence

$$d(\mathrm{Ad}_{u^{-1}} A)_p(v) = \frac{d}{dt}|_{t=0}(\mathrm{Ad}_{u^{-1}(\gamma(t))}) = \frac{d}{dt}|_{t=0}(\mathrm{Ad}_{u^{-1}(p)} \cdot \mathrm{Ad}_{g(t)}) = \mathrm{Ad}_{u^{-1}(p)} \cdot [(u^{-1})^*\theta^L, \cdot].$$

Thus we find that

$$d\,\mathrm{Ad}_{u^{-1}} \wedge A = [\mathrm{Ad}_{u^{-1}}(u^{-1})^*\theta^* \wedge \mathrm{Ad}_{u^{-1}} A].$$

Letting $R : G \to G$ denote right multiplication, observe that

$$\mathrm{Ad}_{u^{-1}}(u^{-1})^*\theta^L = dR_{u^{-1}} \circ dL_u \circ dL_{u^{-1}} \circ du^{-1} = dR_{u^{-1}} \circ du^{-1} = (u^{-1})^*\theta^R,$$

where θ^R is the right invariant Maurer–Cartan form. Recall that if $\mathrm{Inv} : G \to G$ denotes the inversion map $\mathrm{Inv}(g) \equiv g^{-1}$, then $\theta^R = -\,\mathrm{Inv}^*\,\theta^L$. Therefore,

$$(u^{-1})^*\theta^R = -(u^{-1})^*(\mathrm{Inv})^*\theta^L = -u^*\theta^L,$$

and overall, we find that

$$d(\mathrm{Ad}_{u^{-1}}) \wedge A = -[u^*\theta^L \wedge \mathrm{Ad}_{u^{-1}} A],$$

and so $F_{A\cdot u} = \mathrm{Ad}_{u^{-1}} F_A$. □

25th lecture, December 9th 2011

Now, we would like to define a 2D Yang–Mills action analogous to $S[A] = -\frac{1}{2}\int_{\mathbb{R}^4} dA \wedge *dA$. In our case, dA gets replaced by $F_A \in \Omega^2_{\mathrm{hor}}(P,\mathfrak{g})^G$. To be able to do this, we want to define a map

$$\mathrm{Tr} : \Omega^l_{\mathrm{hor}}(P,\mathfrak{g})^G \times \Omega^r_{\mathrm{hor}}(P;\mathfrak{g})^G \to \Omega^{l+r}_{\mathrm{hor}}(P)^G \cong \Omega^{l+r}(\Sigma).$$

In order to do so, let $\langle\cdot,\cdot\rangle \in \mathrm{Sym}^2\,\mathfrak{g}^*$ be an Ad-invariant, negative definite, non-degenerate, $\mathbb{R}$-bilinear form on $\mathfrak{g}$, so

$$\langle \mathrm{Ad}_g\, Y_1, \mathrm{Ad}_g\, Y_2\rangle = \langle Y_1, Y_2\rangle$$

for all $Y_1, Y_2 \in \mathfrak{g}$. By general Lie algebra theory, such a thing exists (assuming maybe(?) that the Lie group is compact and semi-simple). For $\alpha = \sum \alpha_j \otimes Y_j \in \Omega^l_{\mathrm{hor}}(,\mathfrak{g})^G$, and $\beta \in \sum \beta_k \otimes Y_k \in \Omega^r_{\mathrm{hor}}(P;\mathfrak{g})^G$, define

$$\mathrm{Tr}(\alpha \wedge \beta) := \sum_{j,k} \langle Y_j, Y_k\rangle \alpha_j \wedge \beta_k \in \Omega^{l+r}(P). \tag{24}$$

Proposition 6.2.13. *Let $\alpha \in \Omega^l_{\mathrm{hor}}(P;\mathfrak{g})^G$, $\beta \in \Omega^r_{\mathrm{hor}}(P;\mathfrak{g})^G$. Then*

(A) $\mathrm{Tr}(\alpha \wedge \beta) \in \Omega^{l+r}_{\mathrm{hor}}(P)^G \cong \Omega^{l+r}(\Sigma)$,

(B) $\mathrm{Tr}(\alpha \wedge \beta) = (-1)^{lr}\,\mathrm{Tr}(\beta \wedge \alpha)$, *and*

(C) for any $A \in \mathcal{A}_P$, we have $d\,\mathrm{Tr}(\alpha \wedge \beta) = \mathrm{Tr}(d_A\alpha \wedge \beta) + (-1)^l\,\mathrm{Tr}(\alpha \wedge d_A\beta)$

Proof. Exercise. □

Note that we may equivalently view Tr as a map on $\Omega^l(\Sigma, \mathrm{Ad}\,P)$, using our identification from the previous lecture. It can in fact also be defined directly using local trivializations with a formula essentially identical to (24). Similarly, we may extend the Hodge star as

$$* : H^l(\Sigma, \mathrm{Ad}\,P) \to \Omega^{2-l}(\Sigma, \mathrm{Ad}\,P)$$

given a metric $g_\Sigma \in \mathrm{Sym}^2\,T^*\Sigma$. Define this extension as follows: Given a local trivialization $\pi^{-1}(U_J) \cong U_J \times \mathfrak{g}$ for $\mathrm{Ad}\,P$, express $\alpha \in \Omega^l(\Sigma, \mathrm{Ad}\,P)$ locally as

$$\alpha_J := \alpha|_{U_J} \in \Omega^l(\Sigma,\mathfrak{g}) \cong \Omega^l(\Sigma) \otimes \mathfrak{g}.$$

Then $*\alpha$ is defined locally as $*\alpha_J := \sum_j *\alpha_{J,j} \otimes Y_j$, where $\alpha_J = \sum \alpha_{J,j} \otimes Y_j$. One checks that this gives a globally well-defined form as follows:

Let $\pi^{-1}(U_K) \cong U_K \times \mathfrak{g}$ be another trivialization such that $U_K \cap U_J \neq \emptyset$. Let $\varphi_{JK} : U_K \cap U_J \to \mathrm{Aut}\mathfrak{g}$ be the transition map, so that $\varphi_{JK} = \mathrm{Ad}_{\psi_{JK}}$, where $\psi_{JK} : U_K \cap U_J \to G$ is a transition map for P. We need to check that $*\alpha_J = \varphi_{JK} * \alpha_K$. This is easy to see since $\alpha_J = \varphi_{JK}\alpha_K$, and

$$*\alpha_J = *(\varphi_{JK}\alpha_K) = * \left(\sum_i \alpha_{K,i} \otimes \varphi_{JK} Y_i \right) = \sum_i *\alpha_{K,i} \otimes \varphi_{JK} Y_i = \varphi_{JK} * \alpha_k.$$

Thus $\alpha \in \Omega^{2-l}(\Sigma, \mathrm{Ad}\, P)$ is well-defined. Now, we define a 2D Yang–Mills action as follows:

Definition 6.2.14. Given a closed, orientable Riemann surface Σ, a compact connected Lie group G, a principal G-bundle $P \to \Sigma$, and a fixed metric g_Σ of total volume 1, define the *2D Yang–Mills action relative to P* at $A \in \mathcal{A}_P$

$$S^P_{\mathrm{YM}}[A] := -\frac{1}{2} \int_\Sigma \mathrm{Tr}(F_A \wedge *F_A),$$

where $* : \Omega^l(\Sigma, \mathrm{Ad}\, P) \to \Omega^{2-l}(\Sigma, \mathrm{Ad}\, P)$ is the Hodge star for g_Σ.

Recall that A Riemann surface is a smooth 2-manifold equipped with a conformal equivalence class of metrics, denoted $[g_\Sigma]$. We let g_Σ be a fixed metric of $[g_\Sigma]$ of total value $1 = \int_\Sigma *1 = \int_\Sigma d\mathrm{Vol}_{g_\Sigma}$. We want to see how the action varies with metrics in this conformal class.

Note that for any $\tilde{g}_\Sigma \in [g_\Sigma]$, there exists $0 < \lambda \in C^\infty(\Sigma, \mathbb{R})$ such that $\tilde{g}_\Sigma = \lambda^2 \cdot \cdot g_\Sigma$. Recall that if $\tilde{*}$ is the Hodge star for $\tilde{g}_\Sigma = \lambda^2 g_\Sigma$, then acting on l-forms on an m-dimensional Riemannian manifold (say Σ),

$$\tilde{*} = \lambda^{m-2l} * .$$

Thus $\mathrm{Tr}(F_A \wedge \tilde{*}F_A) = \frac{1}{\lambda^2} \mathrm{Tr}(F_A \wedge *F_A)$. In general, $S^P_{\mathrm{YM}}[A]$ becomes $\frac{1}{\varepsilon} S^P_{\mathrm{YM}}[A]$, for some $\varepsilon > 0$, given below, when g_Σ becomes $\tilde{g}_\Sigma = \lambda^2 g_\Sigma$. Since $\mathrm{Vol}_{g_\Sigma}(\Sigma) = 1$, then

$$\mathrm{Vol}_{\tilde{g}_\Sigma}(\Sigma) = \int_\Sigma \tilde{*}1 = \int_\Sigma d\mathrm{Vol}_{\tilde{g}_\Sigma} =: \varepsilon.$$

We explicitly include the ε-dependence in our definition of the 2D Yang–Mills partition function.

Definition 6.2.15 (Heuristic)**.** Given a closed Riemann surface Σ, a principal G-bundle and a metric $g_\Sigma \in [g_\Sigma]$ such that $\mathrm{Vol}_{g_\Sigma}(\Sigma) = 1$, and $\varepsilon > 0$, the *2D Yang–Mills path integral relative to P* is the following (heuristic) quantity:

$$Z^P_{\mathrm{YM}}(\varepsilon) := \frac{1}{\mathrm{Vol}(\mathcal{G}_P)} \left(\frac{1}{2\pi\varepsilon} \right)^{\Delta_{\mathcal{G}_P}/2} \fint_{\mathcal{A}_P} [DA] \exp\left[-\frac{S^P_{\mathrm{YM}}[A]}{\varepsilon} \right].$$

Remark 6.2.16. In the above definition, $\Delta_{\mathcal{G}_P}$ is formally the dimension of $\mathcal{G}_P$, and $\mathrm{Vol}(\mathcal{G}_P)$ is formally the volume of $\mathcal{G}_P$ with respect to a natural "measure" on $\mathcal{G}_P$, induced by the metric g_Σ: Recall that the Lie algebra of the gauge group may be identified as $\mathrm{Lie}\,\mathcal{G}_P = C^\infty(P, \mathfrak{g})^G$. The Lie bracket is defined pointwise as

$$[Y_1, Y_2](p) \equiv [Y_1(p), Y_2(p)],$$

for $Y_1, Y_2 \in C^\infty(P, \mathfrak{g})^G$. Also, $Y \in C^\infty(P, \mathfrak{g})^G$ if and only if $Y(p \cdot g)\, \mathrm{Ad}_{g^{-1}} Y(p)$. To see this description of $\mathrm{Lie}\,\mathcal{G}_P$, let $u_t \equiv u(t) : (-\varepsilon, \varepsilon) \to C^\infty(P, G)^G$, be any short curve through $u_0 \equiv e$, i.e. $u_0(p) = e$ for all $p \in P$. We then see (roughly), that a tangent vector at u_0 in $C^\infty(P, G)^G = \mathcal{G}_P$ is a function on P with values in $\mathfrak{g}$, i.e.

