Topological quantum field theories and moduli spaces

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Disclaimer

These are notes from a course given by Johan Martens and Jørgen Ellegaard Andersen in 2011.[1] They have been written and TeX'ed during the lecture and some parts have not been completely proofread, so there's bound to be a number of typos and mistakes that should be attributed to me rather than the lecturer. Also, I've made these notes primarily to be able to look back on what happened with more ease, and to get experience with TeX'ing live. That being said, feel very free to send any comments and or corrections to `fuglede@imf.au.dk`.

The most recent version of these notes is available at `http://home.imf.au.dk/pred`.

1 Introduction

1st lecture, 25th January 2011

The reference for the course is [Ati90]. Another reference is [Tur94]. Other references are listed on the course webpage.

We begin discussing the mathematical side of topological quantum field theory (TQFT). We have a working definition, which one should think of as an analogy of (co)homology. A homology theory is a functor H from the category of topological spaces (or rather some nice subcategory hereof) to the category of Λ-modules for a given commutative ring Λ. The functor satisfies a bunch of axioms.

- $H(M \times I) \cong H(M)$, where I is an interval
- $H(M_1 \sqcup M_2) \cong H(M_1) \oplus H(M_2)$. In particular, $H(\emptyset) = 0$.

The working definition of a TQFT (often referred to as the Atiyah–Segal axioms) is similar.

A d-dimensional TQFT is a functor from the category of oriented, closed, smooth $(d-1)$-dimensional manifolds to the category of finite dimensional $\mathbb{C}$-vector spaces, associating to a $(d-1)$-manifold Σ a vector space $Z(\Sigma)$. Morphisms between $(d-1)$-manifolds are bordisms; that is, d-manifolds M, such that $\partial M = \Sigma$ (where the boundary is given the induced orientation). To this M, we associate $Z(M) \in Z(\partial M)$. As before we want Z to satisfy a bunch of axioms.

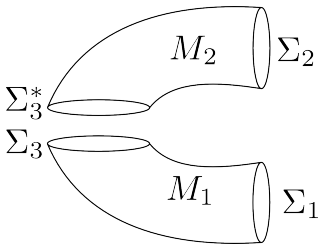


Figure 1: The gluing of M_1 and M_2 along Σ_3.

[1] The course homepage is located at `http://aula.au.dk/courses/IMFTQFTMSF11/` – that probably won't be true forever though.

Figure 2: The d-manifold M as a cobordism between $_1$ and $_2$.

Z is functorial with respect to orientation preserving di eomorphisms of ; M

$Z( ) = Z( )$, where denotes with the opposite orientation.

$Z(_1 \; t \; _2) = Z(_1) \quad Z(_2)$. In particular, this implies that $Z(;) = C$.

If $@M_1 = _1 t \; _3$, and $@M_2 = _2 t \; _3$ (see Fig. 1), we can glue along $_3$ and get a manifold $M = M_1 [_3 M_2$, then $Z(M) = hZ(M_1); Z(M_2)i$, where $h; i : Z(_1) \quad Z(_3) \quad Z(_3) \quad Z(_2) ! \quad Z(_1) \quad Z(_2)$ is the dual pairing.

Note that in particular, if $M = M_1 t M_2$, $Z(M) = Z(M_1) \quad Z(M_2)$. If $@M$ is two unions of components (see Fig. 2), we think of $Z(M) \; 2 \; Z(_1) \quad Z(_2)$ as a morphism of vector spaces $Z(_1) ! \quad Z(_2)$. One further axiom will then be that $Z(\quad I) = id_{Z( )}$ (note that in general, from the above axioms, one can only conclude that this map is an idempotent).

In practice we often allow for tweaking in the above working de nition. For example, one might require the vector spaces to be Hermitian, so that the dual pairing behave nicely with this extra structure.

The axioms are motivated by physics. One should think about as space in a physical theory, and M as a submanifold of space-time. In particular, this is the case for I. One should think about $Z( )$ as the space of quantum states for some quantum theory. One thinks of $Z(M)$ as time evolution on the space of quantum states. In quantum mechanics, time evolution is given by an operation $e^{i(t_1 \quad t_0)\hat{H}}$, where $\hat{H}$ is the quantum operator associated to the Hamiltonian, but in TQFT, $H = 0$ (meaning that the TQFTs carry no dynamics). Another con ict with the quantum theory is in the dimensionality of the vector spaces; one can pick up the the dimension of $Z( )$ as $Z(\quad S^1)$ while in physical quantum theory, the spaces will be in nite-dimensional. One might still use the TQFTs in systems, where a main role is being played by purely topological factors (a la Chern Simons theory, which we will discuss below).

As in physical theories, where one always encounters the two di erent formulations of Lagrangians and Hamiltonians, here physics was in terms of canonical quantization (corresponding to the Hamiltonian version). However, these days, people (such as Witten) tend to take a di erent approach, instead working with path/Feynman integrals, that are more Lagrangian in nature. Here, classical theory is determined by writing down a Lagrangian $L : TN ! \quad R$, where N is a conguration space, and quantizing by using a path integral, which is something like $e^{i \sim \quad L(')d'} [D]$, where here the outer integral is over the space of all elds in the theory, which is mathematically ill-de ned.

The link between canonical quantization and path integrals goes through correlation functions, giving the probability that a given state under time evolution evolve into some other state. In

2nd lecture, January 26th 2011

1.1 Chern–Simons theory

Today, we will talk about Chern–Simons theory (also refered to as Jones–Witten theory) and the Jones polynomial. In the early 1980s, Jones gave a new invariant of knots (i.e. embedded loops in 3-space). Before, people had used the Alexander polynomial, which was constructed in the 1930s; this one could not distinguish a knot and its mirror image – the Jones polynomial can do this. The Jones polynomial assigns a Laurent polynomial in $t^{1/2}$ to any knot, denoted $V(\text{knot})$. It is completely determined by $V(\text{unknot}) = 1$ and the following relationship. If you have 3 knots L_+, L_-, L_0 with the property that they have identical knot diagrams except at one spot, where they differ as in Fig. 4, then their Jones polynomials are related by $(t^{1/2} - t^{-1/2})V(L_0) = t^{-1}V(L_+) - tV(L_-)$. Relations of this kind are known as "skein relations".

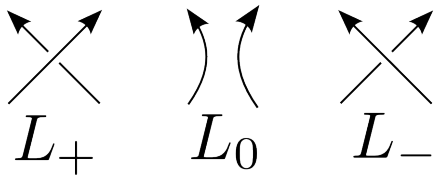


Figure 4: The skein relations for the Jones polynomial.

Chern–Simons theory is a 3-dimensional TQFT (of Schwarz type) that interprets the Jones polynomial. We begin by introducing the path integral of the theory.

1.1.1 Path integral approach

We restrict ourselves to defining it for closed oriented 3-manifolds; according to the Atiyah–Segal axioms, one should really consider manifolds with boundary. To a manifold M with no boundary, we should associate a complex number (i.e. a vector in $Z(\emptyset) = \mathbb{C}$). We will allow not just a 3-manifold but a 3-manifold with an embedded oriented link (i.e. an embedding of several disjoint loops). To define a path integral, we have to first define the space over which we integrate, and we have to define a Lagrangian.

In our situation, the space will be the infinite-dimensional affine space, $\mathcal{A}$, of all connections for the trivial G-bundle on M up to gauge equivalence, where G is a (semi-simple) compact (connected) Lie group. That is, we consider $\mathcal{A}/\mathcal{G}$, where $\mathcal{G}$ is the group of gauge transformations. Note that this space is once again infinite-dimensional.

The Lagrangian is the integral over M of the Chern–Simons form,

$$\mathcal{L}(A) = \int_M \mathrm{CS}(A) = \int_M \frac{1}{4\pi} \operatorname{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

for $A \in \mathcal{A}$. Here Tr is the trace with respect to a representation of G. We will mainly be dealing with the defining representation of $SU(n)$. Note that $d\,\mathrm{CS}(A) = \operatorname{Tr}(F_A^2)$. If you have two different but equivalent gauge connections A_1, A_2, then in general $\mathcal{L}(A_1) - \mathcal{L}(A_2) \in 2\pi\mathbb{Z}$. But when we take the path integral

$$\int_{\mathcal{A}/\mathcal{G}} e^{ik\mathcal{L}(A)} \mathcal{D}(A),$$

this will not make any difference. Here $k \in \mathbb{Z}$ is the "level" of the theory. This integral is the path integral, when no link is present. If an oriented link is present, we associate a representation R_i of G to each component L_i of the link. In this case, we use instead

$$Z_k = \int_{\mathcal{A}/\mathcal{G}} e^{ik\mathcal{L}(A)} \Pi_i(A) \mathcal{D}(A),$$

where $W_i = \mathrm{Tr}(R_i(\text{holonomy of } A \text{ around } L_i))$ is the socalled Wilson line. Thus, for every k we obtain a complex number (dependent on the choices for G, R_i).

Claim 2. *For a single component of the link (i.e. a knot) in S^3, $Z_k = V$ for $t = e^{2\pi i/(k+2)}$.*

1.1.2 Canonical quantization

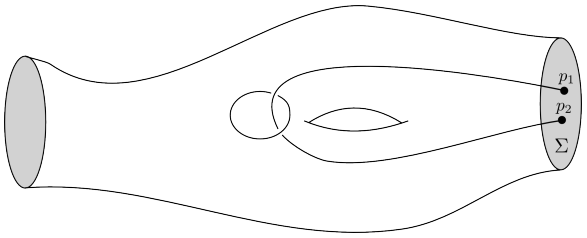


Figure 5: A 3-manifold with boundary components, which may contain marked points. The closed boundary 2-manifolds are depicted as disks.

Consider from now on the cases G is $U(n)$ or $SU(n)$. We now want to associate vector spaces to boundaries of 3-manifolds. The boundary might have marked points: I.e. in this picture, the oriented link in the 3-manifold might have ends in the boundary 2-manifolds (see Fig. 5).

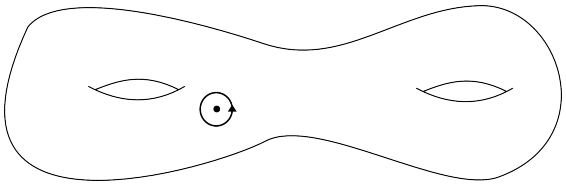


Figure 6: A boundary component with a marked point and a small loop around this point.

We thus look at the moduli space of flat connections on the punctured surface Σ with prescribed holonomy around the marked points p_i (see Fig. 6) – that is, we look at flat connections on $\Sigma \setminus \{p_i\}$ such that the holonomy class around small loops around the marked points will be a fixed conjugacy class. If we choose a holomorphic structure on Σ (i.e. we think of Σ as a Riemann surface). In the algebraic point of view, we can look at holomoprhic vector bundles on Σ. The prescribed structure corresponds to considering vector bundles with *parabolic structure* at the marked points.

Recall that if V is a finite-dimensional $\mathbb{C}$-vector space, a *flag* in V is a choice of nested subspaces,

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_r = V.$$

The *flag type* is given by the numbers $\dim V_i/V_{i-1}$, $i = 1, \ldots, r$. If all these numbers are 1, we have a *full flag*. Given V, the "moduli space" of all flags of a given flag type is a projective variety called the *flag variety*. For example, if the flag type is $(1, \dim V - 1)$, this moduli space will just be the projectivization $\mathbb{P}(V)$ of the vector space. If the flag type is $(n, \dim V - n)$, we get $\mathrm{Gr}(n, V)$.

If we choose a flag $\{0\} \subset V_1 \subset \cdots \subset V_r = V$, we can look at $P_{fl} = \{g \in GL(V) \mid g(V_i) = V_i\}$. This is a *parabolic subgroup*, and the flag variety is isomorphic to $GL(V)/P_{fl}$. E.g. if $V = \mathbb{C}^n$, we have the full flag, $V_i = \{(*, \ldots, *, 0, \ldots, 0)\}$ where there are i *'s and $n-1$ 0's, P_{fl} is the group of upper-triangular matrices, i.e. a Borel or minimal parabolic subgroup.

This is the story for $G = GL(V)$ – for general reductive groups (e.g. $SO(n, \mathbb{C})$), we can talk about G/P for parabolic subgroups P, and we can talk about full flags in G/B. If K is a compact Lie group with $G = K_{\mathbb{C}}$, the complexification of K, and T a maximal torus in K, then $K/T \cong G/B$.

A *parabolic structure* on a holomorphic vector bundle on a Riemann surface with marked points is now a choice of a flag over each marked point – strictly speaking, there's also a notion of

parabolic weights giving rise to of parabolic stability for parabolic vector bundles, and there exists a moduli space $\mathcal{M}_{par}$ of (semi-)stable parabolic bundles (corresponding to the moduli space of flat connections with prescribed holonomy). Under this correspondence, the flag type corresponds to the size of the Jordan blocks of the holonomy.

Now, we want to take the "geometric quantization" of this moduli space $\mathcal{M}_{par}$ (to do this, we need a complex structure on $\mathcal{M}_{par}$, which can be obtained by choosing a complex structure on Σ). It turns out, that we have a natural line bundle $L \to \mathcal{M}_{par}$, and the "geometric quantization at level k" is the space $H^0(\mathcal{M}_{par}, L^k)$ of holomorphic sections in the k'th tensor power of this line bundle. This space depends on the complex structure of Σ, but the existence of Hitchin's connection tells us, that all projectivizations can be identified.

3rd lecture, January 31 2011

Let M be a compact oriented 3-manifold. Let G be a simple simply-connected Lie group. Consider an Ad-invariant pairing $\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ (or $\mathbb{C}$). Let $P \to M$ be a principal G-bundle. Due to the Lie group being simply-connected (and connected) this bundle is trivializable. Assume $s : M \to G$ is a section. Let $\mathcal{A}_P \subseteq \Omega^1(P, \mathfrak{g})$ be the space of connections in P. Let $\mathcal{G}$ be the gauge group; automorphisms of P covering the identity on the base M. For $A \in \mathcal{A}_P$, we can build a number of forms. First off, we have its curvature $F_A \in \Omega^2(P; \mathfrak{g})$. Using the Ad-invariant pairing, we could form things like $\langle A \wedge F_A \rangle \in \Omega^3(P)$, $\langle A \wedge [A \wedge A] \rangle \in \Omega^3(P)$, and $\langle F_A \wedge F_A \rangle \in \Omega^4(P)$.

Definition 3. Let $\alpha(A) = \langle A \wedge F_A \rangle - \frac{1}{6}\langle A \wedge [A \wedge A] \rangle \in \Omega^3(P)$ be the *Chern–Simons form*.

Proposition 4. *The form $\alpha(A)$ satisfies $d\alpha(A) = \langle F_A \wedge F_A \rangle$.*

We want to use this to define the *Chern–Simons functional* $\mathrm{CS} : \mathcal{A}_P \to \mathbb{R}$, depending on the section s. This is defined by

$$\mathrm{CS}_s(A) = \int_M s^* \alpha(A).$$

We want to consider the behavior of this under gauge transformation. Let $\varphi \in \mathcal{G}_P$. Then

$$\alpha(\varphi^* A) = \alpha(A) + d\langle \mathrm{Ad}_{g_\varphi} A \wedge \varphi \rangle - \frac{1}{6}\langle \varphi \wedge [\varphi \wedge \varphi] \rangle.$$

Here $g_\varphi : P \to G$ is defined by $p g_\varphi(p) = \varphi(p)$, and $\theta \in \Omega^1(G; \mathfrak{g})$ is the Maurer-Cartan 1-form, and $\varphi = g_\varphi^*(\theta)$.

From now on, let M be closed. In this case, the second term of the above expression vanishes under integration. Thus

$$\mathrm{CS}(\varphi^* A) = \mathrm{CS}(A) - \frac{1}{6}\int_M s^* \langle \varphi \wedge [\varphi \wedge \varphi] \rangle.$$

The normalization of $\langle , \rangle$ is chosen such that the closed form $\frac{-1}{6}\langle \varphi \wedge [\varphi \wedge \varphi] \rangle$ is an integral generator of $H^3(G; \mathbb{Z})$. (Assume here that $H^3(G; \mathbb{Z}) \cong \mathbb{Z}$ – this is true except when $G = S\tilde{O}(4) = \mathrm{Spin}(4)$.) The upshot of this is that $\mathrm{CS}_s(\varphi^* A) = \mathrm{CS}_{\varphi \circ s}(A) = \mathrm{CS}_s(A) \mod \mathbb{Z}$. We thus have a well-defined map $\mathrm{CS} : \mathcal{A}_P / \mathcal{G}_P \to \mathbb{R}/\mathbb{Z}$.

Witten's idea is now the following: We define an invariant of 3-manifolds,

$$Z_k(M) = \int_{\mathcal{A}_P/\mathcal{G}_P} e^{2\pi i k \,\mathrm{CS}(A)} \mathcal{D}_A.$$

Here $k \in \mathbb{N}_+$ corresponding to taking multiples of the above integral generator of $H^3(G; \mathbb{Z})$.

If we have an oriented link $L = K_1 \sqcup \cdots \sqcup K_m \subseteq M$, K_i knots, and $R_1, \ldots, R_m$ finite dimensional real or complex representations of G, we can define a function $W_L : \mathcal{A}_P / \mathcal{G}_P \to \mathbb{C}$ by

$$W_L(A) = \prod_{r=1}^{m} R_i(\mathrm{hol}(A, K_i)).$$

Then our invariant extends to an invariant

$$Z_k^G(M,L,R) = \int_{\mathcal{A}_P/\mathcal{G}_P} e^{2\pi i k \operatorname{CS}(A)} W_L(A) \mathcal{D}_A.$$

Witten argued the following.

Theorem 5. *Let R_1 be the defining representation of $SU(2)$. Then*

$$Z_k^{SU(2)}(S^3, L, R_1) = J_L(e^{2\pi i/(k+2)}).$$

Witten verified this using the skein relations of the Jones polynomial, corresponding to using "known" properties of the ill-defined path integral. Another way of dealing with the path integral would be to use perturbation theory on the integral; one can then compare the resulting the asymptotic expansions of the integral with the concrete construction of Z_k^G by Reshetikhin and Turaev as a topological quantum field theory.

In the following, assume that M is non-closed, $\partial M = \Sigma$. Our construction should behave well under some kind of gluing law, as in Fig. 1. Physicists here want to assume we have certain boundary conditions on the gluable parts, giving rise to pairings of certain vector spaces. Let $Q = P|_\Sigma$. We then have spaces $\mathcal{A}_Q, \mathcal{G}_Q$. Let $A \in \mathcal{A}_Q$ be a boundary condition. "Define" $s : \mathcal{A}_Q \to \mathbb{C}$ by

$$S(A) = \int_{\{\mathbb{A} \in \mathcal{A}_P | \mathbb{A}_{|\Sigma} = A\}/\mathcal{G}_P^0} e^{2\pi i k \operatorname{CS}(\mathbb{A})} \mathcal{D}_{\mathbb{A}}$$

where $\mathcal{G}_P^0 = \{\varphi \in \mathcal{G}_P \mid \varphi|_\Sigma = \mathrm{id}\}$. Let $\psi \in \mathcal{G}_Q$. As before we want to understand $S(\psi^* A)$. Choose $\Psi \in \mathcal{G}_P$, such that $\Psi|_\Sigma = \psi$. This is always possible in our case; generally the only obstruction will be elements of homotopy groups up to π_2. We have

$$S(\psi^* A) = \int_{\{\mathbb{A} \in \mathcal{A}_P | \mathbb{A}_{|\Sigma} = A\}/\mathcal{G}_P^0} e^{2\pi i k \operatorname{CS}(\Psi \mathbb{A})} \mathcal{D}_{\mathbb{A}}.$$

By Stokes' theorem

$$\operatorname{CS}_s(\Psi A) = \operatorname{CS}_s(A) + \int_\Sigma s^* d\langle \operatorname{Ad}_{g_\psi} A \wedge \varphi|_\Sigma \rangle - \frac{1}{6} \int_M \langle \varphi \wedge [\varphi \wedge \varphi] \rangle,$$

where here $\varphi = (g_\Psi \circ s)^*(\theta)$, and $s : \Sigma \to P$ is a section. Define the Wess–Zumino–Witten term $W : \operatorname{Map}(\Sigma, G) \to \mathbb{R}/\mathbb{Z}$ as follows. For $g \in \operatorname{Map}(\Sigma, G)$ and any extension $\tilde{g} : M \to G$, let $W(g) = -\frac{1}{6} \int_M \langle \varphi \wedge [\varphi \wedge \varphi] \rangle$, for $\varphi = \tilde{g}^*(\theta)$. This is independent of the choice of extension. Define $c_{\Sigma,s} : \mathcal{A}_Q \times \mathcal{G}_Q \to U(1)$ by

$$c^k_{\Sigma, s|_\Sigma}(A, \psi) = \exp(2\pi k i \int_\Sigma s^* \langle \operatorname{Ad}_{g_\psi} A \wedge \varphi \rangle + W(g_\psi \circ s)).$$

We now note that

$$S_k(\psi^* A) = c^k_{\Sigma, s|_\Sigma}(A, \psi) S_k(A).$$

The dependence of s in S is supressed. Also, we will write $s = s|_\Sigma$ So S is not completely invariant, but we want to see it as a section in a certain line bundle. Let $\tilde{\mathcal{L}} = \mathcal{A}_Q \times \mathbb{C}$. For $\psi \in \mathcal{G}_Q$, define $\psi(A, z) = (\psi^* A, c_{\Sigma,s}(A, \psi) \cdot z)$. Then $S_k \in C^\infty(\mathcal{A}_Q, \tilde{\mathcal{L}}^k)^{\mathcal{G}_Q}$.

Formally, we could consider the bundle $\tilde{\mathcal{L}}^k/\mathcal{G}_Q \to \mathcal{A}_Q/\mathcal{G}_Q$, having S_k as a section.

Recall that $T_A \mathcal{A}_Q = \Omega^1(\Sigma, \operatorname{ad} Q)$. Define $\alpha_S(A)(a) = \int_\Sigma S^* \langle A \wedge a \rangle$. This defines a 1-form on the space of connections. We get a connection ∇^s in $\tilde{\mathcal{L}}^s$ defined as $\nabla^s + \nabla^{triv} + \alpha$. The nice thing about this is that ∇^s is invariant under the action of $\mathcal{G}$ defined above, so it really lives in the quotient $\tilde{\mathcal{L}}^k/\mathcal{G}_Q$. Note that the section S is not covariantly constant under this connection, even if that would be nice.

It is easy to find the curvature of this connection, simply computing the exterior derivative of α_S, choosing a path of connections. One obtains that

$$F_{\nabla^s}(a_1, a_2) = \int_\Sigma S^* \langle \alpha_1 \wedge \alpha_2 \rangle,$$

for $a_i \in \Omega^1(\Sigma, \operatorname{ad} Q)$. This is really a symplectic form, $\omega = F_{\nabla^s}$, on $\mathcal{A}_Q$. Thus $\tilde{\mathcal{L}}^k/\mathcal{G}_Q$ is a pre-quantum line bundle over $\mathcal{A}_Q/\mathcal{G}_Q$. Inside the configuration space, we have a nice finite-dimensional model. Namely, $\mathcal{M}_Q$, the moduli space of flat connections, i.e. $\{A \in \mathcal{A}_Q \mid F_A = 0\}/\mathcal{G}_Q$, is nothing but $\operatorname{Hom}(\pi_1(\Sigma), G)/G$, a subquotient of G^{2g}, where g is the genus of Σ. It might have singularities, but $\mathcal{M}_Q \subseteq \mathcal{A}_Q/\mathcal{G}_Q$. Path integral mumbo jumbo tells us that S is determined by its restriction to $\mathcal{M}_Q$. Once we choose a metric on M, we get an induced metric on the boundary Σ and hence a Riemann surface structure σ on Σ. Then $\mathcal{M}_Q$ gets induced the structure of a projective algebraic variety, say $\mathcal{M}_{Q,\sigma}$ and $\mathcal{L}_Q = \tilde{\mathcal{L}}/\mathcal{G}_Q$ becomes a holomorphic line bundle $\mathcal{L}_{Q,\sigma}$ and $S_{k,g} \in H^0(M_{Q,\sigma}, \mathcal{L}^k_{Q,\sigma})$ which depends in some way on the Riemannian metric g. Let $\mathcal{T}$ be Teichmüller space of Σ. We get a vector space $H^{(k)}$ over $\mathcal{T}$ with fibers $H^{(k)}_\sigma = H^0(\mathcal{M}_{Q,\sigma}, \mathcal{L}^k_{Q,\sigma})$. This bundle has a connection, namely the Hitchin connection, and the section $S_{k,g}$ in $H^{(k)}$ is flat with respect to the Hitchin connection (actually, the Hitchin connection is only projectively flat), so really we could write $S_{k,g} = S_{k,[g|_\Sigma]}$. In this picture $Z^G_k(\Sigma)$ is exactly the covariantly constant sections $H^{(k)}$ (with respect to Hitchin's conenction).

4th lecture, February 2nd 2011

We start with a schematic recap from last time. We started with a path integral. Treating this completely formal led us to assigning vector spaces to 2-dimensional closed oriented surfaces, and to 3-manifolds bounding a given surface, we associate a vector in the corresponding vector space. Studying this, Atiyah and Segal came up with the axioms for a topological quantum field theory. A nice written account of this is [Tur94]. What Turaev and Reshetikhin did was to give concrete examples, using quantum groups, of these. What we want in this course is to stay closer to the gauge theory: Let Σ be a closed oriented surface, and let G be a compact simple simply-connected Lie group (think $SU(n)$). We now continue in two steps. Classical gauge theory associates to the surface a moduli space $\mathcal{M}(\Sigma)$ of flat G-connections. On the symmetry side we have the following: If $f : \Sigma_1 \to \Sigma_2$ is a diffeomorphism, we get a map $f^* : \mathcal{M}(\Sigma_2) \to \mathcal{M}(\Sigma_1)$ by pulling back flat connections. This part we will not be too concerned about. The second part is using *geometric quantization* of the moduli space and Hitchin's connection: To the surface we associate the vector space $Z^G_k(\Sigma)$ together with a representation of (a central extension) of the mapping class group of Σ.

1.2 Geometric quantization

We will now extract the necessary differential geometric structure of the moduli space necessary for doing geometric quantization. I.e. we want to consider the following: Let M be a symplectic manifold with a symplectic form $\omega \in \Omega^2(M)$. Assume that $[\omega/2\pi] \in \operatorname{Im}(H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R}))$ and that $\mathcal{T}$ is a complex manifold with a smooth map $I : \mathcal{T} \to C^\infty(M, \operatorname{End}(TM))$ such that (M, ω, I_σ) is Kähler. Smoothness means that the corresponding section $\mathcal{T} \times M \to \operatorname{End}(TM)$ is smooth. In this context, *geometric quantization* consists of the following.

Definition 6. A triple $(L, \nabla, \langle , \rangle)$ is called a pre-quantum line bundle if $L \to M$ is a complex line bundle, such that ∇ is a connection in L, and such that $\langle , \rangle$ is a Hermitian structure on L (i.e. an inner product in each fiber fitting together smoothly over L). Furthermore, we require that for any two sections $s_1, s_2 \in C^\infty(M, L)$ and $X \in C^\infty(M, TM)$, we have the compatibility

$$X\langle s_1, s_2 \rangle = \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle.$$

Finally, we require that the connection has curvature $F_\nabla = i\omega$ (this is the pre-quantum condition).

Remark 7. The symplectic form governs the dynamics of a system in classical mechanics, and the pre-quantum condition ties this together with the line bundle.

Lemma 8. *A pre-quantum line bundle $(L, \nabla, \langle,\rangle)$ exists if and only if $[\omega/2\pi] \in \mathrm{Im}(H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{R}))$ (we say that ω is integral).*

A proof is available in [Woo91].

Assume that $(L, \nabla, \langle,\rangle)$ is a pre-quantum line bundle on M. Pre-quantization is the replacement of M with the Hilbert spaces $\mathcal{H}^{(k)} = C^\infty(M, L^{\otimes k})$. (Note that taking the k'th tensor power of L corresponds to multiplying the symplectic form by k. Here, k corresponds to $1/\hbar$.) We should think of M as the cotangent bundle, which is the usual phase space in mechanics. The space $\mathcal{H}^{(k)}$ then consists of wave functions, which is really not the right space to consider from a physical point of view. Rather, we have to cut this space down somewhat. Take a point $\sigma \in \mathcal{T}$, so (M, ω, I_σ) is a Kähler manifold. We know that $I_\sigma^2 = -\mathrm{id}$, and we have a splitting $TM \otimes \mathbb{C} = T_M \oplus \overline{T}_M$ into the i and $-i$ eigenspaces of I_σ. Here T_M is called the *holomorphic tangent bundle*. This really is a holomorphic bundle over the Kähler manifold M. Dually, we have a similar decomposition $T^*M \otimes \mathbb{C} = T_M^* \oplus \overline{T}_M^*$, taking the dual operator to I, which also squares to $-\mathrm{id}$, also with i and $-i$ eigenspaces. We have projections $\pi_{I_\sigma}^{1,0}$, and $\pi_{I_\sigma}^{0,1}$ to T_M^* and $\overline{T}_M^*$ respectively. Similarly all differential forms decompose into forms of various types, but for now we really only need the 1-forms. We could consider the composition

$$C^\infty(M, L^{\otimes k}) \xrightarrow{\nabla} C^\infty(M, T^*M \otimes L^{\otimes k}) \xrightarrow{\pi_{I_\sigma}^{1,0} \oplus \pi_{I_\sigma}^{0,1}} C^\infty(M, T_M^* \otimes L^{\otimes k}) \oplus C^\infty(M, \overline{T}_M^* \otimes L^{\otimes k}).$$

The corresponding composition is denoted $\nabla_{I_\sigma}^{1,0} \oplus \nabla_{I_\sigma}^{0,1}$. We also write $\nabla_{I_\sigma}^{0,1} = \overline{\partial}_{I_\sigma}$. In other words, we now have a holomorphic structure on all the L^k. We are now in a position to define the appropirate subspace of $\mathcal{H}^{(k)}$.

Definition 9. Let $H_\sigma^{(k)} = \{s \in C^\infty(M, L^{\otimes k}) \mid \nabla_{I_\sigma}^{0,1} s = 0\} \subseteq C^\infty(M, L^k)$ be the space of holomorphic sections of L^k with respect to the structure I_σ.

The process here is known as *Kähler quantization*. Really what we have done above is that we've chosen a *polarization*. A main belief is that quantization should be independent of polarization. Remark that dropping the integrability condition on I might result in $H^{(k)}$ being empty.

Remark 10. Notice also that $g(X,Y) = \omega(X, IY)$ is a Riemannian metric having as its volume form $\omega^n/n!$, and there is an inner product $(s_1, s_2) = \int_M \langle s_1, s_2\rangle \frac{\omega^n}{n!}$. Taking the Riemannian metric on $\overline{T}^*$ we get a Riemannian metric on $\overline{T}^* \otimes L^{\otimes k}$. Thus, we can construct an operator $(\nabla_{I_\sigma}^{0,1})^*$ adjoint to the $\nabla_{I_\sigma}^{0,1}$ above, i.e. $(\nabla_{I_\sigma}^{0,1})^* \circ \nabla_{I_\sigma}^{0,1} : C^\infty(M, L^k) \to C^\infty(M, L^k)$. This composition is a self-adjoint operator and behaves rather nicely: Its spectrum consists of finitely many points around 0 and a continuum above $\sqrt{k}$. The direct sum of the eigenspaces corresponding to the finitely many points around 0 gives a nice candidate to quantization. Note that this applies to any symplectic structure as any such will have an almost complex structure.

Our objective now is to understand how the Hilbert spaces $H_\sigma^{(k)}$ depend on the various complex structures. Being able to identify these vector spaces corresponds to finding a finite-dimensional representation of the symplectomorphism group of the symplectic manifold, which isn't always possible by the Groenewald–van Howe theorem.

1.2.1 The Hitchin connection

Consider the following trivial bundle $\mathcal{T} \times C^\infty(M, L^k) \to \mathcal{T}$. Since it is trivial, it has the trivial connection $\boldsymbol{\nabla}^t$. The idea is to try to find a connection in this trivial bundle $\mathcal{T} \times C^\infty(M, L^k)$ that preserves the subspaces $H_\sigma^{(k)}$, $\sigma \in \mathcal{T}$. Assume from now on that M is compact. We have a pre-Hilbert space structure $C^\infty(M, L^k) \otimes C^\infty(M, L^k) \to \mathbb{C}$. We could complete this to a Hilbert space structure on the L^2-sections.

Fact 11. *We have* $\dim H^{(k)}_\sigma < \infty$.

This is a deep fact whose proof uses Hodge theory and the theory of pseudo-differential operators. In the finite-dimensional case we have projections $\pi^{(k)}_\sigma : C^\infty(M, L^k) \to H^{(k)}_\sigma$. We thus have a connection $\boldsymbol{\nabla}^{L^2} = \pi^{(k)} \circ \boldsymbol{\nabla}^t$. This preserves the subbundle we are considering. One could thus compare the fibers by using parallel transport of the connection, but the above connection has curvature and thus parallel transport is not an isomorphism.