$$Y(p) \equiv \frac{d}{dt}|_{t=0}[u_t(p)] \in T_e G \cong \mathfrak{g},$$

since $u_t(p) : (-\varepsilon, \varepsilon) \to G$ such that $u_0(p) = \varepsilon$. Let $C_g : G \to G$ be the conjugation map $h \mapsto ghg^{-1}$. We then see that the condition $u_t(p \cdot g) = g^{-1}u_t(p)g$ translates into

$$Y(p \cdot g) = \frac{d}{dt}|_{t=0}[C_{g^{-1}} \circ u_t(p)] = dC_{g^{-1}}|_e \circ Y(p) = \mathrm{Ad}_{g^{-1}} Y(p).$$

Thus (roughly) $\mathrm{Lie}\,\mathcal{G}_P = C^\infty(P, \mathfrak{g})^G$. Also, there exists an exponential map $\mathrm{Exp} : \mathrm{Lie}\,\mathcal{G}_P \to \mathcal{G}_P$ defined as

$$\mathrm{Exp}(Y)(p) := \exp(Y(p)).$$

In general, for $\alpha, \beta \in \Omega^l(\Sigma, \mathrm{Ad}\, P)$, we define the L^2-inner product

$$\langle \alpha, \beta \rangle := -\int_\Sigma \mathrm{Tr}(\alpha \wedge *\beta),$$

where $*$ is the Hodge star with respect to g_Σ. Since $C^\infty(P, \mathfrak{g})^G \cong \Omega^0(\Sigma, \mathrm{Ad}\, P)$, we get an inner product on $\mathrm{Lie}\,\mathcal{G}_P$, which we then formally take to define $\mathrm{Vol}\mathcal{G}_P$.

By construction, $S^P_{\mathrm{YM}}[A]$ is invariant under the gauge group action of $\mathcal{G}_P$ on $\mathcal{A}_P$, i.e. for $u \in \mathcal{G}_P$, by a previous proposition, we find that

$$\begin{aligned} S_{\mathrm{YM}}[A \cdot u] &= -\frac{1}{2}\int_\Sigma \mathrm{Tr}(F_{A\cdot u} \wedge *F_{A\cdot u}) = -\frac{1}{2}\int_\Sigma \mathrm{Tr}(\mathrm{Ad}_{u^{-1}} F_A \wedge \mathrm{Ad}_{u^{-1}}(*F_A)) \\ &= -\frac{1}{2}\int_\Sigma \mathrm{Tr}(F_A \wedge *F_A) \equiv S^P_{\mathrm{YM}}[A], \end{aligned}$$

since Tr is Ad-invariant.

The analogue of Maxwell's equations come from stationary points of the action which we will now describe. Recall the following:

Proposition 6.2.17 (Bianchi identity)**.** *For $A \in \mathcal{A}_P$, $d_A F_A = 0$.*

Proof. Exercise. □

Since $\mathcal{A}_P$ is affine, $T_A\mathcal{A}_P \cong \Omega^1(\Sigma, \mathrm{Ad}\, P)$, we compute a variation of $S^P_{\mathrm{YM}}[A]$ as follows: For $\beta \in T_A\mathcal{A}_P$, define

$$\delta_\beta S^P_{\mathrm{YM}}[A] := \frac{d}{dt}|_{t=0} S^P_{\mathrm{YM}}[A + t\beta].$$

We require that $\delta_\beta S^P_{\mathrm{YM}}[A] = 0$ for all $\beta \in T_A\mathcal{A}_P$.

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Since by definition

$$S^P_{\mathrm{YM}}[A] = \frac{1}{2}\langle F_A, F_A \rangle_{L^2},$$

we consider $F_{A+t\beta} = F_A + t d_A\beta + \frac{t^2}{2}[\beta \wedge \beta]$. It is easy to show that $\delta_\beta S^P_{\mathrm{YM}}[A] = \langle d_A\beta, F_A\rangle_{L^2}$. Recall that the adjoint $d_A^* : \Omega^l(\Sigma, \mathrm{Ad}\, P) \to \Omega^{l-1}(\Sigma, \mathrm{Ad}\, P)$ of d_A is defined by $\langle d_A\alpha, \beta\rangle_{L^2} =: \langle \alpha, d_A^*\beta\rangle_{L^2}$. One can show that $d_A^* = - * d_A *$ on $\Omega^2(\Sigma, \mathrm{Ad}\, P)$. Thus, overall $\delta_\beta S^P_{\mathrm{YM}}[A] = 0$ for all $\beta \in \Omega^1(\Sigma, \mathrm{Ad}\, P)$ if and only if $\langle \beta, d_A^* F_A\rangle_{L^2} = 0$ for all β, which holds if and only if $d_A^* F_A = 0$ or, equivalently, $d_A * F_A = 0$.

Definition 6.2.18. The *Yang–Mills equations* are

$$d_A F_A = 0, \quad d_A * F_A = 0.$$

The goal is now to sketch is to compute $Z^P_{\mathrm{YM}}(\varepsilon)$ using some *formal* localization ideas. As observed in [AB83], the space $\mathcal{A}_P$ of connections admits a natural symplectic structure such that the gauge group action is Hamiltonian with respect to a particular moment map. The idea by Witten is therefore to mimic the Duistermaat–Heckman formula and do something analogous for the path integral.

Given $\alpha, \beta \in T_A\mathcal{A}_P \cong \Omega^1(\Sigma, \operatorname{Ad} P)$, and define $\omega \in \Omega^2(\mathcal{A}_P, \mathbb{R})$ by

$$\omega_A(\alpha, \beta) \equiv -\int_\Sigma \operatorname{Tr}(\alpha \wedge \beta).$$

Proposition 6.2.19. *The form $\omega \in \Omega^2(\mathcal{A}_P, \mathbb{R})$ is a symplectic form, i.e. it is non-degenerate and closed.*

Here, in this infinite-dimensional setting, we run into some problems about what to mean by "closed". Note first that ω is non-degenerate since $\omega_A(\alpha, *\alpha) =: \langle \alpha, \alpha \rangle_{L^2}$ and by properties of $\langle \cdot, \cdot \rangle_{L^2}$, $\omega_A(\alpha, *\alpha) = 0$ if and only if $\alpha = 0$. To see that $d\omega = 0$, we use the working definition of the exterior derivative in the infinite-dimensional case as follows:

For $\tau \in \Omega^l(\mathcal{A}_P, \mathbb{R})$, define $d\tau \in \Omega^{l+1}(\mathcal{A}_P, \mathbb{R})$ by

$$\begin{aligned} d\tau(v_0, v_1, \dots, v_l) := &\sum_{r=0}^{l} (-1)^r \mathcal{L}_{v_r}(\tau(v_0, \dots, \hat{v}_r, \dots, v_l)) \\ &+ \sum_{0 \le i \le j \le l} (-1)^{i+j} \tau(\mathcal{L}_{v_i} v_j, v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_l), \end{aligned}$$

where $v_r \in \Gamma(T\mathcal{A}_P)$ for $0 \le r \le l$, and $\mathcal{L}_v \tau := \frac{d}{dt}|_{t=0}(\Phi_t^* \beta)$ is well-defined for $\Phi_t : \mathcal{A}_P \to \mathcal{A}_P$, the flow of $v \in \gamma(T\mathcal{A}_P)$.

To evaluate $d\omega|_A(\beta_0, \beta_1, \beta_2)$ at $A \in \mathcal{A}_P$, for $\beta_i \in T_A\mathcal{A}_P$, we only need vector fields $v_i \in \Gamma(T\mathcal{A}_P)$ such that $v_i|_A = \beta_i$. Therefore, let $v_i|_A := \beta_i$ for $A \in \mathcal{A}_P$, such that the flows generated by v_i are $\Phi_t^i(A) = A + t\beta_i$. One can show that the flows commute, i.e. $\Phi_t^i \circ \Phi_s^j = \Phi_s^j \circ \Phi_t^i$, if and only $\mathcal{L}_{v_i} v_j = 0$, $0 \le i < j \le 2$. Also, since $A \mapsto \omega_A(v_i|_A, v_j|_A) = -\int_\Sigma \operatorname{Tr}(\beta_i \wedge \beta_j)$ is constant, $\mathcal{L}_{v_r}(\omega(v_0, \dots, \hat{v}_r, \dots, 2)) = 0$ and overall, $d\omega = 0$.

Definition 6.2.20. The form $\omega_A \in \Omega^2(\mathcal{A}_P, \mathbb{R})$defined by $\omega_A(\alpha, \beta) = -\int_\Sigma \operatorname{Tr}(\alpha \wedge \beta)$ is called the *Atiyah–Bott symplectic form* .

Next, we want to construct a moment map $\mu : \mathcal{A}_P \to (\operatorname{Lie} \mathcal{G}_P)^*$, for the gauge group action. First we need to understand the vector fields generated by $Y \in \operatorname{Lie} \mathcal{G}_P \cong C^\infty(P, \mathfrak{g})^G$.