Let $\alpha \in C^\infty(\mathcal{T}, T^*\mathcal{T} \otimes \operatorname{End}(\mathcal{H}^{(k)}))$ and consider $\boldsymbol{\nabla}^\alpha = \boldsymbol{\nabla}^t + \alpha$. This will be a new connection, and we could try to find an α such that $\boldsymbol{\nabla}^\alpha$ preserves $H^{(k)}_\sigma$. From a physical point of view, differential operators would be more natural than general sections. The *Hitchin Ansatz* is the following: Try to find $\alpha \in C^\infty(\mathcal{T}, T^*\mathcal{T} \otimes \operatorname{Diff}^{(2)}(M, L^k))$ such that $\boldsymbol{\nabla}^\alpha$ preserves $H^{(k)}_\sigma$.

From now on we use the shorthand $T_\sigma = T_{M_{I_\sigma}}$. Hitchin proposed the following. Consider again $I : \mathcal{T} \to C^\infty(M, \operatorname{End}(TM))$. We know that $I^2 = -\mathrm{id}$, so for $V \in C^\infty(\mathcal{T}, T\mathcal{T})$, we obtain

$$V[I]I + IV[I] = 0.$$

This means that $V[I]_\sigma : T_\sigma \oplus \overline{T}_\sigma \to \overline{T}_\sigma \oplus T_\sigma$, since it anti-commutes with I. Writing this out, we have $V[I]_\sigma \in C^\infty(M, (T^*_\sigma \otimes \overline{T}_\sigma) \oplus (\overline{T}^*_\sigma \otimes T_\sigma))$. We write $V[I]_\sigma = V[I]''_\sigma \oplus V[I]'_\sigma$. From the beginning, we assumed that $\mathcal{T}$ is a complex manifold, so we have a decomposition $T(\mathcal{T}) = T_\mathcal{T} \oplus \overline{T}_\mathcal{T}$. So we could break up the vector field V into $V = V' \oplus V''$. We require that I is holomorphic. That is, we require that $V'[I]_\sigma = V[I]'_\sigma$ and $V''[I]_\sigma = V[I]''_\sigma$.

Recall that $\omega \in T^*_\sigma \otimes \overline{T}^*_\sigma$. Define $G_\sigma : T_\sigma(\mathcal{T}) \to C^\infty(M, S^2(T_\sigma))$ by

$$\omega \dot{G}_\sigma(V) = V[I]'_\sigma,$$

where we contract the $T^*_\sigma \otimes \overline{T}^*_\sigma$. This defines $G_\sigma(V)$ uniquely and G_σ depends linearly on the tangent vector. Define an operator

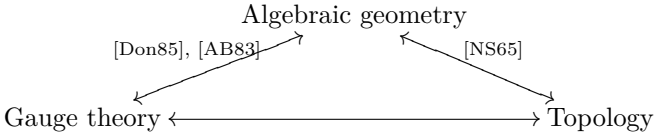
$$\begin{aligned}\Delta_{G(V)_\sigma} : C^\infty(M, L^k) \overset{\nabla^{1,0}_\sigma}{\to} C^\infty(M, T^*_\sigma \otimes L^k) \overset{G_\sigma(V)}{\longrightarrow} C^\infty(M, T_\sigma \otimes L^k) \\ \overset{\hat{\nabla}^{1,0} \otimes 1+1 \otimes \nabla^{1,0}}{\longrightarrow} C^\infty(M, T^*_\sigma \otimes T_\sigma \otimes L^k) \overset{\mathrm{Tr}}{\to} C^\infty(M, L^k).\end{aligned}$$

Here we used that the Levi–Civita connection $\hat{\nabla}$ on a Kähler manifold preserves types. This composition defines a second order differential operator acting on sections of L^k. This is exactly the kind of object, we were looking for. Write $\alpha^0_\sigma(V) = \Delta_{G(V)_\sigma}$. In other words, $\alpha \in C^\infty(\mathcal{T}, T^*\mathcal{T} \otimes \operatorname{Diff}^{(2)}(M, L^k))$. The question now is: Will $\boldsymbol{\nabla}^{\alpha^0} = \boldsymbol{\nabla}^t + \alpha^0$ preserve $H^{(k)}$? To answer this, one should consider the infinitesimal conditions for a section to be in $H^{(k)}$.

5th lecture, February 7th 2011

2 Moduli spaces of surfaces with marked points

Today, we will continue discussing moduli of parabolic vector bundles. Recall that there are correspondences



In the algebraic geometry picture, we consider the moduli space of semi-stable vector bundles over a Riemann surface. In the gauge theory picture, we consider the moduli space of constant scalar curvature Hermitian connections. Finally, on the topological side, one considers unitary representations of (a central extension) of the fundamental group of the Riemann surface.

2.1 Algebraic geometry

We consider now the case where the Riemann surface in question has marked points. We will only be talking about vector bundles, since in the case of parabolic structures, the "correct" notion of stability hasn't quite been developed for principal bundles with general gauge groups. Notation-wise, E will always denote a holomorphic vector bundle and $\mathbb{E}$ the corresponding smooth complex vector bundle underlying E. We will write $r = \operatorname{rk}\mathbb{E}$, $d = \deg\mathbb{E}$.

Definition 12. Let X be a Riemann surface with distinct marked points $D = p_1 + \cdots + p_n$ (and assume $n = 1$ – this turns out to not be a problem). A *parabolic vector bundle* on the Riemann surface on (X, D) is a holomorphic vector bundle E together with a choice of a flag in the fibers over the marked point p: I.e. a nested chain $\{0\} = E_0 \subseteq \cdots \subseteq E_s = E|_p$. The *multiplicity* is $m_i = \deg E_i/E_{i-1}$.

Strictly speaking, this is what people call a pre-parabolic bundle. A parabolic bundle would then be a pre-parabolic bundle together with a *parabolic weight*. The vector spaces E_i in a flag $\{0\} = E_0 \subseteq \cdots \subseteq E_k = E|_p$ have associated real numbers K_i such that $K_0 = 1$, and $1 > K_1 > \cdots > K_s \geq 0$. It is useful to write these with multiplicity, $(\alpha_1, \ldots, \alpha_r)$, where $\alpha_1, \ldots, \alpha_{m_1}$ are all equal to K_1, and so on. These weights are important since they enter the definition of stability for parabolic bundles.

Definition 13. The parabolic degree of E is $\operatorname{pdeg}(E) = \deg(E) + \sum_i \alpha_i = \deg(E) + \sum_j m_j K_j$.

Whereas the degree of a vector bundle is always an integer, this will just be a real number. If $F \subset E$ is a holomorphic subbundle of a parabolic vector bundle E, then F inherits a parabolic structure by intersecting the flag of E_i with the fiber of F at p. This way we get a flag F at p (but maybe with some other flag type, as the inclusions will not necessarily be strict – in this case, we remove repeated subspaces from the flag). For the parabolic weight for F_i, we use $\tilde{K}_i = \max_{E_j \cap F|_p = F_i}\{K_j\}$.

Definition 14. The parabolic vector bundle E is said to be *semi-stable* if for all holomorphic subbundles F of E, we have

$$\frac{\operatorname{pdeg}(F)}{\operatorname{rk}(F)} \leq \frac{\operatorname{pdeg}(E)}{\operatorname{rk}(E)}$$

and it is called semi-stable, if the inequality is strict.

Remark 15. One could always choose the trivial flag $\{0\} \subseteq E_p$ in the fiber over a marked point, but there is still the choice of a weight $1 > K_1 \geq 0$. In this case, E is stable as a holomorphic vector bundle if and only if parabolically stable, independent of the choice of K_1.

Theorem 16 (Mehta–Seshadri)**.** *Fix (X, D) and fix all the flag types (i.e. the multiplicities of the flag) and parabolic weights for all p as well as the rank r and degree d. Then there exists a (coarse) projective moduli space $\mathcal{M}^\alpha$ for (S-equivalence classes of) semi-stable parablic vector bundles. We have $\dim(\mathcal{M}^\alpha) = \dim(\mathcal{M}) + \sum_{p\in D} \dim G/p$, where $\mathcal{M}$ is the moduli space of the non-parabolic vector bundles, $\dim(\mathcal{M}) = r^2(g-1)+1$, and G/p is the corresponding flag variety with the flag types defined by the data at the point p.*

Recall that one distinguishes between coarse and fine moduli spaces: A moduli space $\mathcal{M}$ is called *fine* if it has a "universal bundle" – here, a universal bundle would be a universal vector bundle $\mathcal{E} \to \mathcal{M} \times X$ such that any family of vector bundles on X parametrized by S, i.e. a vector bundle $\mathcal{F} \to S \times X$, there exists a morphism $S \xrightarrow{f} \mathcal{M}$ such that $\mathcal{F} \cong f^*\mathcal{E}$. In the non-parabolic case, if $\operatorname{rk}(E)$ and $\deg(E)$ are co-prime, then all semi-stable vector bundles are stable (since $\deg(E)/\operatorname{rk}(E)$ is non-integer). In this case, the moduli space is fine, and will in fact also be smooth, but the connection between fineness and smoothness is subtle – if $\deg(E)$ and $\operatorname{rk}(E)$ is not co-prime, one could restrict attention to the stable locus of the moduli space, which will be smooth, but it will not be fine.

By S-stability we mean the following: If we have a vector bundle E which is semi-stable but not stable (i.e. strictly semi-stable), we can find a subbundle F_1 of maximal rank, $F_1 \subset E$, such that $\frac{\deg(E)}{\mathrm{rk}(E)} = \frac{\deg(F_1)}{\mathrm{rk}(F_1)}$. Now F_1 is necessarily semi-stable again. If it is not stable, we could find F_2 of maximal rank. Doing this, we obtain a finite sequence $\cdots \subset F_2 \subset F_1 \subset E$ of subbundles of equal slope. This is called the Jordan–Hölder filtration. This is not unique, but we could take the associated graded $\mathrm{Gr}(E) = E/F_1 \oplus F_1/F_2 \oplus F_2/F_3 \oplus \cdots$. Here, each summand will be stable, since each F_i was chosen to have maximal rank. Now, for different Jordan–Hölder filtrations, the corresponding $\mathrm{Gr}(E)$ will be canonically isomorphic.

Definition 17. If $E, \tilde{E}$ are both semi-stable, E and $\tilde{E}$ are called S-equivalent if $\mathrm{Gr}(E) \cong \mathrm{Gr}(\tilde{E})$.

For stable bundles, S-equivalence is the same as isomorphism, but in general, it is something stronger.

The main difference in the parabolic story is the following. Given a pre-parabolic structure, one could consider all the possible ways of putting parabolic weights on the bundle, and one could ask whether the result will be stable or not – this will depend on the choice of weights!

Exercise 18. Let r, d and all flag types be given. Strict semi-stability for parabolic vector bundles only occurs if the α_i satisfy one of a set of affine equations. In particular, a generic choice of weights will not lead to strict semi-stability.

Proposition 19. *If α_i are generic, then $\mathcal{M}^\alpha$ is smooth (and fine).*

Moreover, for the same numerical reasons, as long as the weights α_i stay in a connected component of the complement of the union of these linear inequalities in the space of possible weights, the notion of stability doesn't change, and so the moduli space stays the same. Crossing into other connecting components might give rise to different notions of stability, and the topology of the moduli space changes (but the dimension stays the same, by the above formula).

In some sense, the parabolic moduli spaces are very nice, as (generically) we will not have to worry about singularities.

Remark 20. In the non-parabolic case, we would restrict our attention to Riemann surfaces of genus 2 or higher, to avoid having automorphisms on the surface. In the case with surfaces with marked points, the automorphisms are required to preserve the marked points (and the flag in the fiber), and as such we can talk of $\mathcal{M}^\alpha$ even for genus 0 surfaces (with at least 3 marked points) or genus 1 surfaces (with at least 1 marked point).

2.2 Correspondence results with gauge theory

In the gauge theory picture, we consider the marked points as punctures on the Riemann surface and look at Hermitian connections on $\mathbb{E}|_{X\setminus D}$ (note that every such vector bundle will be trivializable). Recall that in the non-parabolic case, E is called stable, if there exists a Hermitian metric, such that the canonical connection A has constant curvature $F_A = -2\pi i \mu(\mathbb{E})\mathrm{id}_{\mathbb{E}}\omega_X$, where ω_X is the symplectic form on the surface, and $\mu(E) = \deg(\mathbb{E})/\mathrm{rk}(\mathbb{E})$.

6th lecture, February 9th 2011

We begin today considering the chamber structure on the space of stability conditions, mentioned last time.

Example 21. Consider a rank 3, degree d bundle E with a full flag parabolic structure at a single point p. We have weights $0 \leq \kappa_1 < \kappa_2 < \kappa_3 < 1$. In the parabolic slope inequality, when do we have equality for a subbundle $F \subsetneq E$.

If $\mathrm{rk}\, F = 2$, we have $\deg F = \tilde{d}$, the equation reads either

$$(\tilde{d} + \kappa_1 + \kappa_2)/2 = (d + \kappa_1 + \kappa_2 + \kappa_3)/3,$$
$$(\tilde{d} + \kappa_1 + \kappa_3)/2 = (d + \kappa_1 + \kappa_2 + \kappa_3)/3,$$

or

$$(\tilde{d} + \kappa_2 + \kappa_3)/2 = (d + \kappa_1 + \kappa_2 + \kappa_3)/3.$$

This holds for any $\tilde{d} \in \mathbb{Z}$. The chamber structure is then defined by the intersection of the above equations with $0 \leq \kappa_1 < \kappa_2 < \kappa_3 < 1$. Similar things happen for $\operatorname{rk} F = 1$.

Before going into gauge theory, we discuss the so-called Hecke correspondence

2.2.1 Hecke correspondence

Recall that if the rank is $r = 1$, and our vector bundles in question are line-bundles, we have a group $\operatorname{Pic}(X)$ of line-bundles over our Riemann surface X, with multiplication given by tensoring. The degree d defines us a short exact sequence

$$1 \to \operatorname{Jac}(X) = \operatorname{Pic}_0(X) \to \operatorname{Pic}(X) \xrightarrow{d} \mathbb{Z} \to 0.$$

Here the Jacobian $\operatorname{Jac}(X)$ is an Abelian variety. The set $d^{-1}(z)$, $z \neq 0$, is still isomorphic to $\operatorname{Jac}(X)$ as a variety. This isomorphism is not canonical, and we have to make a choice: For $p \in X$, we define maps $\operatorname{Pic}_k \xrightarrow{\otimes \mathcal{O}(p)} \operatorname{Pic}_{k+1}$ and $\operatorname{Pic}_{k+1} \xrightarrow{\otimes \mathcal{O}(-p)} \operatorname{Pic}_k$. This defines our isomorphism. We ask the following question: Can we still do this for higher rank? Pick a point $p \in X$ and assume that $\operatorname{rk}(E) = r$, $\deg(E) = d$. Look at the map $E \mapsto E \otimes \mathcal{O}(p)$. In this case, $\deg(E \otimes \mathcal{O}(p)) = \deg(E) + r$, which follows from a Chern class calculation.

Exercise 22. Tensoring with a $\mathcal{O}(p)$ doesn't change stability of the vector bundle. In other words, a choice of p induces an isomorphism $\mathcal{M}_{r,d} \cong \mathcal{M}_{r,d+r}$.

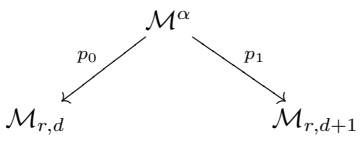
We thus have a certain periodicity when considering the moduli spaces of vector bundles with fixed rank. Often though, we would want to compare the degree d moduli space and the $d+1$ one. This leads naturally to parabolic vector bundles.

Remark 23. Over an algebraic curve/a Riemann surface, we have that for coherent sheaves, torsion free is the same as locally free. In general, torsion free sheaves aren't necessarily locally free. Recall that locally free sheaves are sheaves of holomorphic sections of a vector bundle. Coherent sheaves, in particular are sheaves, where to every open subset we associate a finitely generated module over the structure sheaf(?)

Since a subsheaf of a torsion-free sheaf is torsion free, we have that over a Riemann surface, any subsheaf of a sheaf of sections of a vector bundle is again a sheaf of sections of a vector bundle, but not necessarily of a subbundle of the given vector bundle. It might very well be, that the new vector bundle has the same rank. For example, if for $p \in X$, $\mathcal{O}$ can be viewed as a subsheaf of $\mathcal{O}(p)$, the map $\mathcal{O} \to \mathcal{O}(p)$ mapping s to $\cdots s_p$, where s_p is a section vanishing only at p (to first order).