Lemma 6.2.21. *If $g(t) : (-\varepsilon, \varepsilon) \to G$, and $Y \in \mathfrak{g}$, then*

$$\frac{d}{dt}|_{t=0}[\operatorname{Ad}_{g(t)} Y] = \left[\frac{d}{dt}|_{t=0}(g^{-1}(0) \cdot g(t)), \operatorname{Ad}_{g^{-1}(0)} Y\right].$$

Proof. One uses that if $g(t) : (-\varepsilon, \varepsilon) \to G$ is such that $g(0) = e$, then

$$\frac{d}{dt}|_{t=0}[\operatorname{Ad}_{g(t)} Y] = [g'(0), Y],$$

where $g'(0) = [\frac{d}{dt}|_{t=0} g(t)]^L_e \in \mathfrak{g}$. The details are left as an exercise. □

Lemma 6.2.22. *Let $u_t : (-\varepsilon, \varepsilon) \to \mathcal{G}_P = C^\infty(P, G)^G$ such that $u_0 \equiv e$, and let $Y := \frac{d}{dt}|_{t=0}(u_t) \in \operatorname{Lie} \mathcal{G}_P$. Then $dY = \frac{d}{dt}|_{t=0}[u_t^* \theta^L]$.*

Proof. Suppose $\gamma : (-\varepsilon, \varepsilon) \to P$, and recall that $(u_t^* \theta^L)_{\gamma(0)}(\dot\gamma(0)) = \frac{d}{ds}|_{s=0}[u_t^{-1}(\gamma(0)) \cdot u_t(\gamma(s))]$. We compute

$$\begin{aligned}\frac{d}{dt}|_{t=0}(u_t^*\theta^L)_{\gamma(0)}(\dot\gamma(0)) &= \frac{\partial^s}{\partial s \partial t}|_{s=t=0}[u_t^{-1}(\gamma(0)) \cdot u_t(\gamma(s))] \\ &= \frac{d}{ds}|_{s=0}\left[\left(\frac{d}{dt}|_{t=0}u_t^{-1}(\gamma(0))\right) + u_0(\gamma(s)) + u_0^{-1}\gamma(0)\left(\frac{d}{dt}|_{t=0}u_t(\gamma(s))\right)\right] \\ &= \frac{d}{ds}|_{s=0}\frac{d}{dt}|_{t=0}[u_t(\gamma(s))] = \frac{d}{ds}|_{s=0}Y(\gamma(s)) = dY_{\gamma(0)}(\dot\gamma(0)).\end{aligned}$$

Thus $\frac{d}{dt}|_{t=0}[u_t^*\theta^L] = dY$. □

Theorem 6.2.23. *Let $Y \in \operatorname{Lie}\mathcal{G}_P = C^\infty(P, \mathfrak{g})^G$, then for $A \in \mathcal{A}_P$, we have $Y_A^\sharp = d_A Y \in T_A\mathcal{A}_P$.*

Proof. By definition,

$$Y_A^\sharp = \frac{d}{dt}|_{t=0}[A \cdot \operatorname{Exp}(tY)] = \frac{d}{dt}|_{t=0}\left[\operatorname{Ad}_{\operatorname{Exp}(-tY)} A + \operatorname{Exp}(tY)^*\theta^L\right].$$

By our first lemma, with $g(t) = \exp(tY(p)) =: \operatorname{Exp}(tY)(p) : (-\varepsilon, \varepsilon) \to G$, we have

$$\frac{d}{dt}|_{t=0} \operatorname{Ad}_{\exp(tY(p))} A_p = -[Y(p), A_p] = [A, Y](p),$$

and by the second lemma, with $u_t = \operatorname{Exp}(tY) : (-\varepsilon, \varepsilon) \to \mathcal{G}_P = C^\infty(P, G)^G$, where $Y = \frac{d}{dt}|_{t=0}[\operatorname{Exp}(tY)] \in \operatorname{Lie}\mathcal{G}_P$, we know that $\frac{d}{dt}|_{t=0}[\operatorname{Exp}(tY)^*\theta^L]$. So overall,

$$Y_A^\sharp = dY + [A, Y] = d_A Y.$$

□

Our next goal is to show that the action $\mathcal{G}_P \circlearrowright \mathcal{A}_P$ is Hamiltonian for a particular moment map $\mu : \mathcal{A}_P \to (\operatorname{Lie}\mathcal{G}_P)^*$ relative to the symplectic structure given by the Atiyah–Bott symplectic form. We identify $(\operatorname{Lie}\mathcal{G}_P)^* \cong \Omega^2(\Sigma, \operatorname{Ad} P)$ using the metric g_Σ. That is, we define the pairing $(\cdot, \cdot) : \Omega^0(\Sigma, AdP) \otimes \Omega^2(\Sigma, \operatorname{Ad} P) \to \mathbb{R}$ via

$$(Y, \gamma) = \langle Y, *\gamma\rangle_{L^2}.$$

Then, given $\tilde\varphi \in (\operatorname{Lie}\mathcal{G}_P)^*$ as a map $\tilde\varphi : \mathcal{G}_P \to \mathbb{R}$, there exists a unique $\varphi \in \Omega^2(\Sigma, \operatorname{Ad} P)$ such that $\tilde\varphi(Y) = (Y, \varphi)$ by the Riesz representation theorem. We identify $\tilde\varphi \leftrightarrow \varphi$, i.e. $(\operatorname{Lie}\mathcal{G}_P)^* \cong \Omega^2(\Sigma, \operatorname{Ad} P)$.

Define $\mu : \mathcal{A}_P \to (\operatorname{Lie}\mathcal{G}_P)^* \cong \Omega^2(\Sigma, \operatorname{Ad} P) \cong \Omega^2_{\text{hor}}(P, \mathfrak{g})^G$ as

$$\mu(A) = -F_A.$$

Recall that $\mathcal{G}_P$ acts on $\operatorname{Lie}\mathcal{G}_P$ and $(\operatorname{Lie}\mathcal{G}_P)^*$ via the pointwise adjoint and coadjoint actions respectively, i.e. if $u \in \mathcal{G}_P$, $Y \in \operatorname{Lie}\mathcal{G}_P$, $\tilde\varphi \in (\operatorname{Lie}\mathcal{G}_P)^*$, then

$$(Y \cdot u)(p) \equiv \operatorname{Ad}_{u^{-1}(p)} Y(p), \ (\tilde\varphi \cdot u)(p) \equiv \operatorname{Ad}^*_{u^{-1}(p)} \tilde\varphi(p).$$

By definition, $(Y, \operatorname{Ad}^*_u \tilde\varphi) := (\operatorname{Ad}_{u^{-1}} Y, \tilde\varphi)$. Also by Ad-invariance of $(\cdot, \cdot)$, we know that $(\operatorname{Ad}_{u^{-1}} Y, \varphi) = (Y, \operatorname{Ad}_u \varphi)$. Thus, under our identification, $\operatorname{Ad}^*_u \tilde\varphi = \operatorname{Ad}_u \varphi$. Also, observe that if $u \in \mathcal{G}_P$ and $\psi_u : \mathcal{A}_P \to \mathcal{A}_P$ is the action map $\psi_u(A) := A \cdot u$, then $d\psi_u|_A : T_A\mathcal{A}_P \to T_{A\cdot u}\mathcal{A}_P$ is calculated as

$$\begin{aligned}d\psi_u|_A(\alpha) &= \frac{d}{dt}|_{t=0}[\psi_u(A + t\alpha)] = \frac{d}{dt}|_{t=0}[(A + t\alpha) \cdot u] \\ &= \frac{d}{dt}|_{t=0}[\operatorname{Ad}_{u^{-1}}(A + t\alpha) + u^*\theta^L] = \operatorname{Ad}_{u^{-1}}\alpha\,.\end{aligned}$$

Thus, $u_*|_A(\alpha) = d\psi_u|_A(\alpha) = \operatorname{Ad}_{u^{-1}} \alpha$.

Theorem 6.2.24. *We have that $(\mathcal{A}_P, \omega, \mathcal{G}_P, \mu)$ is a Hamiltonian $\mathcal{G}_P$-space with moment map $\mu(A) \equiv -F_A$.*

Proof. We want to show that

(i) the action $\mathcal{G}_P \circlearrowright (\mathcal{A}_P, \omega)$ is symplectic, that

(ii) for $u \in \mathcal{G}_P$, we have $u^*\mu = \mathrm{Ad}^*_{u^{-1}} \mu$, and that

(iii) $d\mu^Y = \iota_{Y^\sharp}\omega$.

For (i), let $u \in \mathcal{G}_P$, $A \in \mathcal{A}_P$, $\alpha, \beta \in T_{A\cdot u}\mathcal{A}_P$, and compute

$$\begin{aligned}(u^*\omega)_A(\alpha,\beta) = \omega_{A\cdot u}(u_*\alpha, u_*\beta) &= -\int_\Sigma \mathrm{Tr}(u_*\alpha \wedge u_*\beta) = -\int_\Sigma \mathrm{Tr}(\mathrm{Ad}_{u^{-1}}\alpha \wedge \mathrm{Ad}_{u^{-1}}\beta) \\ &= -\int_\Sigma (\alpha\wedge\beta) = \omega_A(\alpha,\beta),\end{aligned}$$

so $u^*\omega = \omega$ for all $u \in \mathcal{G}_P$.

For (ii), let $u \in \mathcal{G}_P$, and $A \in \mathcal{A}_P$. We want to show that $\mu(A\cdot u) = \mathrm{Ad}_{u^{-1}}\mu(A)$. This follows from the fact that $\mu(A) \equiv -F_A$, and also by a previous proposition, $F_{A\cdot u} = \mathrm{Ad}_{u^{-1}} F_A$.

For (iii), let $Y \in \mathrm{Lie}\,\mathcal{G}_P$, $A \in \mathcal{A}_P$, and $\alpha \in T_A\mathcal{A}_P$. We want to show that $d\mu^Y|_A(\alpha) = (\iota_{Y^\sharp}\omega_A)(\alpha) = \omega_A(Y^\sharp, \alpha)$. One one hand,

$$d\mu^Y|_A(\alpha) = \frac{d}{dt}|_{t=0}[\mu^Y(A+t\alpha)] = \frac{d}{dt}|_{t=0}(Y, \mu(A+t\alpha)) = \frac{d}{dt}|_{t=0}(Y, -F_{A+t\alpha}) = -(Y, d_A\alpha),$$

since $\frac{d}{dt}|_{t=0}F_{A+t\alpha} = d_A\alpha$. On the other hand, integrating by parts and using that, in general,

$$d\,\mathrm{Tr}(a\wedge b) = \mathrm{Tr}(d_A a \wedge b) + (-1)^{|a|}\,\mathrm{Tr}(a \wedge d_A b),$$

we find that

$$\omega_A(Y^\sharp, \alpha) = -\int_\Sigma \mathrm{Tr}(d_A Y \wedge \alpha) = \int_\Sigma \mathrm{Tr}(Y \wedge d_A\alpha) = -(Y, d_A\alpha).$$

□

In order to recast the partition function, we introduce additional structure to $\mathcal{A}_P$. Since $*^2 = -1$ on $T_A\mathcal{A}_P \cong \Omega^1(\Sigma, \mathrm{Ad}\,P)$, then $*$ defines a complex structure on $\mathcal{A}_P$. Also, by definition $(\omega, \langle\cdot,\cdot\rangle, *)$ form a compatible triple, i.e. $\langle\cdot,\cdot\rangle_{L^2} = \omega(\cdot, *\cdot)$. In other words, $\mathcal{A}_P$ admits a natural Kähler structure.

Therefore, the "measure" $[DA]$ on $\mathcal{A}_P$, we take formally to be defined by the Riemannian measure associated to $\langle\cdot,\cdot\rangle_{L^2}$, which is equivalent to the formal symplectic measure in the Kähler structure given by e^ω. Note that in finite dimensions, the symplectic measure is given by $\omega^n/n! = [e^\omega]_m$ for $\dim M = m = 2n$.

Also, to define $S^P_{\mathrm{YM}}[A] = -\frac{1}{2}\int_\Sigma \mathrm{Tr}(F_A \wedge *F_A)$, on a Riemann surface, we only need to specify a duality operator $* : \Omega^2(\Sigma, \mathrm{Ad}\,P) \to \Omega^0(\Sigma, \mathrm{Ad}\,P)$. This can be given by a choice of a volume form (or a symplectic form) $\sigma \in \Omega^2(\Sigma, \mathbb{R})$ so that $*1 \equiv \sigma$.

Let $(\cdot,\cdot) : \Omega^2(\Sigma, \mathrm{Ad}\,P) \times \Omega^2(\Sigma, \mathrm{Ad}\,P) \to \mathbb{R}$ denote the operator $(\varphi, \varphi) \equiv -\int_\Sigma \mathrm{Tr}(\varphi \wedge *\varphi)$. We may then write

$$Z^P_{\mathrm{YM}}(\varepsilon) = \frac{1}{\mathrm{Vol}(\mathcal{G}_P)}\left(\frac{1}{2\pi\varepsilon}\right)^{\Delta_{\mathcal{G}_P}/2} \fint_{\mathcal{A}_P} \exp\left[\omega - \frac{1}{2\varepsilon}(\mu,\mu)\right].$$

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We consider the canonical moment map squared form for the "2D Yang–Mills partition function,

$$Z^P_{\mathrm{YM}}(\varepsilon) = \frac{1}{\mathrm{Vol}(\mathcal{G})}\left(\frac{1}{2\pi\varepsilon}\right)^{\Delta_{\mathcal{G}_P}/2} \fint_{\mathcal{A}_P} \exp[\omega - \frac{1}{2\varepsilon}(\mu,\mu)],$$

where $\omega \in \Omega^2(\mathcal{A}_P, \mathbb{R})$ is the Atiyah–Bott symplectic form and $\mu : \mathcal{A}_P \to (\mathrm{Lie}\,\mathcal{G}_P)^*$ maps $A \mapsto -F_A$, and $(\cdot,\cdot) : \Omega^2(\Sigma, \mathrm{Ad}\,P) \times \Omega^2(\Sigma, \mathrm{Ad}\,P) \to \mathbb{R}$ defined by

$$(\varphi, \varphi) = -\int_\Sigma \mathrm{Tr}(\varphi \wedge *\varphi).$$

Here $* : \Omega^2(\Sigma, \mathrm{Ad}\,P) \to \Omega^0(\Sigma, \mathrm{Ad}\,P)$ is defined by a symplectic form $\sigma \in \Omega^(\Sigma, \mathbb{R})$ so $* \cdot 1 \equiv \sigma$ and $*^2 = 1$. We want to study $Z^P_{\mathrm{YM}}(\varepsilon)$ using localization ideas analogous. Let $\tilde{\mathfrak{g}} := \mathrm{Lie}\,\mathcal{G}_P$ and consider $\omega_{\tilde{\mathfrak{g}}} \in \Omega^2_{\mathcal{G}_P}(\mathcal{A}_P, \mathbb{R})$, defined for $Y \in \tilde{\mathfrak{g}}$ by $\omega_{\tilde{\mathfrak{g}}}(Y) = \omega + \mu^Y$, the $\mathcal{G}_P$-equivariant 2-form that extends the Atiyah–Bott form ω. We have $d_{\tilde{\mathfrak{g}}}\omega_{\tilde{\mathfrak{g}}} = 0$ by definition of the equivariant derivative since $(\mathcal{A}_P, \omega, \mathcal{G}_P, \mu)$ is Hamiltonian. Let $\alpha \in \Omega^\bullet_{\mathcal{G}_P}(\mathcal{A}_P, \mathbb{C})$ be defined by $\alpha = (-i)^{\Delta_{\mathcal{A}_P}/2}\exp[i\omega_{\tilde{\mathfrak{g}}}(Y)]$, where $\Delta_{\mathcal{A}_P}$ is formally the dimension of $\mathcal{A}_P$. Then $d_{\tilde{\mathfrak{g}}}\alpha = 0$ formally, since $d_{\tilde{\mathfrak{g}}}\omega_{\tilde{\mathfrak{g}}} = 0$. The Atiyah–Bott-Berline–Vergne theorem applies to the integral

$$\fint_{\mathcal{A}_P} \alpha(Y) = (-i)^{\Delta_{\mathcal{A}_P}/2} \fint_{\mathcal{A}_P} \exp[i\omega + i\mu^Y]. \tag{25}$$

Witten's idea is to relate (25) to the moment map squared form of $Z^P_{\mathrm{YM}}(\varepsilon)$.

Claim 6.2.25. *We have*

$$Z^P_{YM}(\varepsilon) = \frac{1}{\mathrm{Vol}(\mathcal{G}_P)} \fint_{\tilde{\mathfrak{g}}} \left[\frac{dY}{2\pi}\right] \exp[-\frac{\varepsilon}{2}(Y,Y)] \fint_{\mathcal{A}_P} \alpha(Y). \tag{26}$$

Proof. We "prove" this by integrating $Y \in \tilde{\mathfrak{g}}$ out using formal Gaussian integration. Rewrite (26) as

$$\frac{(-i)^{\Delta_{\mathcal{A}_P}/2}}{\mathrm{Vol}(\mathcal{G}_P)} \fint_{\tilde{\mathfrak{g}}\times\mathcal{A}_P} \left[\frac{dy}{2\pi}\right] \exp[i\omega + i(\mu, Y) - \frac{\varepsilon}{2}(Y,Y)].$$

Completing the square,

$$i(\mu, Y) - \frac{\varepsilon}{2}(Y,Y) = -\frac{\varepsilon}{2}[Y - \frac{i}{\varepsilon}\mu]^2 - \frac{1}{2\varepsilon}(\mu,\mu).$$

Changing variables $Y \mapsto Y - \frac{i}{\varepsilon}\mu$ and using the formal shift invariance of $[\frac{dY}{2\pi}]$, we have

$$\frac{(-i)^{\Delta_{\mathcal{A}_P}/2}}{\mathrm{Vol}(\mathcal{G}_P)} \fint_{\tilde{\mathfrak{g}}} \left[\frac{dY}{2\pi}\right] \exp[-\frac{\varepsilon}{2}(Y,Y)] \fint_{\mathcal{A}_P} \exp[i\omega - \frac{1}{2\varepsilon}(\mu,\mu)].$$

Recall, a Gaussian integral has the value $\int_{\mathbb{R}} \exp[-ax^2]dx = \sqrt{\pi/a}$. Let $\{Y_q\}_{q=1}^d$ be an orthonormal basis of $\tilde{\mathfrak{g}}$ and write $Y = \varphi_1 Y_1 + \cdots + \varphi_d Y_d$ for some $\varphi_q \in \mathbb{R}$. Then we can write

$$\begin{aligned}\int_{\tilde{\mathfrak{g}}} \left[\frac{dY}{2\pi}\right] \exp[-\frac{\varepsilon}{2}(Y,Y)] &= \prod_q \left(\int_{\mathbb{R}} \left[\frac{d\varphi_q}{2\pi}\right] \exp[-\frac{\varepsilon}{2}\varphi_q^2]\right) \\ &= \frac{1}{(2\pi)^{\Delta_{\mathcal{G}_P}}}\left[\frac{2\pi}{\varepsilon}\right]^{\Delta_{\mathcal{G}_P}/2} = \left(\frac{1}{2\pi\varepsilon}\right)^{\Delta_{\mathcal{G}_P}/2}.\end{aligned}$$

[2]This lecture was written up while slightly tipsy which should explain the increased number of typos

Since $(-i)^{\Delta_{\mathcal{A}_P}/2} \fint_{\mathcal{A}_P} \exp(i\omega) = \fint_{\mathcal{A}_P} \exp(\omega)$, we obtain

$$\frac{1}{\mathrm{Vol}(\mathcal{G}_P)} \left(\frac{1}{2\pi\varepsilon}\right)^{\Delta_{\mathcal{G}_P}/2} \fint_{\mathcal{A}_P} \exp[\omega - \frac{1}{2\varepsilon}(\mu,\mu)] =: Z^P_{\mathrm{YM}}(\varepsilon).$$

□

Thus, we have now expressed $Z^P_{\mathrm{YM}}(\varepsilon)$ into a form for which localization ideas may apply.

Definition 6.2.26. Given a manifold M with the action of a Lie group $G \circlearrowright M$, and a form $\alpha \in \mathrm{Map}(\mathfrak{g}, \Omega^\bullet(M))$, define the *equivariant integral* of α to be

$$\oint_M^\varepsilon \alpha := \int_{\mathfrak{g}} \exp[-\frac{\varepsilon}{2}(Y,Y]dm_{\mathfrak{g}} \cdot \int_M \alpha(Y),$$

with respect to some measure dm_g on $\mathfrak{g}$ and for a fixed $\varepsilon \in \mathbb{R}^{>0}$.

By definition, $Z^P_{\mathrm{YM}}(\varepsilon) := \frac{1}{\mathrm{Vol}(\mathcal{G}_P)} \oint^\varepsilon_{\mathcal{A}_P} \alpha(Y)$, where $\alpha(Y) := (-i)^{\Delta_{\mathcal{A}_P}/2} \exp[i\omega + i(\mu, Y)]$, and $\alpha \in \Omega^\bullet_{\mathcal{G}_P}(\mathcal{A}_P, \mathbb{C})$ is equivariantly closed. There exists a localization principle for equivariant integrals of the form $\oint^\varepsilon_M \alpha$, when $d_{\mathfrak{g}}\alpha = 0$, analogous to the Duistermaat–Heckamn theroem and the Atiyah–Bott-Berline–Vergne theorem. There is of course a bit of an issue in integrating over all of $\mathfrak{g}$. Thus in some sense, we are capturing all of the Lie group in our case, even if the group is non-abelian. This is why we speak about non-abelian localization.

Assume $G \circlearrowright M$ is a Lie group action on a manifold, not necessarily G to be compact. Given $\theta \in \Omega^\bullet_G(M)$, let

$$\mathcal{C}^\theta := \{p \in M \mid \iota_{Y^\sharp}\theta|_p = 0, \forall Y \in \mathfrak{g}\}.$$

Write $C^\theta = \bigcup_{\chi \in I} \mathcal{C}^\theta_\chi$, where $\mathcal{C}^\theta_\chi$ are connected components of C^θ induced by $\chi \in I$.

Claim 6.2.27 (General non–abelian localization)**.** *Given an equivariantly closed form $\alpha \in \mathrm{Map}(\mathfrak{g}, \Omega^\bullet(M))$ that has polynomial dependence on $Y \in \mathfrak{g}$ (or even exponential dependence) and $\theta \in \Omega^\bullet_G(M)$, then the equivariant integral of α can be written as a sum over contributions from the sets $\mathcal{C}^\theta_\chi$,*

$$\oint_M^\varepsilon \alpha = \sum_{\chi \in I} Z^\theta_\chi(\varepsilon),$$

where $Z^\theta_\chi(\varepsilon) \in \mathbb{C}$ depend only on $\mathcal{C}^\theta_\chi$ and α, for each $\varepsilon \in \mathbb{R}^{>0}$.

This is different from the Berline–Vergne in that the localization depends on θ.