Now, pick a rank r degree $d+1$ vector bundle E. Choose a point $p \in X$ and a pre-parabolic structure in E, with a choice of flag $0 = E_0 \subseteq E_1 \subseteq E_2 = E|p$, where $\dim E_1 = m_1 = r - 1$. Let $\mathcal{E} = \mathcal{O}(E)$, and consider the map $\mathcal{E} \to E|_p/E_1 \to 0$, where we t hink of $E|_p$ is the scyscraber sheaf at p. The map is given by evaluating a section in the point p. This map has kernel $\mathcal{F}$, the sheaf of sections of E that lie at E_1 at p. Note that $\mathcal{O} \to \mathcal{O}(p)$ becomes a special case of this picture. Since $\mathcal{F} \subseteq \mathcal{E}$, and $\mathcal{E}$ is locally free, so is $\mathcal{F}$, and therefore it determines a vector bundle F over X having $\mathcal{F}$ as its sheaf of sections. The degree is given by $\deg(E) = \deg(F) + 1$. So, by choosing a particular point p and a particular subspace of $E|_p$, we have obtained a vector bundle of degree one lower than the one we started with.

This leads to the Hecke correspondence



Here $\mathcal{M}$ denotes the moduli space of rank r degree $d+1$ parabolic vector bundles with flag type $(r-1,1)$ at p.

Exercise 24. One can find (small and generic) parabolic weights such that p_0 and p_1 in this picture are morphisms between the moduli spaces. Here p_1 is given by forgetting the parabolic structure, and p_0 is given by taking the kernel of evaluation at the point p, $\mathcal{E} \to E|_p/E_1 \to 0$.

Even if one is interested in non-parabolic vector bundles, this construction is motivation for considering parabolic ones.

Assuming that the stability conditions go through, we could consider the p_1 fibers of a stable vector bundle of degree $d+1$, which correspond to all possible choices of codimension 1, so the fiber will be a projective space $\mathbb{P}^r$. Using p_0 on this, we get a copy $p_0(p_1^{-1}(E))$ of this projective space in $\mathcal{M}_{r,d}$.

2.2.2 Back to gauge theory

We now turn back to the gauge-theoretic picture, and the correspondence between parabolic bundles and connections on punctured curves. Main references for this are [Biq91], [Por93] and [DW97], all of them were written in the early 90s. The representation theoretical/topological version of the correspondence was earlier and was due to Mehta and Seshadri.

Now, look at $X \setminus p$ and consider flat unitary connections ∇ on $X \setminus p$ (one could also consider non-flat connections, but in this case life is more complicated). In this case, we can consider the holonomy of ∇ around each puncture. In general, holonomy is determined up to conjugacy, but all elements of $U(r)$ and $\mathfrak{u}(r) = \mathrm{Lie}(U(r))$, in our case the holonomy will be the conjugacy class of a diagonal matrix, and so we assume that holonomy is given by $\exp(2\pi i A)$, where $A = \mathrm{diag}(\alpha_1, \dots, \alpha_r)$, $\alpha_i \in \mathbb{R}$.

Lemma 25. *After a suitable gauge transformation, there exists a trivialization around p, such that ∇ can be written $\nabla = d + Ad\theta$, having a singularity at 0, where θ is the angular coordinate for polar coordinates near p. This is refered to as having the ∇ being in ("temporal gauge").*

Assume that $1 > \alpha_1 \geq \cdots \geq \alpha_r \geq 0$. Doing this completely determines a holonomy matrix. In temporal gauge, glue in the trivial bundle on a neighborhood around p.

Lemma 26. *Now ∇ determines a holomorphic structure of this new vector bundle $\tilde{E}$.*

Equip $\tilde{E}$ with the flag of eigenspaces of A, and use the α_i as parabolic weights.

Theorem 27. *The vector bundle $\tilde{E}$ is stable as a parabolic bundle.*

Theorem 28. *Every stable parabolic bundle arises in this way.*

Theorem 29. *There exists a symplectomorphism between the moduli space of irreducible flat connections on a punctured Riemann surface and the stable locus of the moduli space of parabolic vector bundles on the surface.*

7th lecture, February 14th 2011

3 Geometric quantization of $\mathcal{M}_{\lambda,r}$

Today we will discuss geometric quantization of moduli spaces and fill in some of the details, that we've been hand-waving about so far. Now we will be looking at a (semi-)simple compact gauge group G, e.g. $G = SU(r)$. Let $\mathcal{M}_{\lambda,r}$ denote the moduli space of semi-stable rank r bundles with fixed determinant λ, for some line bundle λ on X. Assume that $\deg(\lambda)$ and r are coprime. In this case, $\mathcal{M}_{\lambda,r}$ is a smooth projective variety. Note that this is a subspace of $\mathcal{M}_{d,r}$, as $\mathcal{M}_{d,r}$ maps to the degree d line bundles Pic_d by taking the determimant (i.e. the top exterior power), and $\mathcal{M}_{\lambda,r}$ is a pre-image of some element in Pic_d.

In order to do geometric quantization, we need a line bundle. Recall that in the gauge theoretic approach, $\mathcal{M}_{\lambda,r} = \mathcal{A}^{(\text{stable})}/\mathcal{G}^{\mathbb{C}} = F^{-1}(\text{central curvature})/\mathcal{G}$, where $\mathcal{A} = \{\overline{\partial}_E\}$ is the space of all compatible $\overline{\partial}_E$ operators, that make the given complex vector bundle into a holomorphic one. Recall that there is a symplectic (Kähler) form on $\mathcal{A}$ given by

$$\omega_{\mathcal{A}}(\alpha, \beta) = \int_X - \operatorname{Tr}(\alpha \wedge \overline{\beta}),$$

where α and β are tangent vectors at a point $\overline{\partial}_E$ given by elements of $\Omega^{0,1}(\operatorname{End}_0 \mathbb{E})$, where End_0 denotes trace-less endomorphisms.

We construct now the *so-called Quillen line bundle* $\mathcal{L}$ on $\mathcal{A}$. The reference for this is [Qui85]. Let $\overline{\partial}_E$ be a point in $\mathcal{A}$. The fiber of the line bundle will be the following: Write $\overline{\partial}_E : \Omega^0(\mathbb{E}) \to \Omega^{0,1}(\mathbb{E})$. Complete $\Omega^0(\mathbb{E})$ to the Hilbert space $\mathcal{H}^0$ of $\square$-integrable sections, and $\Omega^{0,1}$ to the Sobolov space $\mathcal{H}^1$ of $\square$-integrable sections with $\square$-integrable first derivatives. Then $\overline{\partial}_E$ is Fredholm (i.e. the kernel and cokernel have finite dimensions). Put

$$\mathcal{L}|_{\overline{\partial}_E} \cong \Lambda^{\text{top}}(\operatorname{Ker}\overline{\partial}_E)^* \otimes \Lambda^{\text{top}}(\operatorname{Coker}\overline{\partial}_E).$$

Note that this could also be caleld a determinant line bundle: In general if $f : V_0 \to V_1$ is a linear map between vector spaces of equal finite dimension, then $\det f \in \Lambda^{\text{top}} V_0^* \otimes \Lambda^{\text{top}} V_1$. If $V_0 = V_1$, then these two spaces are naturally dual, and $\det f \in \mathbb{C}$.

Now, the $\mathcal{L}|_{\overline{\partial}_E}$ will form a holomorphic bundle. Let $\mathcal{F}$ be the space of Fredholm operators $T : \mathcal{H}^0 \to \mathcal{H}^1$. By general arguments, $\mathcal{F}$ is an infinite-dimensional holomorphic Banach manifold. Cover this Banach manifold by open subsets; let $F \subseteq \mathcal{H}^1$ be a finite-dimensional subspace, and let $U_F = \{T \in \mathcal{F} \mid \operatorname{Im}(T) + F = \mathcal{H}^1\}$. Note that U_F in $\mathcal{F}$ (in the topology coming from the operator norm, probably?), and that they cover $\mathcal{F}$. Given $T \in U_F$, we have an exact sequence,

$$0 \to \operatorname{Ker} T \to T^{-1}(F) \xrightarrow{T} F \to \operatorname{Coker} T \to 0.$$

Exercise 30. We get an isomorphism

$$\Lambda^{\text{top}}(\ker T)^* \otimes \Lambda^{\text{top}}(\operatorname{Coker} T) \cong \Lambda^{\text{top}}(T^{-1}F)^* \otimes \Lambda^{\text{top}}(F).$$

Note that $\Lambda^{\text{top}}(F)$ in the above exercise is just the trivial line bundle, and on U_F, $\Lambda^{\text{top}}(T^{-1}F)^*$ is a holomorphic line bundle. Since on the left hand side of the isomorphism of the exercise, F doesn't occur, the line bundles glue together to form a line bundle $\mathcal{L} \to \mathcal{F}$. Pull back this line bundle by $\mathcal{A} \to \mathcal{F}$ to obtain a line bundle over $\mathcal{A}$.

We define now a Hermitian product $\langle , \rangle_{\mathcal{L}}$. One way to go about this is to simply notice that $\ker \overline{\partial}_E$ is a subspace of the Hermitian space $\mathcal{H}^0$; similarly one could consider $\operatorname{Coker}\overline{\partial}_E$ as a Hermitian subspace of $\mathcal{H}^1$. Rather than doing this, we consider the following scaling:

Definition 31 (Quillen)**.** Take the ζ-function regularization of the induced (as above) inner product $\langle , \rangle$ on $\Lambda^{\text{top}}(\operatorname{Ker}\overline{\partial}_E)^* \otimes \Lambda^{\text{top}}(\operatorname{Coker}\overline{\partial}_E)$.

That is, given an operator $\overline{\partial}_E$, we could take the associated ζ-function mapping $\zeta_{\overline{\partial}_E} : s \mapsto \sum_{\lambda \neq 0 \text{ eigenvalue of } \overline{\partial}_E} \lambda^{-s}$ (if $\operatorname{Re} s > 1$, and take the analytic continuation in general; then $\zeta_{\overline{\partial}_E}$ is regular at 0 – it is holomorphic but has a singularity). Then the ζ-regularization is the following:

$$\langle , \rangle_{\mathcal{L}} = \exp(-\zeta'_{\overline{\partial}_E}(0)) \langle , \rangle_{\text{induced}}.$$

For example, if $T : V \to V$ is a linear operator on a finite dimensional vector space V, and if T is invertible, so all eigenvalues are non-zero. In this case, $\zeta_T(s) = \sum \lambda^{-1} = \sum \exp^{-(\ln(\lambda s))}$, so $\zeta'_T = \sum(-\ln \lambda)\lambda^{-s}$, and so $\exp(-\zeta'(0)) = \exp(\sum \ln \lambda) = \prod \lambda = \det T$. In general, if T is not invertible, we get instead the determinant of T acting on the orthogonal complement of $\ker T$.

Theorem 32 (Quillen)**.** *If $\nabla_{\mathcal{L}}$ is the canonical connection on $\mathcal{L}$ with respect to $\langle , \rangle_{\mathcal{L}}$, we get $F_{\nabla_{\mathcal{L}}} = \omega_{\mathcal{A}}$.*

The line bundle $\mathcal{L} \to \mathcal{A}$ defines a line bundle $\mathcal{L} \to \mathcal{M}_{\lambda,r}$ such that $F_{\mathcal{L}}$ is the symplectic form on $\mathcal{M}_{\lambda,r}$, and we can do geometric quantization and consider $H^0(\mathcal{M}_{\lambda,r}, \mathcal{L}^k)$. We began by assuming $\deg(\lambda)$ and r coprime, so that the resulting moduli space was smooth, but it turns out that the smoothness isn't necessary. If $(\deg \lambda, r) \neq 1$ instead we have semi-stable bundles (or equivalently, reducible connections), and the moduli space $\mathcal{M}_{\lambda,r}$ denotes the moduli space of S-equivalence classes and has singularities. As an algebraic variety, however, the moduli space is still normal; normality is a local property in algebraic geometry. A point $p \in M$ in a variety is normal if the ring of regular functions at p is integrally closed in the function field of the variety; a variety is normal if every point is normal. In general, smoothness always implies normality. The converse isn't necessarily true but is true, when the complex dimension is 1. There is always a canonical way to normalize a variety (while there is no canonical way to desingularize one): Given a variety M, if M is not normal, one can find a canonical normalization $M^{\text{norm}} \to M$. Also, Hartog's theorem as stated below is still true on normal varieties.

Theorem 33 (Hartog's theorem)**.** *Holomorphic functions extend (uniquely) over submanifolds of codimension 2 or greater.*

The upshot of this discussion is this (see Hitchin's article "flat connections and geometric quantization "): By using Hartog's theorem, everything still goes through in the singular case, since all of them live in high enough co-dimension.

3.1 The space of sections and the Verlinde formula

We can ask the following question: Is $\dim_{\mathbb{C}} H^0(\mathcal{M}_{\lambda,r}, \mathcal{L}^k)$ independent of the choice of the complex structure on X? In general, we have the Hirzebruch–Riemann–Roch theorem. Put $\chi(M, \mathcal{L}^k) = \sum(-1)^i \dim H^i(M, \mathcal{L}^k)$. The theorem tells us that this expression is topological, given by the integral over M of some class multiplied some Chern character. By Serre duality, $H^i(M, L) \cong H^{n-i}(M, L^{-1}K)^*$, and by the Kodaira vanishing theorem, if KL^{-1} is negative, then $\dim H^i(M, L) = 0$ for all i. In general, if we take L^k for $k \gg 0$, we always have that KL^{-k} is negative, and $\dim_{\mathbb{C}} H^0(M, L^k)$ is independent of Kähler polarization. In our case, this isn't even necessary: If $M = \mathcal{M}_{\lambda,r}$ and $L = \mathcal{L}$, it works for all k. Consider the determinant line bundle for the operator $\overline{\partial}|_{\text{End}_0 E} : \Omega^0(X, \text{End}_0 E) \to \Omega^{0,1}(X, \text{End}_0 E)$ induced by $\overline{\partial}_E$. This line bundle will be $\mathcal{L}^\lambda$ for some integer $\lambda \geq 0$, but on the other hand, if G is simple, it will be equal to $\Lambda^{\text{top}}(T\mathcal{M}_{\lambda,r})$. As a result, the canonical bundle, which is the dual to this last line bundle, will be $K = L^{-\lambda}$. As a result, $KL^{-k} = L^{-(\lambda+k)}$, which is always negative, and so $\dim H^0(\mathcal{M}_{\lambda,r}, \mathcal{L}^k)$ will always be independent of the complex structure.

This dimension is given by given by the so-called Verlinde formula (conjectured by physicists around 1987 and proven mathematically in the early 90s). Let $Z_k^{SU(r)}(g) = \dim_{\mathbb{C}} H^0(\mathcal{M}_{0,SU(r)}, \mathcal{L}^k)$, where g is the genus of X. Then

$$Z_k^{SU(2)}(g) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\sin^2 \frac{j\pi}{k+2}\right)^{1-g},$$

$$Z_k^{SU(n)}(g) = \left(\frac{r}{k+r}\right)^g \sum_{S \subseteq \{1,\ldots,k+2\}, |S|=2} \left(\prod_{s \in S, t \notin S, 1 \leq g \leq k+2} \left| 2 \sin\left(\pi \frac{s-t}{r+k}\right) \right|^{g-1} \right).$$

8th lecture, January 16th 2011

Remark 34. Last time we defined $\mathcal{M}_{\lambda,r} = \det^{-1}(\lambda)$, for some $\lambda \in \text{Pic}_d$, where $\det : \mathcal{M}_{d,r} \to \text{Pic}_d$. For correspondence results, we really want the case, where $\lambda = 0$ is trivial: For a $U(r)$-principal bundle coming from an $SU(r)$-principal bundle as an extension of structure groups, we must have

$\lambda = 0$. This really comes from the fact that we have short exact sequences

$$1 \to SU(r) \to U(r) \stackrel{\det}{\to} U(1) \to 1,$$
$$1 \to SL(r) \to GL(r) \to \mathbb{C}^x \to 1.$$

From now on we use the notation $\mathcal{M}_{0,SU(2)} = \mathcal{M}_{SU(2)}$.