Sketch of proof. We follow [Wit92]. For all $t \in \mathbb{R}$, we may write

$$\oint_M^\varepsilon \alpha = \oint_M^\varepsilon (\alpha \wedge e^{itd_{\mathfrak{g}}\theta}).$$

By definition,

$$\oint_M^\varepsilon \alpha \wedge e^{itd_{\mathfrak{g}}\theta} = \int_{\mathfrak{g}} \exp[-\frac{\varepsilon}{2}(Y,Y)]dm_{\mathfrak{g}} \int_M \alpha(Y) \wedge e^{itd_{\mathfrak{g}}\theta(Y)}.$$

We consider $\int_M \alpha(Y) \wedge e^{itd_{\mathfrak{g}}\theta}$ and observe that

$$e^{itd_{\mathfrak{g}}\theta(Y)} = 1 + itd_{\mathfrak{g}}\beta(Y)$$

for some $\beta \in \Omega^\bullet_G(M)$, using Taylor expansion. Since $\alpha(Y) \wedge d_{\mathfrak{g}}\beta(Y) = d_{\mathfrak{g}}(\alpha \wedge \beta)(Y)$ is equivariantly exact (since $d_{\mathfrak{g}}\alpha = 0$), we get

$$\int_M \alpha(Y) \wedge e^{itd_{\mathfrak{g}}\theta(Y)} = \int_M \alpha(Y) + it \int_M d_{\mathfrak{g}}(\alpha \wedge \beta)(Y) = \int_M \alpha(Y)$$

by Stokes' theorem. At this point, we make the simplifying assumption that $\theta \in \Omega^1(M)^G$, so that, as a map $\theta : \mathfrak{g} \to \Omega^1(M)$, θ is independent of $Y \in \mathfrak{g}$. Let $\{Y_q\}_{q=1}^d$ be an orthonormal basis for $\mathfrak{g}$ and $\varphi_1, \ldots, \varphi_d$ coordinates so that $Y = \varphi_1 Y_1 + \cdots + \varphi_d Y_d$ in general. Then

$$\circ_M^\varepsilon \alpha = \int_{\mathfrak{g}\times M} dm_{\mathfrak{g}} \cdot \alpha(Y) \cdot \exp[itd\theta - it\sum_q \varphi_q \theta(Y_q^\sharp) - \frac{\varepsilon}{2}\sum_q \varphi_q^2]$$

which follows from $d_{\mathfrak{g}}\theta(Y) \equiv d\theta - \iota_{Y^\sharp}\theta(Y) = d\theta - \sum_q \xi_q(Y)\theta(Y_q^\sharp)$, when $\theta \in \Omega^1(M)^G$. Consider completing the square,

$$\frac{\varepsilon}{2}\varphi_q^2 + it\theta(Y_q^\sharp)\varphi_q = \frac{\varepsilon}{2}(\varphi + \frac{it\theta(Y_q^\sharp)}{\varepsilon})^2 + \frac{t^2\theta(Y_q^\sharp)^2}{2\varepsilon}.$$

Changing variables $\varphi_q \mapsto \varphi_q + \frac{it\theta(Y_q^\sharp)}{\varepsilon}$, and using shift invariance, we have

$$\oint_M^\varepsilon \alpha = \int_{\mathfrak{g}\times M} dm_{\mathfrak{g}} \cdot \alpha(Y) \exp\left[itd\theta - \frac{\varepsilon}{2}\sum_q \varphi_q^2 - \frac{t^2}{2\varepsilon}\sum_q \theta(Y_q^\sharp)^2\right].$$

We take $dm_{\mathfrak{g}}$ to be a Haar measure,

$$dm_{\mathfrak{g}} = \left[\frac{dY}{2\pi}\right] \frac{d\varphi_1\, d\varphi_2\, \cdots\, d\varphi_d}{(2\pi)^d}.$$

We can perform the Gaussian integral,

$$\int_{\mathbb{R}} \frac{d\varphi_q}{2\pi}\varphi_q^{n_q} \exp[-\frac{\varepsilon}{2}\varphi_q^2] = \begin{cases} \sqrt{\frac{2\pi}{\varepsilon^{n_q+1}}}(1\cdot 3\cdot 5\cdot \cdots\cdot (n_q+1)), & n_q \in 2\mathbb{N}_0, \\ 0, & n_q \text{ odd.} \end{cases}$$

For simplicity, we assume that α is independent of Y. Then

$$\oint_M^\varepsilon \alpha = \left(\frac{1}{2\pi\varepsilon}\right)^{d/2} \int_M \alpha \exp[itd\theta - \frac{t^2}{2\varepsilon}\sum_q \theta(Y_q^\sharp)^2].$$

Clearly,

$$\mathcal{C}^\theta = \{p \in M \mid \iota_{Y_q^\sharp}\theta(p) = \theta(Y_q^\sharp)(p) = 0,\, 1 \le q \le d\}.$$

If $W \subseteq M$ is a compact subset of M with $\dim W = \dim M$, such that $W \cap \mathcal{C}^\theta = \emptyset$, then $\sum_q \theta(Y_q^\sharp)^2 > 0$ on W. This implies that

$$\lim_{t\to\infty}\left(\frac{1}{2\pi\varepsilon}\right)^{d/2} \int_W \alpha \exp[itd\theta - \frac{t^2}{2\varepsilon}\sum_q \theta(Y_q^\sharp)^2] = 0.$$

since the term of the form $\exp(-at^2)$ will dominate the convergence to 0 as $t \to \infty$ when integrated against $\exp(itd\theta)$. Let U_χ^θ be a G-equivariant tubular neighbourhood of $\mathcal{C}_\chi^\theta$ and define

$$Z_\chi^\theta(\varepsilon) := \lim_{t\to\infty}\left(\frac{1}{2\pi\varepsilon}\right)^{d/2} \int_{U_\chi^\theta} \alpha \exp[itd\theta - \frac{t^2}{2\varepsilon}\sum_q \theta(Y_q^\sharp)^2].$$

Overall then, we have shown that

$$\oint_M^\varepsilon \alpha = \sum_{\chi\in I} Z_\chi^\theta(\varepsilon).$$

□

Claim 6.2.28. *In the case that (M, ω, G, μ) is Hamiltonian, and*

$$\alpha(Y) := (-i)^{\Delta_M/2} \exp[i(\omega + \mu^Y)]$$

then by judicious choice of $\theta \in \Omega^1(M)^G$, we can show that $\oint_M^\varepsilon \alpha$ localizes to the critical points of the function $f_\mu := \frac{1}{2}(\mu, \mu) : M \to \mathbb{R}$.

Proof. Recall that $\mathcal{C}^\theta = \{p \in M \mid \iota_{Y_q^\sharp}\theta(p) = 0,\ 1 \le q \le d\}$, where $\{Y_q\}_{q=1}^d$ is an orthonormal basis for $\mathfrak{g}$. We show for a good choice of θ that $\mathcal{C}^\theta = \operatorname{Crit} f_\mu$. First, let $J : TM \to TM$ be an almost complex strucutre on M that is positive, i.e. so that the metric $g \in \operatorname{Sym}^2 T^*M$, defined by

$$g(u, v) := \omega(u, Jv)$$

is positive definite (such a J always exists and is in fact unique up to homotopy). Let $\tilde{J} : T^*M \to T^*M$ be the induced map $\tilde{J}(\tau)(v) := \tau(Jv)$ for $\tau \in T_p^*M$, $v \in T_pM$. We can identify $\tilde{J}$ as the transpose of J and locally write

$$\tilde{J}(a_i dx^i) = \sum_{i,k} J_{i,k} a_i dx^k.$$

Also, $\tilde{g} : TM \to T^*M$ be the isomorphism defined by $\tilde{g}(v) := g(v, \cdot)$ and let $\tilde{\omega} : TM \to T^*M$ be defined by $\tilde{\omega}(v) := \omega(v, \cdot)$. Define $\theta := \tilde{J} df_\mu$. We want to show that $\mathcal{C}^\theta := \operatorname{Crit} f_\mu$. Clearly, $\operatorname{Crit} f_\mu \subseteq \mathcal{C}^\theta$ since if $p \in \operatorname{Crit} f_\mu$, then $df_\mu|_p = 0$, so $\mu|_p = 0$ and $\iota_{Y_q^\sharp}\theta(p) = 0$. To see that $\mathcal{C}^\theta \subseteq \operatorname{Crit} f_\mu$, let $\mu_q := \mu^{Y_q} = (\mu, Y_q) \in C^\infty(M, \mathbb{R})$ and define $V := \sum_q \mu_q Y_q^\sharp \in \Gamma(TM)$. We may also express $\mu = \sum_q \mu_q \xi_q$, where ξ_q is the dual basis to Y_q, and also, $(\mu, \mu) = \sum_q \mu_q^2$.

The moment map condition is $d\mu_q = \iota_{Y_q^\sharp}\omega = \tilde{\omega}(Y_q^\sharp)$, or $Y_q^\sharp = \tilde{\omega}^{-1} d\mu_q$, where $\tilde{\omega}^{-1} : T^*M \to TM$. So

$$V = \sum_q \mu_q Y_q^\sharp = \sum_q \mu_q \tilde{\omega}^{-1} d\mu_q = \tilde{\omega}^{-1}[\sum_q \mu_q d\mu_q] = \tilde{\omega}^{-1}[\frac{1}{2} d(\mu, \mu)] = \tilde{\omega}^{-1}(df_\mu).$$

Now, by definition of θ and $\tilde{\omega}$,

$$\theta(V) = (\tilde{J} df_\mu)(\tilde{\omega}^{-1} df_\mu) = \omega^{-1}(df_\mu, \tilde{J} df_\mu). \tag{27}$$

Since $V = \sum_q \mu_q Y_q^\sharp$ and assume that $\theta(Y_q^\sharp)(p) = 0$. Then $\theta(V)(p) = 0$. And so $\omega^{-1}(df_\mu, \tilde{J} df_\mu)(p) = 0$, so $df_\mu(p) = 0$ by positivity of the associated metric g. Hence $p \in \mathcal{C}^\theta$ and $p \in \operatorname{Crit} f_\mu$. □

Witten uses the non-abelian localization technique applied to $Z_{\mathrm{YM}}^P(\varepsilon)$ in moment map squared form to obtain formulas for intersection pairings on $\mathcal{M}(n, d)$, the moduli space of semi-stable, holomorphic, rank n, degree d bundles over Σ.

As an example, one can compute the intersection pairing of the Atiyah–Bott-Goldman symplectic form on $\mathcal{M}(n, d)$, i.e. the symplectic volume of $\mathcal{M}(n, d)$, for example, for a surface of odd genus g,

$$\operatorname{Vol}(\mathcal{M}(n, d)) = 2\frac{(-1)^{g-1}}{(2\pi^2)^{g-1}} \left(1 - \frac{1}{2^{2g-3}}\right) \zeta(2g - 2),$$

where $\zeta(s) := \sum_{n \ge 1} 1/n^2$ is the Riemann zeta function.

A Classifying spaces

This appendix was written by Jens Kristian Egsgaard as part of his student seminar.

Let **HT** the category of CW-complexes as objects and homotopy classes of maps as morphisms. I will use word contractible for a space X such that $\pi_i(X) = 0$ for all i. This is good enough for constructing classifying spaces of bundles over CW-complexes, but the stronger meaning of contratible is nessecary for obtaining the results in the category of paracompact spaces.

A.1 Moduli spaces

Recall the following definition:

Definition A.1.1 (Fine moduli space)**.** Assume we have something that can vary in families over objects in a category $\mathcal{C}$, and assume that these families can be pulled back along morphims in $\mathcal{C}$. We then define a contravariant functor $F : \mathcal{C} \to \mathbf{Sets}$ that assigns to an object $A \in \mathcal{C}$ the set of all equivalence classes of families over A, and to a morphism $f : A \to B$ the function $f^* : F(B) \to F(A)$ induced from the pullback. A *fine moduli space* of families is an object $\mathcal{M} \in \mathcal{C}$ and a family U over $\mathcal{M}$ (called the universal family over $\mathcal{M}$) such that $u : \mathrm{Hom}(\cdot, \mathcal{M}) \to F$ is an natural equivalence, where u is the image of $U \in F(\mathcal{M})$ under the isomorphism from the Yoneda lemma (this is just a fancy way to say that the bijection is given by pulling back U).

We will see that classifying spaces often turn out to be fine moduli spaces for G-bundles (thought of as families of G-torsors).

A.2 General properties

First, we recall some basic facts about principal G-bundles:

Definition A.2.1. A principal G-bundle is a fiber bundle $P \to X$ with a free and transitive action of a topological group G on the fiber

A morphism of principal G-bundles $P \xrightarrow{p_1} X$ and $Q \xrightarrow{p_2} X$ is a G-equivariant map $f : P \to Q$ commuting with the maps to X:

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ X & = & X \end{array}$$

and any morphism is an isomorphism.

The pullback of a principal G-bundle $Q \to Y$ by a map $f : X \to Y$ is the bundle

$$f^*(Q) = \{(x, q) \in X \times Q \mid f(x) = p(q)\},$$

with the map to X given by projection on the first coordinate. Notice that this is just the pullback of of the diagram

$$\begin{array}{ccc} & & Q \\ & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

in the category of G-spaces, with the trivial G action on X and Y.

We need the following theorem from homotopy theory:

Lemma A.2.2. *If X is paracompact and $P \to X \times I$ is a principal bundle, then $\pi_0^*(P) \cong \pi_1^*(P)$.*

A proof of this fact might be found in [Hat10]. This lemma tells us that the isomorphism classes of principal bundles is the same thing as concordance classes. This implies that any bundlei $P \to X$ over a contractible paracompact space is trivial; just look at the pullback of P along a homotopy $X \times I \to X$ from the identity to a constant map.

A G-space is a topological space with a continuous right action by G. By EG we denote a contractible space with a free right action such that $BG = EG/G$ is a principal G-bundle. We will later see that EG and BG exist, and is unique up to homotopy, and we can choose BG to be a CW complex.

Theorem A.2.3. *For any $X \in \mathbf{HT}$ the map $f \mapsto f^*(EG)$ between morphisms $X \to BG$ and isomorphism classes of G-bundles over X is an isomorphim.*

Proof. Let $P \to X$ and $E \to Y$ be G-bundles. Assume $\tilde{f} : P \to E$ is a G-map, and let $f : X \to Y$ be the induced map of base spaces. Then $f^*E \cong P$. This follows either by the universal property of pullback or by observing that since $f^*E \cong \{(x, v) \in X \times E | f(x) = \pi(v)\}$, we have the isomorphism $P \to f^*E$ given by $p \mapsto (\pi(p), \tilde{f}(p))$.

Next, we observe that a morphism between principal bundles is the same thing as a section of the fiber bundle $P \times_G E \to X$, where $P \times_G E = P \times E/(pg, v) \sim (p, vg)$ where the projection to X is just projection of the first coordinate which is well defined since P is a G-bundle. Let $s : X \to P \times_G E$ be a section. To construct a G-equivariant map, its enough to know what to do to a point in each fiber of P. For the fiber over $x \in X$, pick a element (p, q) in $s(x)$ and let $p \mapsto q$. This gives the desired G-equivariant map.

Finally, we observe that the bundle $P \times_G EG \to X$ has fiber EG – a contractible space! So there is no obstructions to a section. We can simply construct one inductivly over the skeleta, first lifting the 0-skeleton, then trivialize the bundle over the 1-cells (this can be done since 1-cells are contratible). We now need to find a function from the 1-cells to E that agrees with the values on the boundary; but this is possible since E is path connected. So assign such paths to the 1-cells, and trivialize over the 2-cells. But we can extend the map over the 2-cells, since $\pi_2(EG) = 0$, etc.

Next, we have to show injectivity. So assume that $f_0, f_1 : X \to BG$ and $\psi : f_0^*(EG) \cong f_1^*(EG)$. We have to show that $f_0 \cong f_1$. Recall that the pullback-bundle comes with an equivariant map $\tilde{f}_i : f_i^*(EG) \to EG$ covering f_i. First we show that $\tilde{f}_0 \cong \tilde{f}_1 \circ \psi$. But, since EG is contratible, we can extend the section of $(f_0^*(EG) \times I) \times_G EG \to X \times I$ given by $\tilde{f}_0$ at $X \times 0$ and by $\tilde{f}_1 \circ \psi$ at $X \times 1$, to all of $X \times I$. Going from this section to a G-equivariant map to E and then to maps from $X \times I \to BG$ we get the desired homotopy ($\tilde{f}_1$ and $\tilde{f}_1 \circ \psi$ induces the same maps from X to BG). □

Corollary A.2.4. *There exist a CW-complex BG that classifies principal G-bundles over spaces in* **HT***; Such a BG is a fine moduli space for G-bundles in* **HT**.

Proof. By the above theorem, there exist a space $\overline{BG}$ that classifies these bundles. By CW-approximation (prop. 4.13 in [Hat01]) there exist a CW-complex BG and a weak homotopy equivalence $g : BG \to \overline{BG}$. We can pull the universal bundle over $\overline{BG}$ back to BG by g, and it is now possible to show that the pullback is weakly contractible by applying the long exact sequence in homotopy theory and the 5-lemma on the maps between the homotopy groups induced by g. □

Corollary A.2.5. *If BG and $\overline{BG}$ are two CW-complexes with universal bundles, then $BG \cong \overline{BG}$ by a unique isomorphism.*

Proof. Both BG and $\overline{BG}$ represent the functor F that maps a object in **HT** to the set of equivalence classes of G-bundles over the object. But the Yonda lemma tells us that there is a unique isomorphism between any two representing objects. □

Theorem A.2.6. *There exist a CW complex X and a natural isomorphism $F \cong Hom(\cdot, X)$, where F is the functor that assigns to a CW complex the set of equivalence classes of principal G-bundles over X.*

Proof. This follows from Browns representability theorem, since F plays nicely with wedge sum and finite coverings of CW-complexes. □

This proof dosn't show the existence of the universal bundle, and the proof is not very helpful. However, there exist concrete constructions of BG and EG for any group.

A.3 Construction for a topological group

In this section we will sketch the construction of a contractible G space for any toplogical group G; this was first done by Milnor in [Mil56], a 7-page paper that dosn't even cite any references. Another construction was given by Segal in [Seg68], which we will now sketch. Consider the following construction: given a small topological category $\mathcal{C}$, define the nerve $\mathcal{N}(\mathcal{C})$ to be the simplical set with 0-simplices the set of objects, 1-simplices the set of arrows, 2-simplices the set of commutative triangles, 3-simplices the set of commutative diagrams shaped like a 3-simplex, etc. This is naturally a simplicial set, and we can define the geometric realization by gluing together simplices according to the structure maps:

$$\operatorname{norm}\mathcal{N}(\mathcal{C}) = \left(\coprod_i \mathcal{C}_i \times \Delta_i\right) / \sim .$$

A topological group G can be considered as a topological category with one point, and a morphism for each element in G. For nice topological groups (ie. groups with a contratible neighbourhood of the identity), $\operatorname{norm}\mathcal{N}(\mathcal{G})$ is a model for BG; for non-nice G, we need to make a small modification to the category before forming the nerve. This generalizes to the construction of a $B\mathcal{C}$ for any category $\mathcal{C}$.

A.4 Construction for compact Lie groups

We will give a construction in terms of grassmanians, which will give classifying spaces that are alot slimmer than the one constructed by Milnor. Recall that any compact lie group embeds into $\mathrm{U}(n)$ for some n. This is a corrollary to the Peter-Weyl theorem, see p. 133 in [BtD95]. This means that we only have to construct $EU(n)$ for all n:

Lemma A.4.1. *Let $P \to X$ be a principal G-bundle and let $H \subseteq G$ a subgroup such that $G \to G/H$ is a principal H-bundle. Then $P \to P/H$ is a principal H-bundle.*

Proof. There are isomorphisms: $P \cong P \times_G G$, $P/G \cong P \times_G *$ and $P/H \cong P \times_G (G/H)$. Under these isomorphisms, the quotient map $P \to P/H$ is the natural map $P \times_G G \to P \times_G (G/H)$. But the map $G \to G/H$ har local sections. □

A.4.1 $B\,\mathrm{U}(1)$

First we will give the construction of $E\,\mathrm{U}(1)$, as the limit of odd-dimensional spheres. $\mathrm{U}(1)$ acts on $S^{2n+1} \subseteq \mathbb{C}^{2n+2}$. Let S^∞ be the CW-complex with two cells in each dimension, constructed inductively in the following way: the n-skeleton is a n-sphere, and the $n+1$ skeleton consist of two $n+1$-cells glued to the n-sphere by the identity map – but this gives us a $n+1$ sphere. The free $\mathrm{U}(1)$-actions on the odd dimensional spheres are clearly compatible, so $\mathrm{U}(1)$ acts freely on S^∞. Let $f : S^n \to S^\infty$ be a map. By cellular approximation, we can replace f by a homotopic map into the n-skeleton. But the n-skeleton deformation retracts to a point inside the $n+1$ skeleton, so f must be nullhomotopic. This proves that $S^\infty = E\,\mathrm{U}(1)$. Furthermore, $B\,\mathrm{U}(1) = \mathbb{C}P^\infty$ – its easy to see this since the quotions of the $2n+1$ skeleton by the $\mathrm{U}(1)$ action gives $\mathbb{C}P^n$. We get a principal $\mathrm{U}(1)$ bundle since we can take the direct limit of local sections (which gives us a local trivialization).

A.4.2 Direct limits of inclusions

First, recall that the direct limit of a sequence of inclusions of topological spaces $X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots$ is is $X_\infty = \bigcup X_i$ (where X_i is idenitfied with its image in X_{i+1}) with the cofinal topology (the finest toplogy making the inclusions $X_i \hookrightarrow X_\infty$ continious, ie a set is open if and only if its intersection with every X_i is open). Assume now that each X_n has the property that points are closed. We want to show that a compact subset K is contained in some X_j. So assume that there exist a sequence $x_n \in K$ such that $x_n \in X_{i_n} \setminus X_{i_n - 1}$, where i_n is a strictly increasing sequence of

natural numbers. But every X_k only contains finitly many x_n's so they form a closed subset. So the set $\cup\{x_n\}$ is closed in X_∞. But the same argument works for any subset of this set, so it has the discrete topology. But its a closed subset of the compact set K, so it must be compact. But this is a contradiction: a infinte discrete set is not compact.