Recall from last time the Verlinde formulas for $Z_k^{SU(2)}(g)$ and $Z_k^{SU(n)}(g)$. Remark in particular that these formulas were independent of the complex structure on X. Hitchin argues that in the general picture, this will also happen for k high enough. Recall from last time that in our picture this followed from an argument using Kodaira's vanishing theorem, describing the dimension in term of Euler characteristic; i.e. using the index theorem or Hirzebruch–Riemann–Roch, $\chi(\mathcal{M}_{SU(r)}, \mathcal{L}^k) = \sum(-1)^i \dim H^i(\mathcal{M}, \mathcal{L}^k) = \dim H^0(\mathcal{M}, \mathcal{L}^k)$. This, on the other hand, is a particular integral over $\mathcal{M}$, and one could ask why the formulas were surprising, when one could just compute this integral; this turns out to be a hard job but gives one proof of the Verlinde formula above. The method is as follows: Kirwan surjectivity says that given a compact finite-dimensional symplectic manifold (M, ω) with a Hamiltonian action G and momentum map μ, there is a surjective Kirwan map $K : H_G^*(M) \to H^*(\mu^{-1}(0)/G)$ from the G-equivariant cohomology of M to the cohomology of the symplectic reduction. There is a general method due to Lisa Jeffrey and Francis Kirwan that tells us that $\int_{\mu^{-1}(0)/G} K(\alpha)$ is given by a socalled "residue formula" in terms of equivariant cohomology for α on M. In particular, one can see $\mathcal{M}_{SU(r)}$ as symplectic reduction of an "extended moduli space". This space is the space of all flat connection modulo gauge transformations that are trivial at a point $p \in X$. This is interesting since the extended moduli space is already finite dimensional. Using this, Jeffrey and Kirwan were able to give a proof of the Verlinde formula.

Notice in the formulas that the k's always appear as $k + r$. One way to interpret this fact is through what is called "metaplectic correction": It turns out that, in many cases, the "naive" geometric quantization, i.e. taking $H^0(M, \mathcal{L}^k)$ doesn't give the expected vector spaces from a physical point of view. Recall that on an Kähler manifold M, giving a spin structure is the same thing as giving a square root $\sqrt{K}$ of the canonical bundle K, i.e. $L^2 = K$ (in particular, finding a spin structure will only be possible, if the degree of K is even, which in particular is true for Riemann surfaces). Metaplectic correction now says the following: Instead of $H^0(M, \mathcal{L}^k)$, one should use $H^0(M, \mathcal{L}^k \otimes \sqrt{K})$.

In general, the metaplectic correction would be $H^0(M, \mathcal{L}^k \otimes \sqrt{K})$, and in our case, $M = \mathcal{M}_{SU(2)}$, we have $K_{\mathcal{M}_{SU(r)}} \cong \mathcal{L}^{-2r}$, and a candidate square root $\sqrt{K}$ exists, namely $\mathcal{L}^{-r}$, and metaplectic quantization will be $H^0(\mathcal{M}_{SU(2)}, \mathcal{L}^{k-r})$, giving some explanation to the $k + r$'s in the Verlinde formulas.

3.2 The parabolic case

Consider now the space $\mathcal{M}^{\alpha}_{SU(r)}$ of S-equivalence classes of semi-stable parabolic vector bundles with parabolic weights α fixed. Recall that in the space of possible weights we have a chamber decomposition, and as long as we're inside of a chamber, the topology and even the complex geometry of the moduli space will not change when α changes; the Kähler structure changes though, and there is a wall crossing phenomenon.

Recall also that the parameter α also referred to fixing the holonomy of connections around marked points in the gauge theory viewpoint. Only for specific α the Kähler form will be integral though, and only in these cases should we expect to be able to find our pre-quantum line bundle. Recall that parabolic semi-stability for E means that

$$\frac{\deg(F) + \sum \alpha_j}{\mathrm{rk}(F)} \leq \frac{\deg(E) + \sum_i \alpha_i}{\mathrm{rk}(E)}$$

for subbundles $F \subseteq E$ (and note that for $SU(r)$, $\deg(F) = \deg(E) = 0$). So morally, only if $\{k\alpha_i\}$ are all integers, there exists a pre-quantum line bundle at level k.

3.3 Review of spin

As we might need it for the second part of the course, and since we already used it, we'll give a review of what spin means. The fact is that $\pi_1(SO(2)) = \mathbb{Z}$, and for $n \geq 3$, $\pi_1(SO(n,\mathbb{R})) = \mathbb{Z}_2$; recall that $SO(3)$ are all orientation preservering transformations of Euclidean 3-space, and topologically, $SO(3) \cong \mathbb{RP}^3$; a rotation is given by an oriented axis in 3-space and giving an angle of rotation, and furthermore one can limit the angles to being between 0 and π, since adding 2π doesn't mean anything, and the angles between π and 2π can be obtained by rotating in the other direction. That is, one can think of $SO(3)$ as $B^3_\pi/\sim$, where p and $-p$ in $\partial B^3_\pi = S^2$ are identified. This space is homeomorphic to $\mathbb{RP}^3$ (which is probably easiest seen considering the 2-dimensional picture). The homotopically non-trivial loop in $SO(3)$ is going from the origin to the boundary, coming out on the other side and going back to the origin; the magic thing is that squaring this loop gives something trivial cf. the belt trick which is hard to live TeX.

Since $\pi_1(SO(n)) \cong \mathbb{Z}_2$, its universal cover will be a double cover $1 \to \mathbb{Z}_2 \to \widetilde{SO(n)} \to SO(n) \to 1$, and we define $\mathrm{Spin}(n) = \widetilde{SO(3)}$. For $n = 3$ it turns out that $\mathrm{Spin}(3) \cong SU(2)$. Topologically, $SU(2) \cong S^3$, and we have a map $SU(2) \to SO(3)$ having as its kernel the center of $SU(2)$ which is a copy of $\mathbb{Z}_2$.

Let M be an oriented Riemannian manifold. In general, the frame bundle of a Riemannian manifold is an $O(n)$-bundle and in the oriented case, it will be an $SO(n)$-bundle. We can now ask for a $\mathrm{Spin}(n)$ bundle such that this frame bundle arises from the $\mathrm{Spin}(n)$-bundle by extension of structure group; this will be called a spin structure on M. Topologically, M is orientable if and only if $\omega_1(TM) = 0$, where ω_1 is the first Stiefel–Whitney class, and moreover M has a spin structure if and only if $\omega_2(TM) = 0$. As mentioned earlier, if M is complex, then M has a spin structure if and only a square root of the canonical K exists.

9th lecture, February 21th 2011

4 The Hitchin connection revisited

Let (M,ω) be a symplectic manifold with a pre-quantum line-bundle $(\mathcal{L}, \langle,\rangle, \nabla) \to (M,\omega)$. Let $I : \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$ be a smoothly varying family of Kähler structures on (M,ω). I.e. $M_\sigma = (M,\omega,I_\sigma)$ is a Kähler manifold for all $\sigma \in \mathcal{T}$.

Define $\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty(M, \mathcal{L}^k)$. Inside of this we have $H^{(k)}_\sigma = \{s \in \mathcal{H}^{(k)}_\sigma \mid \nabla^{0,1}_\sigma s = 0\}$. We want to find conditions for this space to be a bundle over $\mathcal{T}$ and for when it admits a projectively flat connection. In the cases where $H^{(k)} \subseteq \mathcal{H}^{(k)}$ is a subbundle (which is actually true if and only if the dimensions of the $H^{(k)}$ do not change), we have the L^2-connection: Let $\pi^{(k)}_\sigma : C^\infty(M,\mathcal{L}^k) \to H^{(k)}_\sigma$ denote the orthogonal projection with respect to the L^2-inner product and put $\boldsymbol{\nabla}^{L^2}_V = \pi^{(k)}_\sigma \boldsymbol{\nabla}^t_V$. Typically, this connection is not (projectively) flat.

Instead we want to consider the Hitchin connection. The idea is to write $\boldsymbol{\nabla}^H_V = \boldsymbol{\nabla}^t_V - u(V)$. We want $u(V)_\sigma \in \mathrm{Diff}^{(2)}(M,\mathcal{L}^k)$. Let us find conditions on u so that it preserves the subbundle $H^{(k)}$. Take a vector field V and a section s of $\mathcal{L}^k$. Recall that $V(\nabla^{0,1}_\sigma s) = V(\frac{1}{2}(\mathrm{id} + iI)\nabla s) = \frac{i}{2}V[I]\nabla s + \nabla^{0,1}_\sigma V[s] = 0$. Assume that s is covariant constant; i.e. that $V[s] = u(V)s$. We then obtain that $\frac{i}{2}V[I]\nabla^{1,0}_\sigma s + \nabla^{0,1}_\sigma u(V)s = 0$. Notice that $u(V)$ does not take a holomorphic section to a holomorphic section. In order to solve this equation, we must have the following be true.

Exercise 35. Prove that $\nabla^{0,1}_\sigma(V[I]\nabla^{1,0}_\sigma s) = 0$ for all $s \in H^{(k)}_\sigma$.

Recall that we want $\mathcal{T}$ to be a complex manifold such that $\mathcal{T} \times M \to \mathcal{T}$ is holomorphic with respect to the complex structure on $\mathcal{T} \times M$ induced by I (at a given point of $\mathcal{T}$, we have an endomorphism in the tangent space, and we use the complex structure coming from I at that point in the M part). If we write $V = V' + V''$ for a vector field, the condition of holomorphicity is equivalent to $V'[I] = V[I]' \in C^\infty(M_\sigma, T^*_\sigma \otimes \overline{T}_\sigma)$ (recall that we introduced $G(V)_\sigma \in C^\infty(M, S^2(T_\sigma))$ defined by $\omega.G(V)_\sigma = V'[I]_\sigma$, where the . means contracting; we have $\omega \in T^*_\sigma \otimes \overline{T}^*_\sigma$). In our

notation, $\nabla^{1,0}s \in T^*_\sigma \otimes \mathcal{L}^k$, and acting with $V[I]$ is contracting in the first part. In particular, if we use a vector field with $V = V''$, the first term of our equation vanishes, and we can require that $u(V'') = 0$. Note altso that $H^{(k)}$ as a bundle has a $\overline{\partial}$ operator induced from the trivial connection. In other words, we have now defined our connection on the anti-holomorphic part; here it's really nothing but the trivial connection. In general, we have

$$u(V) = \frac{1}{k + n/2}o(V) - V'[F],$$
$$o(V) = -\frac{1}{4}(\Delta_{G(V)} + 2\nabla_{G(V)dF} - 2uV'[F]).$$

Here, we assume that $c_1(M,\omega) = n[\frac{\omega}{2\pi}]$, $n \in \mathbb{Z}$, where $c_1(M,\omega) = c_1(M,\omega,I_\sigma)$ is the first Chern class with respect to the Levi–Civita connection coming from any metric induced by one of the I_σ, and we assume that $k + n/2 \neq 0$. From previously, remember that $\Delta_{G(V)} : C^\infty(M,\mathcal{L}^k) \to C^\infty(M,\mathcal{L}^k)$. Let $\mathrm{Ric}_\sigma \in \Omega^{1,1}(M)$ be the Ricci 1-1 form for (M,ω,I_σ) (defined below). Then $[\mathrm{Ric}_\sigma/2\pi] = c_1(M,\omega)$. Thus $\mathrm{Ric}_\sigma - n\omega$ is exact, and any exact 1-1 form can be written $2i\partial\overline{\partial}F$ for some function F, well-defined up to some constant ambiguity, so let F_σ be the function arising this way; assume here that M is compact. One way of normalizing F_σ then is by requiring that $\int_M F_\sigma \frac{\omega^n}{n!} = 0$. We thus get a function $F : \mathcal{T} \to C^\infty(M)$ (called the *Ricci potential*). Considering types, we obtain that $G(V)_\sigma dF_\sigma \in C^\infty(M,T_\sigma)$. Note that in the expressions for u and o above, $\Delta_{G(V)}$ is a second order operator, $\nabla_{G(V)dF}$ is first order, and the $V'[F]$ operator will be zeroth order.

Let us recall a bit about the Ricci form: Let (M,ω,I_σ) be a Kähler manifold with the metric $g_\sigma(X,Y) = \omega(X,I_\sigma Y)$ and the induced Levi–Civita connections ∇. Then $\nabla I_\sigma = 0$. The curvature tensor is $R_\sigma(X,Y)(Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. This endormorphism satisfies $R_\sigma(X,Y)(I_\sigma Z) = I_\sigma(R(X,Y)(Z))$, and $R_\sigma(I_\sigma X, I_\sigma Y) = R_\sigma(X,Y)$. From this we can define the Ricci curvature $r_\sigma(X,Y) = \mathrm{Tr}(Z \mapsto R(Z,X)Y)$. Define $\mathrm{Ric}_\sigma(X,Y) = r(I_\sigma X, Y)$. This is the Ricci 1-1 form $\mathrm{Ric}_\sigma \in \Omega^{1,1}(M)$.

Exercise 36. Prove that the curvature of the induced connection in the canonical bundle $K_\sigma = \Lambda^{\mathrm{top}}T_\sigma$ of M is Ric_σ.

Recall the equation for u. On one side we had our first order operator $\Delta_{G(V)}$. This operator has symbol G and has to satisfy $\overline{\partial}_\sigma(G(V)_\sigma) = 0$. We write this as $G(V)_\sigma \in H^0(M_\sigma, S^2(T_\sigma))$. We say that the family of complex structures is *rigid*.

Theorem 37. *If we have that M is compact, that we have the above rigidity, that there exists n such that $n[\omega/2\pi] = c_1(M,\omega)$, and that $H^1(M,\mathbb{R}) = 0$ (which might not be necessary), then u as given before satisfies the equation $\frac{i}{2}V[I]_\sigma\nabla^{1,0}_\sigma s_\sigma + \nabla^{0,1}_\sigma u(V)_\sigma s_\sigma = 0$ for all vector fields V on $\mathcal{T}$ for all $\sigma \in \mathcal{T}$ and all $s_\sigma \in H^{(k)}_\sigma$.*

The proof of this theorem is the result of three lemmas.

Lemma 38. *We have*

$$\nabla^{0,1}(\Delta_{G(V)}(s_\sigma) + 2\nabla_{G(V)dF}(s_\sigma)) = -i(2k+n)\omega G(V)\nabla(s_\sigma) - 2ik(G(V)dF)\omega s - ik\delta_\sigma(G)\omega s,$$

where $\delta_\sigma : C^\infty(M,S^2(T)) \to C^\infty(M,T_\sigma)$ is given by $\delta_\sigma(G) = \mathrm{Tr}(\nabla G)$, and the right hand side above contains a bunch of contractions.

Note that we like the first term on the right hand side, since $\omega.G(V) = V'[I]$. The pre-factor here explains the corresponding one in the expression for u. Note also that we have an error term.

Lemma 39. *We have*

$$4i\overline{\partial}(V'[F]) = 2(G(V)dF)\omega + \delta_\sigma(G(V))\omega.$$

Lemma 40. *We have*

$$2(V'[\mathrm{Ric}])^{1,1} = \partial(\delta(G(V)).\omega)$$

Proof of Lemma 39 (assuming Lemma 38 and 40). This proof is where we will use that $H^1(M,\mathbb{R}) = 0$. On a general Kähler manifold, we have

$$\mathrm{Ric}_\sigma = n\omega + 2i\partial\overline{\partial}F_\sigma = n\omega + 2id\overline{\partial}F_\sigma.$$

Then

$$V'[\mathrm{Ric}]_\sigma = 2i\tfrac{1}{2}(\tfrac{i}{2}\partial V'[I]dF_\sigma + \partial\overline{\partial}V'[F]_\sigma).$$

Writing $d = \partial + \overline{\partial}$, and taking the (1,1) part of this, noting again that $V'[I]$ is a section of $\overline{T}_\sigma \otimes T_\sigma$, we obtain

$$(V'[\mathrm{Ric}]_\sigma)^{1,1} = -\partial V'[I]\partial F_\sigma + 2i\partial\overline{\partial}V'[F]_\sigma.$$

Using Lemma 40, we see from this that

$$\begin{aligned} 2i\partial\overline{\partial}V'[F] &= \partial V'[I]\partial F_\sigma + (V'[\mathrm{Ric}])^{1,1} \\ &= \partial V'[I]\partial F_\sigma + \tfrac{1}{2}\partial(\delta_\sigma(G(V))\omega). \end{aligned}$$

This expression is clearly ∂-closed, and it follows from Lemma 38 that it is also $\overline{\partial}$-closed (since the LHS and first term of the RHS is), so since $H^1(M,\mathbb{R}) = 0$, it equals df for some f, and since $df = \partial f + \overline{\partial}f$. Conjugating this expression, we obtain that $\overline{\partial}f = 0$, and using compactness, f is constant and thus df is 0, which gives us the expression of Lemma 39. □

10th lecture, February 23rd 2011

We are considering $\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty(M, \mathcal{L}^k)$ with $H^{(k)}_\sigma \subseteq \mathcal{H}^{(k)}_\sigma = C^\infty(M,\mathcal{L}^k)$. We are looking for the Hitchin connection $\boldsymbol{\nabla}^H_V = \boldsymbol{\nabla}^t_V - u(V)$. The equation on u was

$$\frac{i}{2}V[I]_\sigma\nabla^{1,0}s_\sigma + \nabla^{0,1}_\sigma u(V)_\sigma s_\sigma = 0,$$

for all vector fields V vector fields on $\mathcal{T}$, all $\sigma \in \mathcal{T}$ and all $s_\sigma \in H^{(k)}_\sigma$. Last time we were looking at

$$\begin{aligned} u(V) &= \frac{1}{k+n/2}o(V) - V'[F], \\ o(V) &= -\frac{1}{4}(\nabla_{G(V)} + 2\nabla_{G(V)dF} - 2uV'[F]), \end{aligned}$$

where n is such that $c_1(M,\omega) = n[\omega/2\pi] \in H^2(M,\mathbb{Z})$. We were requiring that $\overline{\partial}_\sigma(G(V)_\sigma) = 0$.