A.4.3 $B\,\mathrm{U}(n)$

Observe that $B\,\mathrm{U}(1)$ is the limit of projective spaces – spaces of lines in $\mathbb{C}^k$. We will construct $B\,\mathrm{U}(n)$ as the limit of spaces of n-dimensional subspaces of $\mathbb{C}^k$. Let $E_k\,\mathrm{U}(n)$ be the set of n-frames in $\mathbb{C}^k$. $\mathrm{U}(n)$ clearly acts freely on this space, and we have compatible maps $E_k\,\mathrm{U}(n) \hookrightarrow E_{k+1}\,\mathrm{U}(n)$ for all $k \geq n$, and we use these to define the direct limit $\varinjlim_k E_k\,\mathrm{U}(n)$

We have the fibration:

$$E_k\,\mathrm{U}(n) \to E_{k+1}\,\mathrm{U}(n+1) \to S^{2k+1},$$

where the map to the sphere is just forgetting the first n basis vectors, ie $(e_0, \ldots, e_n) \mapsto e_n$. We can use the long exact sequence of homotopy groups to get:

$$0 = \pi_{i+1}(S^{2k+1}) \to \pi_i(E_{k+1}\,\mathrm{U}(n+1)) \cong \pi_i(E_k\,\mathrm{U}(n)) \to \pi_i(S^{2k+1}) = 0$$

for $i < 2k$. But $E_{k+1}\,\mathrm{U}(1)$ is homeomorphic to S^k, so for $i < 2k - 2n$ we have that

$$\pi_i(E_{k+n-1}\,\mathrm{U}(n)) \cong \pi_i(E_k\,\mathrm{U}(1)) \cong \pi_i(S^{2k-1}) = 0$$

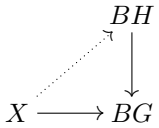
Since a map $f : S^j \to E\,\mathrm{U}(n)$ has compact image, the image is contained in some $E_k\,\mathrm{U}(n)$, with $j < 2k - n$. This implies that f is nullhomotopic, and thus $\pi_i(E\,\mathrm{U}(n)) = 0$ for all i.

A.5 Reduction of structure group

If $H \subseteq G$ is a subgroup, we say that the structure group of a principal G-bundle $P \to X$ can be reduced to H if there exist a prinicpal H-bundle Q such that $P \cong Q \times_H G$.

Assume $H \subseteq G$ is a subgroup such that $G \to G/H$ is a principal H-bundle. We get a map $BH \to BG$ by dividing out with the remaining G/H action. Under this hypothesis, we have the following theorem:

Theorem A.5.1. *The the structure group of the principal G bundle $P \to X$ can be reduced to H if and only if the classifying map to BG can be lifted to BH. In pictures: if and only if the dotted arrow exist in* **HT***:*



One important example of this is when $G = \mathrm{GL}(n)$ and $H = \mathrm{O}(n)$. The quotient $\mathrm{GL}(n)/\mathrm{O}(n)$ is contratible, so we can always find a lift (this is similar to the proof of that BG classifies G-bundles). This says that, for a vector bundle, we can always choose the transistion functions in $\mathrm{O}(n)$ – which is just another way to say that the vector bundle admits a metric. Manifolds of dimension ≥ 5 give another interresting example of this. By definition, a topological manifold comes equipped with a $\mathrm{Homeo}(\mathbb{R}^n)$ bundle (the "locally euclidian" part of the definition). These a classified by maps into $B\mathrm{Homeo}(\mathbb{R}^n)$. But we have an important subgroup $\mathrm{Diff}(\mathbb{R}^n) \subseteq \mathrm{Homeo}(\mathbb{R}^n)$, and we can now reduce the question "does the topologocial manifold X have a smooth structure (and how many?)" by studying the the possible lift of classifying map.

A.6 Characteristic classes

Let $\alpha \in H^i(BG, \mathbb{Z})$. For any G-bundle $P \to X$, we can associate an element of $H^i(X, \mathbb{Z})$ in the following way: Find a map $f : X \to BG$ such that $f^*(EG) \cong P$, and use the cohomology-pullback

to find an element $f^*(P) \in H^i(X, \mathbb{Z})$. This becomes a natural transformation between the functor from the category of principal bundles to **Sets** that assigns the equivalence class to a bundle, to cohomology functor composed with the forgetfull functor from **Ab** $\to$ **Sets** – such a natural transformation is known as a characteristic class. In fact this construction gives all characteristic classes, for, by naturality, if c is a characterstic class:

$$c(P) = c(f^*(EG)) = f^*(c(EG)).$$

If we are really lucky, BG is a $K(G, n)$ space for some abelian group G. Recall that the Eilenberg-Maclane spaces represent cohomology, so that $H^n(X, G) = [X, K(G, n)]$. But this gives an identification of the set of equivalence classes of G-bundles over X and the n'th cohomology group of X with coeffecients in G. For the universal bundles $\mathrm{U}(1) \to S^\infty \to \mathbb{C}P^\infty$ and $\mathbb{Z}/2\mathbb{Z} \to S^\infty \to \mathbb{R}P^\infty$ we are in this fortunate situation, as the long exact sequence of homotopy groups show. This gives the usual identification of complex, respectively real, line bundles over X with $H^2(X, \mathbb{Z})$ and $H^1(X, \mathbb{Z}/2\mathbb{Z})$ – and shows that the first Chern class, respectively the first Steifel-Whitney, are a complete invariant of such bundles.

However, its easy to calculate the homotopy groups of BG in term of the homotopy groups of G. Applying the long exact sequence of homotopy group to the fibration $G \to EG \to BG$, we find that $\pi_{i+1}(BG) = \pi_i(G)$, $i \geq 0$. If G is connected, we only get a $K(H, n)$ if G is a $K(H, n-1)$. If G has the discrete topology, we observe that $\pi_i(G) = 0$ for all $i > 0$, and $\pi_0(G) \cong G$. So BG is a $K(G, 1)$.

B The Abelian Localization Theorem

This section was written by Jens-Jakob Kratmann Nissen as part of his student seminar.

Theorem B.0.1 (Berline-Vergne; Atiyah-Bott)**.** *Let T be a torus acting on a manifold M, and let $\mathcal{F}$ index the components F of the fixed point set M^T of the action of T. Let $\eta \in H_T^*(M)$. Then*

$$\int_M \eta(X) = \sum_{F \in \mathcal{F}} \int_F \frac{\eta(X)}{e_F(X)}$$

We want to give (at least a sketch of) a proof for this theorem.

B.1 Properties of the pushforward

Recall the following definitions:

Definition B.1.1. If $i : A \subset X$ is a pair of spaces then A is called a retract of X if there is a map $r : X \to A$ such that $r \circ i = \mathrm{id}$.

Definition B.1.2. If (X, A) is a pair as above then A is called a neighborhood retract in X if A has a neighborhood in X of which it is a retract.

Definition B.1.3. If $X \subset \mathbb{R}^m$ is a neighborhood retract and $Y \subset \mathbb{R}^n$ is homeomorphic with X then Y is a neighborhood retract. Therefore we define the following: A topological space Y is called an Euclidian neighborhood retract (ENR) if a neighborhood retract $X \in \mathbb{R}^n$ exists which is homeomorphic with Y.

[Dol72, IV 8.5]

Proposition B.1.4. *If $Y \subset X$ is a pair of ENR's then $\check{H}(X, Y) \cong H^*(X, Y)$*

[Dol72, VIII 6.12]

Proposition B.1.5. *If M^n is a compact oriented manifold we have*

$$\check{H}^i(M;G) \cong H_{n-i}(M;G).$$

The coefficients G are arbitrary when M is oriented.

[Dol72, VIII 8.1]

If $f : M' \to M$ is a map between oriented manifolds then we can transform the induced (co-)homology homomorphisms f_* and $\check{f}$ respectively by PoincarÃ© Duality. The resulting maps $f^! = D^{-1} f_* D'$ and $f_! = D' \check{f} D^{-1}$ are called Umkehr-homomorphisms.

We begin with a naturality property of $\frown$-products. Let $f : M' \to M$ be a map of manifolds, let M', M be compact (ENR). We have homomorphisms

$$\check{f} : \check{H}(M) \to \check{H}(M'),$$

$$f_* : H(M') \to H(M)$$

and we have from naturality of ordinary $\frown$-products by passing to limits that

$$f_* \left((\check{f}x) \frown \eta \right) = x \frown (f_*\eta), \quad \text{for} \quad x \in \check{H}(M), \eta \in H(M').$$

Proposition B.1.6. *If $f : M' \to M$ is a map of oriented manifolds of dimension m' resp. m, such that r-times the fundamental cycle $\alpha \in H_m(M)$ is the f_*-image of some $\eta \in H_{m'}(M')$ then there is a sequence of homomorphisms*

$$\check{H}^i(M) \xrightarrow{\check{f}} \check{H}^i(M') \to \check{H}_c^{i+(m'-m)}(M') \xrightarrow{f^!} \check{H}^i(M)$$

whose composite equals r-times the identity map.

Proof. The composition

$$\check{H}^i(M) \xrightarrow{\check{f}} \check{H}^i(M') \xrightarrow{\frown \eta} H_{m-i}(M') \xrightarrow{f_x} H_{m-i}(M)$$

sends $x \in \check{H}^i(M)$ to $f_*((\check{f}x) \frown \eta) = x \frown (f_*\eta) = x \frown (r\alpha) = r(x \frown \alpha)$. Composing further with $(\frown \alpha)^{-1} : H_{m-i}(M) \cong \check{H}^i(M)$ takes x into rx. If we now replace $H_{m-i}(M')$ by the isomorphic group $\check{H}_c^{i+(m'-m)}(M')$ we get the required sequence. □

Definition B.1.7. The homomorphism

$$f^! : \check{H}_c^i(M') \to \check{H}_c^{i+m'-m}(M)$$

which appears above depends only on f, not on η. It is obtained by composing

$$\check{H}_c^i(M') \xrightarrow{\frown \alpha'} \check{H}_{m'-i}(M') \xrightarrow{f_*} \check{H}_{m'-i}(M) \xrightarrow{(\frown \alpha)^{-1}} \check{H}_c^{m-m'+i}(M)$$

Proposition B.1.8. *Let $M'' \xrightarrow{f'} M' \xrightarrow{f} M$ be maps of oriented manifolds. We then have $(ff')_* = f_* f'_*$*

Proof. This follows immediately from the previous because f_* compose functorially. □

Proposition B.1.9. *For $x \in \check{H}(M)$ and $y \in \check{H}(M')$ we have following multiplicative rule for the transfer*

$$f^!(\check{f}x \smile y) = x \smile f^! y$$

which tells us that it is a homomorphism of $\check{H}^*(M)$-modules.

Proof. We look at the diagram, where the vertical arrows are the PoincarÃ© Duality map.

$$\begin{array}{ccc} H^i(M') & \xrightarrow{f^!} & H^{m-(m'-i)}(M) \\ \downarrow{\scriptstyle \frown\alpha'} & & \downarrow{\scriptstyle \frown\alpha} \\ \check{H}_{m'-i}(M') & \xrightarrow{f_*} & \check{H}_{m'-i}(M) \end{array}$$

We want to show that $f^!(\check{f}x \smile y) = x \smile f^!y$ we do that by showing that $f_*\left((\check{f}x \smile y) \frown \alpha\right) = (x \smile f^!y) \frown \alpha$ in $\check{H}_{i-m'}(M)$. This shows the equality since $H^{i-m'+m}(M) \cong \check{H}_{i-m'}(M)$. We first look at the left hand side:

$$f_*\left((\check{f}x \smile y) \frown \alpha'\right) = f_*\left(\check{f}x \frown (y \frown \alpha')\right) \tag{28}$$

$$= x \frown f_*(y \frown \alpha') \tag{29}$$

(1) is by "associativity" of cap product

And now the right hand side:

$$\left(x \smile f^!y\right) \frown \alpha = x \frown \left(f^!y \frown \alpha\right) \tag{30}$$

$$= x \frown \left((f_*(y \frown \alpha'))\,(\frown \alpha)^{-1} \frown \alpha\right) \tag{31}$$

$$= x \frown f_*(y \frown \alpha'). \tag{32}$$

□

If f is a fibering then $f^!$ corresponds to integration over the fiber.

B.2 Thom isomorphism

Let M^{n+k}, N^n be compact oriented manifolds with inclusion map $e : N \to M$, and we assume, that N is closed in M. The transfer $e_!$ is the composite

$$H_{i+k}(M, M-N) \xrightarrow{(\frown\alpha_M)^{-1}} \check{H}_c^{n-i}(M) \xrightarrow{\frown\alpha_N} H_i(N). \tag{33}$$

Proposition B.2.1 (VIII-7.14, [Dol72])**.** *If $L \subset K \subset X$ are topological spaces such that L is closed in K, $K - L$ is closed in $X - L$ and $X - L$ is an n-manifold which is oriented along $K - L$ then*

$$\check{H}_c^i(K, L) \cong \check{H}_c^i(K - L) \cong H(X - L, X - K).$$

As an application of this proposition, let N^n be a submanifold of M^{n+k}. Let $K = N, L = \emptyset$. Applying first B.2.1 in N and then in M gives the isomorphism

$$H_i(N) \cong \check{H}_c^{n-i}(N) \cong H_{i+k}(M, M - N).$$

From this we see, that the transfer $e_!$ is an isomorphism.

For small dimensions (33) implies that

$$H_q(M, M - N) = 0 \quad \text{for} \quad q < k = \dim M - \dim N \tag{34}$$

$$H_k(M, M - N; \mathbb{Z}) \cong H_0(N; \mathbb{Z}) \tag{35}$$

which is generated by elements ν_λ which corresponds to the components N_λ of N. We call ν_λ the transverse class of N_λ. In the composition

$$H_k(M, M - N) \cong \oplus_\lambda H_k(M, M - N_\lambda)$$

it is a generator of $H_k(M, M - N_\lambda; \mathbb{Z}) \cong \mathbb{Z}$.

Proposition B.2.2. *The isomorphism $e_!$ commutes with inclusion. In more detail, if X is closed in N then we have*

$$\begin{array}{ccc} H_q(M, M-N) & \xrightarrow{e_!} & H_{q-k}(N) \\ \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} \\ H_q(M, M-X) & \xrightarrow{e_!} & H_{q-k}(N, N-X) \end{array}$$

If V is an open set in M then

$$\begin{array}{ccc} H_q(V, V-N) & \xrightarrow{e_!|V\cap N} & H_{q-k}(V \cap N) \\ \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} \\ H_q(M, M-X) & \xrightarrow{e_!} & H_{q-k}(N, N-X) \end{array}$$

Proof. For a proof, see [Dol72, VIII 10.9] □

In particular, if $q = k$ $X = N$ we see that $j_* : H_k(V, V-N) \to H_k(M, M-N)$ takes transverse classes into transverse classes.

$$j_*(\nu_{\tilde{\lambda}}) = \nu_\lambda \tag{36}$$

where $\tilde{\lambda}, \lambda$ are components of $V \cap N$ respectively N.

Apply now the Universal coefficient formula, we find

$$H^q(M, M-N) = 0 \quad \text{for} \quad q < k = \dim M - \dim N;$$
$$H^k(M, M-N; G) \cong \mathrm{Hom} H_0(N; \mathbb{Z}), G$$

Proposition B.2.3. *Using (35) the elements of*

$$H^k(M, M-N; G) = Hom H_0(N; Z), G \tag{37}$$

can be described as follows: For every component N_λ of N choose an element $g_\lambda \in G$; then there is a unique class $y \in H^k(M, M-N; G)$ such that $\langle y, v_\lambda \rangle = g_\lambda$ for every λ.

In particular, there is a unique class $\tau = \tau_N^M \in H^k(M, M-N; \mathbb{Z})$ such that $\langle \tau, v_\lambda \rangle = 1$ for every λ. We call this the *Thom class* of N in M.

Proposition B.2.4. *Let $e : N^n \to M^{n+k}$ as above, let $X \subset N$ be closed and $W \subset M$ open such that $(N - X) \subset W \subset (M - X)$. Then the composition*

$$H_q(M, M-X) \xrightarrow{e_!} H_{q-k}(N, N-X) \xrightarrow{i_*} H_{q-k}(M, W)$$

agrees with $\tau \frown$,

Proof. See [Dol72, VIII, 11] □

Proposition B.2.5. *Let $e : N^n \to M^{n+k}$ be as before. Assume N is a neighborhood retract in M, say $r : M' \to N$, $r'e = id$, M' open in M. Then the composition*

$$H_q(M, M-N) \cong H_q(M', M'-N) \xrightarrow{\tau_N^{M'} \frown} H_{q-k}(M', M'-r^{-1}N) \xrightarrow{r_*} H_{q-k}(N)$$

agrees with $e_!$ therefore it is an isomorphism. Dually the composition

$$H^{q-k}(N) \xrightarrow{r^*} H^{q-k}(M', M'-r^{-1}N) \xrightarrow{\tau_N^{M'} \smile} H^q(M', M'-N) \cong H^q(M, M-N)$$

is isomorphic. These are what we call the Thom isomorphisms.

Now when $f : N \hookrightarrow M$ is the inclusion of a submanifold then f_* factors through the Thom isomorphism, that is, in the diagram

$$\begin{array}{ccccc} H^{*-1}(M-N) & \xrightarrow{\delta} & H^*(M, M-N) & \xrightarrow{j^*} & H^*(M) \\ & & \downarrow \cong & \nearrow_{f^!} & \\ & & H^{*-q}(N) & & \end{array} \tag{38}$$

we have

$$f^! = j^* \circ \Phi_N^{-1}$$

With Φ_N the Thom isomorphism.

$H^*(M, M-N)$ is by excision a purely N-local quantity, so that in the differentiable category this group can be identified with $H^*(v_N, v_N - N)$, where v_N is the normal bundle to N in M, and then finally with the compactly supported cohomology of v_N. We call Φ_N the Thom class of the normal bundle and its restriction to N is precisely the Euler class of v_N, therefore we have

$$f^* f^! = \text{Euler}(v_N) \tag{39}$$

B.2.1 The Equivariant Theory

[AB84] The equivariant cohomology of a G-space M is defined as the ordinary cohomology of the space M_G obtained from a fixed universal G-bundle, by mixing construction

$$M_G = EG \times_G M$$

Here we have G acting on EG on the right and on the left of M and the notation means that we identify $(pg, q) \sim (p, gq)$. Thus M_G is the bundle with fibre M over the classifying space BG associated to the universal bundle $EG \to BG$, and π denotes the corresponding natural projection $\pi : M_G \to BG$.

We also have a natural map $\sigma : M_G \to M/G$.

Definition B.2.6. The equivariant cohomology denoted by $H^*_G(M)$, is defined by

$$H^*_G(M) = H_*(M_G)$$

and constitutes a contravariant functor from G-spaces to modules over the base ring

$$H^*_G = H^*_G(pt) = H^*(BG)$$

The module structure is induces by π, and σ defines a natural arrow

$$\sigma^* : H^*(M/G) \to H^*_G(M)$$

which is an isomorphism if G acts freely.

And there is a natural homomorphism

$$\iota^* : H^*_G(M) \to H^*(M)$$

from equivariant theory to ordinary theory which corresponds to inclusion

$$\iota : M \to M_G$$

of M as the fibre over the basepoint of BG.

The functorial nature of the construction $M \mapsto M_G$ enables one to extend all concepts of ordinary cohomology to the equivariant one.

B.3 Berline-Vergne

[Jef]

Proof. We use the functorial properties of the pushforward in equivariant cohomology under the map ι_F including F in M. We have seen that $\iota_F^*(\iota_F)_* = \text{Euler}(v_F) := e_F$ is multiplication by the equivariant Euler class e_F of the normal bundle to F. Furthermore the map

$$\sum_{F\in\mathcal{F}} \iota_F^* : H_T^*(M) \mapsto \oplus_{F\in\mathcal{F}} H^*(F)\otimes H_T^*$$

is injective. Thus we see that each class $\eta \in H_T^*(M)$ satisfies

$$\eta = \sum_{F\in\mathcal{F}} (\iota_F)_* \frac{1}{e_F} \iota_F^* \eta \tag{40}$$

Just apply ι_F^* to both sides of the equation:

$$\iota_f^*\eta = \sum_{F\in\mathcal{F}} \iota_F^*(i_F)_* \frac{1}{e_F} \iota_F^*\eta = \sum_{F\in\mathcal{F}} e_F \frac{1}{e_F} \iota_f^*\eta = \iota_f^*\eta.$$

Now $\int_M \eta = \pi_*\eta$ where the map $\pi : M \to \text{pt}$ and $\pi_* : H_T^*(M) \to H_T^*$ is the pushforward in equivariant cohomology. Now we apply π_* to both sides of (40)

$$\begin{aligned}\int_M \eta(X) = \pi_* \circ \eta(X) = \pi_* \sum_{F\in\mathcal{F}} (\iota_F)_* \frac{1}{e_F(X)} \iota_F^*\eta(X) = \sum_{F\in\mathcal{F}} \pi_* \circ (\iota_F)_* \frac{1}{e_F(X)} \iota_F^*\eta(X) \\ = \sum_{F\in\mathcal{F}} (\pi_F)_* \frac{1}{e_F(X)} \iota_F^*\eta(X) = \sum_{F\in\mathcal{F}} \frac{\int_F \eta(X)}{e_F(X)}\end{aligned}$$

□

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