Proving that u satisfies the above equation was broken into the three lemmas 38, 39, and 40. Last time we saw how 39 followed from the two others.

Proof of Lemma 38. We have

$$\begin{aligned} \nabla^{0,1}(\mathrm{Tr}(\nabla^{1,0}G(V)\nabla^{1,0}s)) &= \mathrm{Tr}(\nabla^{0,1}\nabla^{1,0}G(V)\nabla^{1,0}s) \\ &= -ik\,\mathrm{Tr}(\omega G(V)\nabla^{1,0}s) - i\,\mathrm{Tr}(\mathrm{Ric}\,G(V)\nabla^{1,0}(s)) \\ &\quad + \mathrm{Tr}(\nabla^{1,0}\nabla^{0,1}G(V)\nabla^{1,0}s). \end{aligned}$$

Since the family $G(V)_\sigma$ was holomorphic, and since ω is parallel, the last term here becomes

$$\begin{aligned} \mathrm{Tr}(\nabla^{1,0}\nabla^{0,1}G(V)\nabla^{1,0}s) &= -ik\,\mathrm{Tr}(\nabla^{1,0}(G(V)\omega s)) \\ &= -ik\,\mathrm{Tr}(\nabla^{1,0}(G(V))\omega)s - ik\,\mathrm{Tr}(G(V)\omega\nabla^{1,0}s). \end{aligned}$$

Recall that $\mathrm{Ric} = n\omega + 2i\partial\overline{\partial}F$. We then see that

$$\begin{aligned} \nabla^{0,1}(\mathrm{Tr}(\nabla^{1,0}G(V)\nabla^{1,0}s)) &= -2ik\,\mathrm{Tr}(G(V)\omega\nabla^{1,0}s) - i\,\mathrm{Tr}(G(V)\,\mathrm{Ric}\,\nabla^{1,0}s) - ik\delta(G(V))\omega s \\ &= -i(2k+n)\,\mathrm{Tr}(G(V)\omega\nabla^{1,0}s) - ik\delta(G(V))\omega s + 2\,\mathrm{Tr}(G(V)\partial\overline{\partial}F)\nabla^{1,0}s. \end{aligned}$$

Note that $\nabla^{0,1}\nabla_{G(V)dF}s = G(V)\overline{\partial}\partial F\nabla^{1,0}s - ikG(V)dF\omega$. This implies the lemma. □

Proof of Lemma 40. We prove that $(V'[\mathrm{Ric}])^{1,1} = -\frac{1}{2}\partial(\operatorname{Tr}\nabla^{1,0}(G(V))\omega)$. The Kähler curvature is an element $R \in C^\infty(M, S^2(T \wedge \overline{T}))$. Then $\mathrm{Ric} = R(\omega) = R((\Lambda^2 i_g)^{-1}(\omega))$. We claim that $i_g = -Ii_\omega$. We have $\Lambda^2 i_g = \Lambda^2(Ii_\omega)$, and so $(\Lambda^2 i_g)^{-1}(\omega) = (\Lambda^2 i_\omega)^{-1}(\omega)$. It follows that $V'[\mathrm{Ric}] = V'[R]((\Lambda^2 i_\omega)^{-1}(\omega))$. By Theorem 1.174(c) of [Bes87], the variation of the curvature is given by

$$\begin{aligned} V'[R](X,Y,Z,U) =& \frac{1}{2}(\nabla^2_{Y,Z}(V'[g])(X,U) + \nabla^2_{X,U}(V'[g])(Y,Z) \\ &- \nabla^2_{X,Z}(V'[g])(Y,U) - \nabla^2_{Y,U}(V'[g])(X,Z) \\ &+ V'[g](R(X,Y)Z,U) - V'[g](R(X,Y)U,Z)), \end{aligned}$$

where

$$\nabla^2_{X,Y} = \nabla_X\nabla_Y - \nabla_{\nabla_X Y}$$

is function linear in the two variables. If we define $\tilde{G}(V) = G(V) + \overline{G(V)}$, we find that $V[g] = i_\omega \otimes i_\omega(\tilde{G}(V))$ for V a real vector field, and in particular $V[g]$ has pure type. If $V = V'$, $\tilde{G}(V') = G(V')$.(?) The Levi–Civita connection preserves types, and so does the curvature tensor. Plugging in X, Y, Z, U from $T, \overline{T}, T, \overline{T}$ respectively, the last two and first two terms of the above expression for $V'[R]$ will vanish. Thus

$$\begin{aligned} (V'[R]((\Lambda^2 i_\omega)^{-1})(\omega))^{1,1} =& -\frac{1}{2}(\operatorname{Tr}\nabla^{1,0}\nabla^{1,0}(V'[g]) \otimes (\Lambda^2 i_\omega)^{-1}(\omega) \\ &+ \operatorname{Tr}\nabla^{0,1}\nabla^{0,1}(V'[g]) \otimes (\Lambda^2 i_\omega)^{-1}(\omega)). \end{aligned}$$

From this the lemma follows. □

Remark 41. Several remarks are in order.

1. The condition $b_1(M) = 0$ would be nice to remove, but it hasn't been done.
2. The Fano condition $c_1(M, \omega) = m[\omega/2\pi]$ can be removed: One goes to half-form quantization/metaplectic quantization, where instead of using $\mathcal{L}^k$, we use $\mathcal{L}^k \otimes K^{-1/2}$, where $K = \Lambda^{\mathrm{top}} T*M$, and $(K^{-1/2})^{\otimes 2} = K^*$. A sqaure root of K does not necessarily exist, but it does when the second Stiefel–Whitney class $w_2(M)$ vanishes. Note that in this case, one has to come up with a reference connection in $\mathcal{L}^k \otimes K^{-1/2}$. The condition $c_1(M, \omega) = m[\omega/2\pi]$ implies that $K = \mathcal{L}^m$ as complex line bundles (not necessarily of Hermitian line bundles, although this can be obtained by multiplying the isomorphism by $e^{F/2}$).
3. What can we do if $\overline{\partial} G(V) \neq 0$. For example, this is the case for the universal bundle over Teichmüller space; here, however, the condition will be satisfied when going to the moduli space picture.
4. The moduli space of irreducible $SU(n)$ representations of $\pi_1(\Sigma)$, that is, the space $M = \mathrm{Hom}'(\pi_1(\Sigma), SU(n))/SU(n)$, is a smooth symplectic manifold, and the construction applies (a fact that is due to Hitchin).

In both the non-metaplectic and metaplectic case, we have the following:

Theorem 42. *The Hitchin connection $\boldsymbol{\nabla}^H$ is projectively flat. That is, there is a 2-form ω on $\mathcal{T}$ such that $\omega \otimes id = R_{\boldsymbol{\nabla}^H}$.*

If Γ is a group which acts on $(\mathcal{L}, \nabla, \langle,\rangle)$ and $\mathcal{T}$, such that I is Γ-equivariant. Then Γ acts projectively on the space $\mathbb{P}V_k$ of global covariantly constant sections of $(\mathbb{P}(H^{(k)}), \boldsymbol{\nabla}^H) \to \mathcal{T}$. That is, we obtain representations $\rho_k : \Gamma \to \mathrm{Aut}(\mathbb{P}V_k)$. In the moduli space case, these are the *quantum representations* of the mapping class groups $\Gamma_g = \mathrm{Diff}_+(\Sigma_g)/\mathrm{Diff}_0(\Sigma_g)$.

Recall that $\boldsymbol{\nabla}^H_V = \boldsymbol{\nabla}^t_V - u(V)$. Consider now $\pi^{(k)} : \mathcal{H}^{(k)} \to H^{(k)}$. If $s : \mathcal{T} \to H^{(k)}$ is a section, then $\boldsymbol{\nabla}^H_V s = \pi^{(k)}\boldsymbol{\nabla}^H_V s = \pi^{(k)}\boldsymbol{\nabla}^t_V s - \pi^{(k)}u(V)s$, so we conclude that $\boldsymbol{\nabla}^H_V - \boldsymbol{\nabla}^{L_2}_V = \pi^{(k)}u(V)$, and so $\pi^{(k)}u \in \Omega^1(\mathcal{T}, \mathrm{End}(H^{(k)}))$. For a vector field V, $u(V)_\sigma \in \mathrm{Diff}^2(M, \mathcal{L}^k)$, and $\pi^{(k)}u(V)_\sigma : C^\infty(M, \mathcal{L}^k) \to H^{(k)}_\sigma$.

Definition 43. If $D \in \mathrm{Diff}^{(l)}(M, \mathcal{L}^k)$, then we define the associated *Toeplitz operator* is $\pi_\sigma^{(k)} \circ D : C^\infty(M, \mathcal{L}^k) \to H_\sigma^{(k)}$.

In particular, we could consider operators D_σ parametrized by $\mathcal{T}$, and we would obtain sections of $\mathrm{End}(H^{(k)})$, and we could ask how these would relate to the Hitchin connection.

Definition 44. The *Toeplitz operator* $T_f^{(k)}$ associated to a function $f \in C^\infty(M)$ (corresponding to zero order differential operators) is the operator $T_{f,\sigma}^{(k)} = \pi_\sigma^{(k)} \circ M_f$, where $M_f(s) = f \cdot s$.

Recall that the operator norm on $\mathrm{End}(H_\sigma^{(k)})$ is given by $\|A\| = \sup_{|s_\sigma|=1} |As_\sigma|$ for $A \in \mathrm{End}(H_\sigma^{(k)})$.

Theorem 45 (BMS)**.** *The operator norms of the Toeplitz operators satisfy*

$$\lim_{k\to\infty} \|T_{f,\sigma}^{(k)}\| = \sup_{x\in M} |F(x)|.$$

This also implies that the map $f \mapsto (T_{f,\sigma}^{(k)})$ is faithful, since $(f-g) \mapsto (T_{(f-g),\sigma}^{(k)}) = T_{f,\sigma}^{(k)} - T_{g,\sigma}^{(k)} = 0$ implies that $f = g$.

5 Student seminars

5.1 Jens Kristian

Jens Kristian wrote his own notes on geometric quantization; prequantization, polarization, and metaplectic correction:

5.1.1 Goal

Given a symplectic manifold (M, ω) we want to associate a Hilbert space $\mathcal{H}$ and and for (some) $f \in C^\infty(M, \mathbb{R})$ an operator $\hat{f}$ on $\mathcal{H}$ such that:

1. The map $f \mapsto \hat{f}$ is linear over $\mathbb{R}$.
2. If f is constant of value a, $\hat{f} = a\mathrm{id}$.
3. $i\hbar\widehat{f} = [\hat{f}_1, \hat{f}_2]$ for $f = \{f_1, f_2\}$.

Remark 46. Let $M = T^*\mathbb{R}$ and $\omega = dq \wedge dp$ the standard symplectic form. This is the phase space for the harmonic oscillator and the functions $p, q : M \to \mathbb{R}$ corresponds to position and momentum. The hamiltonien vector field associated to p is $X_p = 2\frac{\partial}{\partial q}$, so the poisson bracket of the position and momentum observables is: $\{p, q\} = -dq(X_p) = -2$. By (2) and (3) from above, this tells us that $[\hat{p}, \hat{q}] = i\hbar\widehat{(-2)} = -2i\hbar\mathrm{id}$, ie. that position and momemtum doesn't commute in the quantization.

5.1.2 Prequantization

A prequantization of a symplectic manifold (M, ω) is a complex line bundle $B \to M$ with hermitian structure, and a connection ∇_B with curvature $F^{\nabla_B} = i\frac{\omega}{\hbar}$ (this makes sense: The curvature of a connection in a vector bundle is a two-form with coeffecients in the endomorphism bundle; but for a line bundle the endomorphism bundle is just $\mathbb{C}$, so the curvature is a complex-valued 2-form on M). The cohomology class of the curvature is the first chern class of the line bundle, which is an element of $H^2(M, 2\pi i\mathbb{Z})$. So we can hope to prequantize a symplectic manifold if $\frac{\omega}{\hbar} \in H^2(M, 2\pi\mathbb{Z})$. In fact, this is the only obstruction:

Theorem 47. *For all closed two-forms ω such that $[\omega] \in H^2(M, 2\pi i\mathbb{Z})$ there exist a hermitian line bundle $B \to M$ and a connection ∇_B in B compatible with the metric, such that $\omega = F^{\nabla_B}$.*

Proof. Let $\{U_j\}$ be an open cover of M consisting of open contractible sets with contractible intersections. For each j, choose a 1-form $\theta \in \Omega^1(U_j)$, such that $d\theta_j = \omega|_{U_j}$, which is possible since $H^2(U_j, \mathbb{R}) = 0$ and ω is closed. For each par (i, j) such that $U_{ij} = U_i \cap U_j \neq \emptyset$ let $f_{ij} : U_{ij} \to \mathbb{R}$ be a smooth function such that $df_{ij} = \Theta_i - \Theta_j$ on U_{ij} which is possible since $\Theta_i - \Theta_j$ is closed. Let δ denote the Chech differential. We see that $d(\delta f)_{ijk} = 0$ and thus $(\delta f)_{ijk}$ is constant, and therefore in the check cohomology group of locally constant functions. But since its closed, and the cohomology group is 0, this implies that there exist g_{ij} locally constant, such that $\delta g = \delta f$. Let $u_{ij} = f_{ij} - g_{ij}$. Then $du_{ij} = \Theta_i - \Theta_j$ and $\delta u_{ij} = 0$. Let $c_{ij} = e^{u_{ij}}$, and we see that $dc_{ij} = d(u_{ij})c_{ij} = (\Theta_i - \Theta_j)c_{ij}$. The c_{ij}'s define a Check 2-cocycle and therefore they are the transistions functions for a line bundle B on M: $c_{ij}c_{ji} = 1$ and $c_{jk}c_{ik}^{-1}c_{ij} = e^{u_{jk}-u_{ik}+u_{ij}} = e^{(\delta u)_{ijk}} = 1$. In this line bundle we can define a connection as $\nabla_X s_i = \Theta(X)s_i$ where s_i is the constant section 1 in the trivialization over U_i, and it follows that the curvature $F^\nabla = d\Theta = \omega$. □

Assume now that $B \to M$ is a prequantum line bundle for (M, ω) with connection ∇. We can define a inner product of the space of smooth sections of B by:

$$\langle s, s' \rangle = \int_M (s, s')\varepsilon$$

where $\varepsilon = \frac{\omega^n}{n!}$ is the volume form from from the symplectic form. The prequantization Hilbert space $L^2(B, \varepsilon)$, the completion of the space of square integrable section of B wrt. ε. We define, for $f \in C^\infty(M)$:

$$\hat{f} \cdot s = -i\hbar\nabla_{X_f} s + fs.$$

symmetric:

$$\begin{aligned}
\langle \hat{f}s, s' \rangle &= \int_M (\hat{f}s, s')\varepsilon \\
&= \int_M (-i\hbar\nabla_{X_f} s, s') + (fs, s')\varepsilon \\
&= \int_M -i\hbar(\nabla_{X_f} s, s') + f(s, s')\varepsilon \\
&= \int_M -i\hbar(-(s, \nabla_{X_f} s') + X_f(s, s')) + (s, fs')\varepsilon \\
&= \int_M (s, -i\hbar\nabla_{X_f} s' + fs') - i\hbar d((s, s'))(X_f)\varepsilon \\
&= \int_M (s, \hat{f}s') - i\hbar d((s, s'))(X_f)\varepsilon \\
&= \int_M (s, \hat{f}s')\varepsilon - i\hbar \int_M d((s, s'))(X_f)\varepsilon \\
&= \langle s, \hat{f}s' \rangle
\end{aligned}$$

The last equality follows since

$$\begin{aligned}
\int_M d((s, s'))(X_f)\frac{\omega^n}{n!} &= \int_M \left(\mathcal{L}_{X_f}(s, s')\right)\frac{\omega^n}{n!} \\
&= \int_M \mathcal{L}_{X_f}\left((s, s')\frac{\omega^n}{n!}\right) - \int_M (s, s')\mathcal{L}_{X_f}\left(\frac{\omega^n}{n!}\right) \\
&= 0 + 0
\end{aligned}$$

Because the integral of the Lie derivative of a top form over a manifold without boundary is 0, from Cartans formula, and because the vector field preserves the ω. If f is constant, $X_f = 0$, and

thus $\hat{f}$ becomes the appropiate multiplication operator.

$$\begin{aligned}\hat{f}\hat{g} &= \hat{f}(-i\hbar\nabla_{X_g}s + gs)\\ &= -\hbar^2\nabla_{X_f}\nabla_{X_g}s - fi\hbar\nabla_{X_g}s - i\hbar\nabla_{X_f}gs + fgs\\ &= -\hbar^2\nabla_{X_f}\nabla_{X_g}s - fi\hbar\nabla_{X_g}s - i\hbar X_f gs - i\hbar g\nabla_{X_f}s + fgs\\ &= -\hbar^2\nabla_{X_f}\nabla_{X_g}s - fi\hbar\nabla_{X_g}s - i\hbar\omega(X_g, X_f)s - i\hbar g\nabla_{X_f}s + fgs\end{aligned}$$

We can calculate the commutator:

$$\begin{aligned}\hat{f}\hat{g} - \hat{g}\hat{f} &= i\hbar\left(i\hbar(\nabla_{X_f}\nabla_{X_g}s - \nabla_{X_f}\nabla_{X_g}s) + 2\{f,g\}s\right)\\ &= i\hbar\left(i\hbar\nabla_{[X_f,X_g]}s + i\hbar F^{\nabla}(X_f,X_g)s + 2\{f,g\}s\right)\\ &= i\hbar\left(i\hbar\nabla_{[X_f,X_g]}s + i\hbar\frac{i}{\hbar}\{f,g\} + 2\{f,g\}\right)\\ &= i\hbar(\nabla_{[X_f,X_g]}s + \{f,g\}s)\\ &= i\hbar(i\hbar\nabla_{\{f,g\}}s + \{f,g\}s)\\ &= i\hbar\widehat{\{f,g\}}s\end{aligned}$$

There is however a problem. When we try to quantize the simplest possible system, $T^*\mathbb{R}$, the harmonic oscillator, we obtain $\hat{p} = -\hbar\frac{\partial}{\partial q}$ and $\hat{q} = i\hbar\frac{\partial}{\partial p} + q$ – an operator that does depend on both position and momentum, which it shouldn't because of physics. This leads us to consider polarization, as it allows us to restrict the numbers of variables the observarbles can depend upon.

5.1.3 Polarization

The Hilbert space $\mathcal{H}$ we obtained from prequantization is too big. We do this by require that the sections should be constant along some distribution in the tangent bundle of M. We will only discuss complex polarizations:

Definition 48. A complex polarization P is a distribution in the complexified tangent bundle of M satisfying:

1. For each $p \in M$: $P_p \subset (T_pM)_{\mathbb{C}}$ is a Lagrangian subspace
2. The dimension of $D = P \cap \overline{P} \cap TM$ is constant
3. P is integrable

P is called a Kähler polarization if $P \cap \overline{P} = 0$.

Thus every Kähler polarization determines a Kähler structure on M, compatible with ω, and a Kähler structure determines a Kähler polarization, by choosing P to be the i-eigenspace of $TM_{\mathbb{C}}$, and we call this the holomorphic Kähler polarization, and its conjugate for the antiholomorphic polarization. Given a polarization P and a prequantum line bundle $B \to M$, we define $\mathcal{H}_P \subseteq \mathcal{H}$ to be the completion of the space of square integrable sections s such that $\nabla_{\overline{X}}s = 0$ for all $X \in P$ – such sections are called polarized.

Two nonvanishing polarized sections s, s' are related by a function $\varphi : M \to \mathbb{C}$ by $s = \varphi s_2$. Thus: $0 = \nabla_{\overline{X}}s = \nabla_{\overline{X}}\varphi s' = \overline{X}\varphi s + \nabla_{\overline{X}}s = \overline{X}\varphi$ for every $X \in P$. For the holomorphic polarization of a Kähler manifold, this tells us that φ are holomorphic. We can define local trivilizations of B that are polarized sections, and thus the transitions functions becomes holomorphic, and thus gives a holomorphic structure on the prequantum line bundle B, and the polarized sections becomes the holomorphic sections. It can be shown that the space of square integrable holomorphic sections is closed, and thus we dont need to pass to the completion.

We still define $\hat{f}s = -i\hbar\nabla_{X_f}s + fs$, but there is a problem: for $\hat{f}$ to act on $\mathcal{H}_P$, we must have that $\nabla_{\overline{X}}\hat{f}s = 0$ for all $s \in \mathcal{H}_P$ and $X \in P$:

$$\begin{aligned}
0 = \nabla_{\overline{X}}\hat{f}s &= \nabla_{\overline{X}}\left(-i\hbar\nabla_{X_f}s + fs\right) \\
&= -i\hbar\nabla_{\overline{X}}\nabla_{X_f}s + df(\overline{X})s + f\nabla_{\overline{X}}s \\
&= -i\hbar\left(\nabla_{X_f}\nabla_{\overline{X}} + \nabla_{[\overline{X},X_f]}s - \Omega(X_f,\overline{X})\right)s + df(\overline{X})s + f\nabla_{\overline{X}}s \\
&= -i\hbar\left(\nabla_{X_f}\nabla_{\overline{X}} + \nabla_{[\overline{X},X_f]}s - \frac{i}{\hbar}\omega(X_f,\overline{X})\right)s + \omega(f,\overline{X})s + f\nabla_{\overline{X}}s \\
&= \hat{f}\nabla_{\overline{X}}s - i\hbar\nabla_{[\overline{X},X_f]}s
\end{aligned}$$

We can be sure this is true if $[\overline{X}, X_f] \in \overline{P}$ or, as $X_f = \overline{X_f}$, that $[X, X_f] \in P$ for all $X \in P$ – ie, that the flow of f preserves P. This is a big restriction on the class of quantizable observables: If $z^1, \cdots, z^n$ and $\overline{z}^1, \cdots, \overline{z}^n$ are local coordinates on a Kähler manifold with Kähler polarization $P = \text{span}\{\frac{\partial}{\partial z^1}, \cdots \frac{\partial}{\partial z^n}\}$ we get:

$$\begin{aligned}
\mathcal{L}_{X_f}\left(\frac{\partial}{\partial z^j}\right) = \left[X_f, \frac{\partial}{\partial z^j}\right] &= \frac{\partial}{\partial z^j}X_f \\
&= 2i\left(\frac{\partial^2 f}{\partial z^j \partial \overline{z}^k}\right)\frac{\partial}{\partial z^k} - 2i\left(\frac{\partial^2 f}{\partial z^j \partial z^k}\right)\frac{\partial}{\partial \overline{z}^k}
\end{aligned}$$

This implies that $\frac{\partial^2 f}{\partial z^i \partial z^j} = 0$. Since f is real-valued, we also have that $\frac{\partial^2 f}{\partial \overline{z}^i \partial \overline{z}^j} = 0$. So the only real-valud functions on of the local coordinates whose flow preserves P are of the form

$$f = w_i z^i - \overline{w}_i z^i + W_{ij} z^i \overline{z}^j + c$$

where w_i, x are complex constants and W_{ij} entries in a hermitian matrix.

Example 49 (The harmonic oscillator)**.** The state space for the harmonic oscillator is just $\mathbb{R}$ and the phase space is $M = T^*\mathbb{R}$ with symplectic form $\omega = \frac{i}{2}dz \wedge d\overline{z}$ where $z = \frac{1}{2}(p + iq)$, and the Hamiltonian is $H = \frac{1}{2}z\overline{z}$. This is a Kähler manifold with holomorphic Kähler polarization $P = \mathbb{C}\frac{\partial}{\partial z}$. Since the phase space is contractible, $H^2(M, \mathbb{Z}) = 0$ so any line bundle can be used as a prequantum line bundle, so lets use $M \times \mathbb{C}$. We need to define a metric and a connection in this line bundle. It is enough to make a choice for a singe nonvanishing section – ie. $s(z, \overline{z}) = 1$, the constant section. Define $\nabla_X s = \frac{i}{\hbar}\Theta(X)s$ where $\Theta = \frac{i}{2}(zd\overline{z} - \overline{z}dz)$ is the symplectic potential – this connection has curvature $\frac{i}{\hbar}d\Theta = \frac{i}{\hbar}\omega$. Let the hermitian product be given by $(s, s) = 1$. This is compatible with the connection:

$$\begin{aligned}
(\nabla_X s, s) + (s, \nabla_X s) &= (\frac{i}{\hbar}\Theta(X)s, s) + (s, \frac{i}{\hbar}\Theta(X)s) \\
&= \frac{1}{\hbar}(i - i)\Theta(X)(s, s) = 0 = X1 \\
&= X(s, s)
\end{aligned}$$

Assume φs is a polarized, nonvanishing section:

$$\begin{aligned}
0 = \nabla_{\frac{\partial}{\partial \overline{z}}}\varphi s &= d\varphi(\frac{\partial}{\partial \overline{z}})s + \varphi\nabla_{\frac{\partial}{\partial \overline{z}}}s \\
&= \left(\frac{\partial}{\partial \overline{z}}\varphi\right)s + \varphi\frac{-i}{\hbar}\Theta(\frac{\partial}{\partial \overline{z}})s \\
&= \left(\frac{\partial}{\partial \overline{z}}\varphi\right)s + \varphi\frac{-i}{\hbar}\frac{i}{2}zs
\end{aligned}$$

So

$$-\frac{z}{2\hbar} = \frac{\frac{\partial}{\partial \overline{z}}\varphi}{\varphi}$$

So we see that $\varphi s = e^{-\frac{z\overline{z}}{2\hbar}} s$ is such a section. All sections f can be written as $f = \psi\varphi s$. Assume f is polarized and compute:

$$0 = \nabla_{\frac{\partial}{\partial \overline{z}}} \psi\varphi s = (\frac{\partial}{\partial \overline{z}}\psi)\varphi s + \psi \nabla_{\frac{\partial}{\partial \overline{z}}} \varphi s = (\frac{\partial}{\partial \overline{z}}\psi)\varphi s$$

Since φs is nonvanishing, this implies that ψ is a holomorphic function. On the other hand its clear that $\psi\varphi s$ is polarized when ψ is holomorphic. Thus we have identified the space of holomorphic sections with the space of holomorphic functions on M. Now we want to identify the square integrable ones. $(\psi\varphi s, \psi\varphi s) = |\psi|^2\varphi^2(s,s) = |\psi|^2 e^{-|z|^2/\hbar}$ so we can identify $\mathcal{H}_P$ with the space of holomorphic functions integrable with respect to the gaussion measure scaled by $\hbar$.

We want to quantize the functions z and $\overline{z}$ (for some reason we are suddely considering complex functions –why?). We find that $X_z = -i\frac{\partial}{\partial \overline{z}}$ and $X_{\overline{z}} = i\frac{\partial}{\partial z}$. Thus we optain the operators:

$$\begin{aligned}\hat{z}s &= -i\hbar\nabla_{X_z} s + zs = -\hbar\nabla_{\frac{\partial}{\partial \overline{z}}} s + zs \\ &= zs\end{aligned}$$

Since s is a polarized section, and if ψ is a holomorphic function:

$$\begin{aligned}\hat{\overline{z}}\psi\varphi s &= -i\hbar\nabla_{X_{\overline{z}}}\psi\varphi s + \overline{z}s = \hbar\nabla_{\frac{\partial}{\partial z}}\psi\varphi s + \overline{z}\psi\varphi s \\ &= \hbar\frac{\partial\psi}{\partial z}\varphi s + \psi\hbar\left(\nabla_{\frac{\partial}{\partial z}}\varphi s\right) + \overline{z}\psi\varphi s \\ &= \hbar\frac{\partial\psi}{\partial z}\varphi s + \psi\hbar\left(\frac{\partial\varphi}{\partial z}s + \frac{i}{\hbar}\Theta(\frac{\partial}{\partial z})\varphi s\right) + \overline{z}\psi\varphi s \\ &= \hbar\frac{\partial\psi}{\partial z}\varphi s + \psi\hbar\left(-\frac{1}{2\hbar}\varphi\overline{z}s + \frac{-1}{2\hbar}\overline{z}\varphi s\right) + \overline{z}\psi\varphi s \\ &= \hbar\frac{\partial\psi}{\partial\varphi}s\end{aligned}$$

(I should get a factor of 2 more). Thus, under our identification of polarized sections with holomorphic functions, $\hat{z}f = zf$ and $\hat{\overline{z}}f = \hbar\frac{\partial f}{\partial z}$

The Hamiltonian vector field for the Hamiltonian is $X_H = \frac{1}{2}\left(zX_{\overline{z}} + \overline{z}X_z\right)$ and the operator corresponding to the Hamiltonian is:

$$\begin{aligned}\hat{H}\psi\varphi s &= -i\hbar\nabla_{X_H}\psi\varphi s + H\psi\varphi s \\ &= -iz\frac{\hbar}{2}\nabla_{X_{\overline{z}}}\psi\varphi s + -i\frac{\hbar}{2}\overline{z}\nabla_{X_z}\psi\varphi s + H\psi\varphi s \\ &= \frac{\hbar}{2}z\frac{\partial\psi}{\partial z}\varphi s - \frac{z\overline{z}}{2}\psi\varphi s + H\psi\varphi s \\ &= \frac{\hbar z}{2}\frac{\partial\psi}{\partial z}\varphi s\end{aligned}$$

We see that the sections corresponding to z^n is the eigenstates for the Hamiltonian. From a physics point of view, $\hat{H}$ is not the right quantized Hamiltonian: it should be $\hat{H} + \frac{\hbar}{2}$. The solution to this is metaplectic correction.

5.1.4 Metaplectic correction

Let δ_P be a square root of the canonical bundle of M with the complex structure from the holomorphic Kähler polarization P. We can now define the Hilbert space as $\mathcal{H}_P^{\delta_P} = \mathcal{H} \otimes \delta_p$ – half–forms

with coeffecients in the Hilbert space of square integrable polarized sections of B. We now define the quantization of observables as $\tilde{f}(s\otimes\nu) = (\hat{f}s)\otimes\nu - i\hbar s\otimes\mathcal{L}_{X_f}\nu$, where the Lie derivative of a half-form is defined such that $\mathcal{L}_X\delta(\nu\otimes\nu) = 2\nu\mathcal{L}_X\nu$, where $\delta : \delta_P^{\otimes 2} \to K_p$ is the isomorphism. For an operator of the form (5.1.3) we can calculate $\tilde{f}$. Let $\mu = dz^1\wedge dz^2\wedge\cdots\wedge dz^n$ – since this is nonvanishing, all forms in the canonical bundle can be expressed as the product of this and a holomorpic function. Let ν be a half form that is the square root of μ. We compute:

$$\begin{aligned}\mathcal{L}_{X_f}\mu_0 = d(i_{X_f}\mu) &= 2id\left(\frac{\partial f}{\partial\overline{z}^i}i_{\frac{\partial}{\partial z}}\mu\right)\\ &= \overline{w}^i d\left(i_{\frac{\partial}{\partial z^i}}\mu\right) + id\left(W_{ij}z^j i_{\frac{\partial}{\partial z^i}}\mu\right)\\ &= i(W_{ij}\delta^{ij})\mu = i\operatorname{tr}W\mu\end{aligned}$$

So $\mathcal{L}_{X_f}\nu = \frac{i}{2}\operatorname{tr}W\nu$, and

$$\tilde{f}(s\otimes\nu) = (\hat{f}s)\otimes\nu - i\hbar s\otimes\frac{i}{2}\operatorname{tr}W\nu = (\hat{f}s + \frac{\hbar}{2}\operatorname{tr}Ws)\otimes v$$

For the hamiltonian of the harmonic oscillator, this is indeed what we want.

5.2 Jens-Jakob

This talk is on Edward Witten's *Supersymmetry and Morse theory*, [Wit82].

We will show how to derive *Morse inequalities*.

We start with a quantum field theory: We have a Hilbert space $\mathcal{H}$, which we decompose into two parts $\mathcal{H} = \mathcal{H}^+\oplus\mathcal{H}^-$ called the *bosonic* and *fermionic* states respectively.

Definition 50. Supersymmetry consists of the following. Consider operators Q_i, $Q_i : \mathcal{H}^+\oplus\mathcal{H}^- \to \mathcal{H}^-\oplus\mathcal{H}^+$, $i = 1,\dots,N$. Define an operator $(-1)^F$, which can detect whether $\psi\in\mathcal{H}$ is in $\mathcal{H}^+$ or $\mathcal{H}^-$, i.e. $(-1)^F(\psi) = -\psi$ for $\psi\in\mathcal{H}^-$, and $(-1)^F(\psi) = \psi$ for $\psi\in\mathcal{H}^+$. We have a couple of basic conditions for the Q_i's. Namely

1. $Q_i(-1)^F + (-1)^F Q_i = 0$.
2. If H denotes the Hamiltonian, then $Q_iH - HQ_i = 0$.
3. We have the following algebraic structure: $H = Q_i^2$ for all i, and if $i\neq j$, we have $Q_iQ_j + Q_jQ_i = 0$.

One example of the above would be letting $\mathcal{H}$ being the forms on a manifold splitting into those with odd or even parity, and the Q_i would be taking the wedge product with some 1-form and $H = 0$.

We change now the algebraic structure, going to the relativistic world. We assume that we have 1 time dimension, and so we have a momentum operator P. If $N = 2$, we require now that $Q_1^2 = H + P$, $Q_2^2 = H - P$ and $Q_1Q_2 + Q_2Q_1 = 0$. It follows that $2H = Q_1^2 + Q_2^2$, and $2P = Q_1^2 - Q_2^2$. We also have $[Q_i, H] = [Q_i, P] = 0$: For example, $2HQ_i = (Q_1^2+Q_2^2)Q_1 = Q_1^3 + Q_2^2Q_1 = Q_1^3 + Q_1Q_2^2 = 2Q_iHH$. The Hamiltonian is positive semi-definite, $H = \frac{1}{2}(Q_1^2+Q_2^2)$. Also, $[H,(-1)^F] = [P,(-1)^F] = 0$. We conclude that the basic conditions of the definition of supersymmetry are still satisfied by these Q_i and H.

Witten now asks the following question: Does there exist a state $|\Omega\rangle\in\mathcal{H}$ such that $Q_i|\Omega\rangle = 0$, meaning that Ω is a zero-energy state. From a physical point of view, this would mean that we have fermions and bosons of equal mass.

If $Q_i|\Omega\rangle = 0$, then $P|\Omega\rangle = 0$. Consider the subspace $\mathcal{H}_0\subseteq\mathcal{H}$of states annihilated by P. This means that we are back to the setup of supersymmetry in the definition above. Thus, $Q_i^2 = H$ for all (i.e. both) i, and $Q_1Q_2 + Q_2Q_1 = 0$. For $|\Omega\rangle\in\mathcal{H}_0 = \mathcal{H}_0^+\oplus\mathcal{H}_0^-$ annihilated by both Q_i, i.e. $Q_i|\Omega\rangle = 0$, and we simply write $Q = Q_i$ for some i. We want to know if $Q|_{\mathcal{H}_0}$ has any zero

eigenvalue, and we write $Q = Q^+ + Q^-$. This will be the case if the index of Q^+, which is the trace of $(-1)^F$ on $\mathcal{H}^+$ is non-zero or something like that.

Morse theory gives a relation between the topology of a compact manifold M and critical points of a Morse function $f : M \to \mathbb{R}$. One way of describing these relations is by Morse inequalities.

Definition 51. Given a field F and a natural number $k \in \mathbb{N}$, the *k'th Betti number* of M is the rank of $H_k(M, F)$.

The weak Morse inequality is given as follows: If we let C_k denote the number of critical points with index k, then $B_k(M) \leq C_k$. Here, the *index* of a critical point is the number of negative eigenvalues of the Hessian in the given point.

We will try to prove the weak Morse inequality using supersymmetry considerations.

Sketch of proof. Let M be a Riemannian manifold, $\dim_{\mathbb{R}}(M) = n$. Let $\Omega^p(M)$ denote the space of p-forms on M, $p = 1, \ldots, n$, and let d be the usual exterior derivative, $d : \Omega^p(M) \to \Omega^{p+1}(M)$, $dd = 0$. Let d^* be the adjoint of d: The Riemannian metric $\gamma \in T^2(M)$ induces an inner product $\langle , \rangle$ on T_pM, which gives an inner product $\langle , \rangle_p$ on Ω^p, and the adjoint satisfies

$$\int_M \langle d\alpha, \beta \rangle_{p+1} dV = \int_M \langle \alpha, d^*\beta \rangle_p dV.$$

This implies that $d^*d^* = 0$. Now define $Q_1 = d + d^*$ and $Q_2 = i(d - d^*)$ with $H = dd^* + d^*d$. We obtain that $Q_1^2 = Q_2^2 = H$ and $Q_1Q_2 + Q_2Q_1 = 0$.

Let h be a Morse function, $h : M \to \mathbb{R}$, and take $t \in \mathbb{R}$. Define $d_t = e^{-ht}de^{ht}$ and $e_t^* = e^{ht}d^*e^{-ht}$. These are still differentials. Define $Q_1^t = d_t + d_t^*$, $Q_2^t = i(d_t - d_t^*)$, and $H_t = d_td_t^* + d_t^*d_t$. Define the Betti number $B_p(M, t)$ to be the number of linearly independent p-forms ψ such that $d_t\psi = 0$ and ψ cannot be written as d_tx, $\psi \in \Omega^p$, $x \in \omega^{p-1}$. It is now a fact that $B_p(M, t) = B_p(M)$. It follows that the number of zero eigenvalues of H_t acting on p-forms equals the Betti number B_p.

We want to find an upper bound on $B_p(M)$. We do this by looking at the spectrum of H_t. For a point $\tilde{p} \in M$, we choose an orthornormal basis of tangent vectors $a^k(\tilde{p})$. Consider these as operators on the exterior algebra (by inner multiplication). The adjoint operators to these are the 1-forms $a^k(\tilde{p})^*$ acting by exterior multiplication. Direct computation gives that

$$H_t = dd^* + d^*d + t^2(dh)^2 + \sum_{i,j} \frac{D^2h}{D\varphi^i D\varphi^j}[(a^i)^*, a^j],$$

where the D denotes the covariant derivative, and $(dh)^2 = \gamma^{i,j}(\partial h/\partial\varphi^i)(\partial h/d\varphi^j)$ is the square of ∇h evaluated with respect to the Riemannian metric.

We want an asymptotic expansion of $1/t$ in terms of local data around the critical points. Since h is Morse, (dh) is zero only at isolated points of M. Let M_p be the number of critical points of index p. We then have an expansion

$$\lambda_p^{(n)}(t) = t\left(A_p^{(n)} + \frac{B_p^{(n)}}{t} + \frac{C_p^{(n)}}{t^2} + \cdots\right),$$

where $\lambda_p^{(n)}(t)$ is the n'th smallest eigenvalue of H_t. Now $B_p^{(n)}$ is the number of $\lambda_p^{(n)}(t)$ that are zero. For large t, the number of $\lambda_p^{(n)}$ that vanish is no larger than $A_p^{(n)}$. We need to show that the number of $A_p^{(n)}$ that vanish is equal to M_p. For t large, the low-lying (whatever that means) eigenvalues of H_t can be calculated about critical points x of h.

Around these critical points, we choose coordinates φ_i such that $\varphi_i(x) = 0$, $h(\varphi_i) = h(0) + \frac{1}{2}\lambda_i\varphi_i^2 + O(\varphi^3)$ for some real numbers λ_i. Whether these are positive or not depends on the index of the critical point. Near these points, we approximate H_t with

$$\begin{aligned}\overline{H}_t &= \sum_i \left(-\frac{\partial^2}{\partial(\varphi^i)^2} + t^2\lambda_i^2\varphi_i^2 + t\lambda_i[(a^i)^*, a^j]\right) \\ &= \sum_i H_i + t\lambda_i K_i,\end{aligned}$$

where $H_i = -\frac{\partial^2}{\partial(\varphi^i)^2} + t^2\lambda_i^2\varphi_i^2$ and $K_i = [(a^i)^*, a^j]$. Then H_i and K_i mutually compute, and they can be diagonalized simultaneously. The H_i are just the Hamiltonians for the harmonic oscilator. The eigenvalues of the H_i's are $t|\lambda_i|(1+2N_i)$, where $N_i = 0, 1, 2, \ldots$. Eigenvalues of K_i are ± 1, so the eigenvalues for H_t will be $t\sum_i |\lambda_i|(1+2N_i) + \lambda_i n_i$, $N_i = 0, 1, 2, \ldots$, and $n_i = \pm 1$. When H_t acts on p-forms, we need at least p positive n_i's.

If this expression equals 0, we must choose $N_i = 0$, and n_i is positive if and only if λ_i is negative. Expanding around a critical point, we see that there is precisely one zero eigenvalue, and the eigenvector is a p-form only if the critical point has index p. We can get the leading coefficients of the $A_p^{(n)}$'s around a critical point. Consider the A coming from considering all critical points. Now $\overline{H}_t$ has one eigenstate $|a\rangle$, which is a p-form if A has Morse index p, which means that $M_p \geq B_p$.

□

Another result is that

$$\sum M_p t^p - \sum B_p t^p = (1+t)\sum Q_p t^p,$$

where the Q_p are nonnegative integers. At this point, we don't have time to prove that.

5.3 Søren

The reference for the following will be [Tur94]. We will describe how to obtain operator invariants from ribbon graphs. These will be used to create an invariant of closed 3-manifolds via the process of surgery. In the second part (Jakob's part) we will see how this ties together with the notion of TQFT.

5.3.1 Ribbon categories and graphical calculus

In order to obtain our operator invariants, we first set up the relevant categorical framework. The notion of a *ribbon category* will encompass the structures relevant for our constructions.

Definition 52. A *ribbon category* is a monoidal category $\mathcal{V}$ with a braiding c, a twist θ, and a compatible duality $(*, b, d)$.

Let us discuss these concepts one at a time.

Definition 53. A *monoidal category* is a category $\mathcal{V}$ with a covariant functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ associating to two objects V and W an object $V \otimes W$ and to morphisms $f : V \to V'$ and $g : W \to W'$ a morphism $f \otimes g : V \otimes W \to V' \otimes W'$, such that the following properties are satisfied:

1. There exists a unit object $\mathbf{1}$ satisfying $V \otimes \mathbf{1} = V$, and $\mathbf{1} \otimes V = V$ for all objects V in $\mathcal{V}$.
2. For triples U, V, W of objects, we have $(U \otimes V) \otimes W = U \otimes (V \otimes W)$.
3. For morphisms f in $\mathcal{V}$, $f \otimes \mathrm{id}_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}} \otimes f = f$.
4. For triples f, g, h of morphisms, we have $(f \otimes g) \otimes h = f \otimes (g \otimes h)$.

Remark 54. The above definition is really that of a strict monoidal category. More generally one could consider categories, where the above equalities are replaced by fixed isomorphisms. By general abstract nonsense, nothing is lost by requiring equality, and we will avoid the more general case completely.

The connection with ribbon graphs, as will be explained later, is best understood in terms of graphical calculus, which will be introduced now and discussed in parallel with the axioms of a ribbon category. A morphism $f : W_1 \otimes \cdots \otimes W_m \to V_1 \otimes \cdots \otimes V_n$ in $\mathcal{V}$ is represented by a diagram with a box with several downward arrows as in Fig. 7. The cases where either m or n is 0 (or both are) will be allowed as well; here, the corresponding tensor product will be $\mathbf{1}$ by

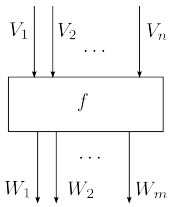


Figure 7: A morphism represented by a diagram.

convention. Composing morphisms is represented by stacking diagrams on top of each other, and tensoring is represented by placing them next to each other; an example is illustrated in Fig. 8 where $g : U \to W'$ and $f : W \otimes W' \to V$ are two morphisms, and the diagram represents the morphism $f \circ (\mathrm{id}_W \otimes g) : W \otimes U \to V$.

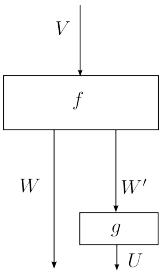


Figure 8: Composition and tensor products of morphisms.

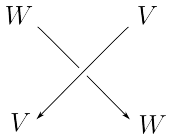


Figure 9: The braiding isomorphisms $c_{V,W}$.

Definition 55. A *braiding* in $\mathcal{V}$ is a family $c = \{c_{V,W} : V \otimes W \to W \otimes V\}$ of isomorphisms, represented in graphical calculus by the diagram in Fig. 9, satisfying the following properties, where U, V, W, V', W' are objects of $\mathcal{V}$, and $f : V \to V'$, $g : W \to W'$ are morphisms:

1. $c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W)$.
2. $c_{U\otimes V,W} = (c_{U,V} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W})$.
3. $(g \otimes f)c_{V,W} = c_{V',W'}(f \otimes g)$.

Example 56. It follows from the definitions that

$$(\mathrm{id}_W \otimes c_{U,V})(c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}) = (c_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W).$$

This isn't obvious from the algebraic statements above but is a natural consequence of the existence of graphical calculus.

Definition 57. A *twist* in $(\mathcal{V}, c)$ is a family $\theta = \{\theta_V : V \to V\}$ of isomorphisms, represented diagrammatically as in Fig. 10, satisfying the relations $\theta_{V\otimes W} = c_{W,V}c_{V,W}(\theta_V \otimes \theta_W)$ and $\theta_V f = f\theta_U$ for objects V, U, W and morphisms $f : U \to V$ in $\mathcal{V}$.

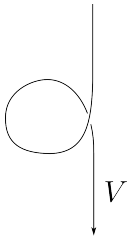


Figure 10: The twist isomorphisms θ_V.

Figure 11: The duality morphisms b_V and d_V respectively.

Definition 58. A *duality* in $\mathcal{V}$ associates to every object V of $\mathcal{V}$ an object V^* of $\mathcal{V}$ and two morphisms $b_V : \mathbf{1} \to V \otimes V^*$ and $d_V : V^* \otimes V \to \mathbf{1}$ satisfying the following:

1. $(\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V) = \mathrm{id}_V$.
2. $(d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}$.

A duality $(*, b, d)$ in a category with braiding and twist, $(\mathcal{V}, c, \theta)$ is called *compatible* with the braiding and twist (when defined) if furthermore $(\theta_V \otimes \mathrm{id}_{V^*})b_V = (\mathrm{id}_V \otimes \theta_{V^*})b_V$.

In graphical calculus, a downward arrow coloured by an object V^* will be used interchangeably with upward arrows coloured by V, and we will represent b_V and d_V by the diagrams in Fig. 11.

5.3.2 The Reshetikhin–Turaev functor

The graphical calculus hints that we might be able to associate to an oriented graph in $\mathbb{R}^2 \times [0,1]$, where strings are coloured by objects of $\mathcal{V}$, and possibly containing a number of coupons, a pair of objects in $\mathcal{V}$, and a morphism between them. The existence of the Reshetikhin–Turaev functor tells us that this is indeed more or less the case, and that the resulting morphism. In this subsection, this is made somewhat more precise, but most details are omitted.

A *band* is a homeomorphic image in $\mathbb{R}^2 \times [0,1]$ of $[0,1] \times [0,1]$, an *annulus* is a homeomorphic image of $S^1 \times [0,1]$, and a *coupon* is a band with distinguished bases $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$, where k bands meet $\mathbb{R}^2 \times \{0\}$, and l bands meet $\mathbb{R}^2 \times \{1\}$.

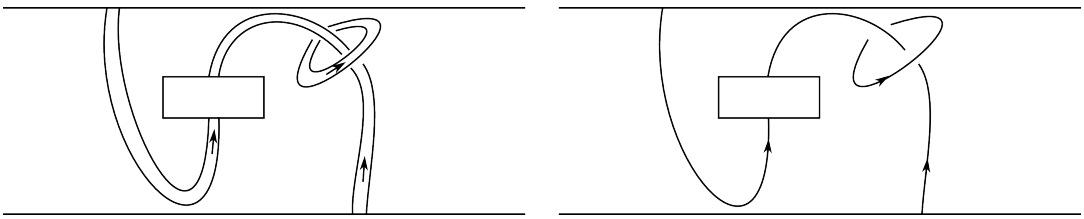


Figure 12: An example of a ribbon graph and how to represent it by a 1-dimensional graph.

Definition 59. A *ribbon* (k,l)*-graph* (or a *ribbon graph* for short) is an oriented surface embedded in $\mathbb{R}^2 \times [0,1]$, decomposed into a union of bands, annuli, and coupons, such that something like Fig. 12 occurs (see [Tur94] for a more precise statement). Here, k bands meet $\mathbb{R}^2 \times \{0\}$, and l bands meet $\mathbb{R}^2 \times \{1\}$.

To simplify drawings in the following, we will always picture bands and annuli in ribbon graphs as being strings rather than two-dimensional objects, see 12; here, the 1-dimensional graph represents the ribbon graph obtained by letting bands and annuli be parallel to the plane of the picture. This is possible since ribbon graphs are always assumed to be orientable. Note that the theory of ribbon graphs contains in particular the theory of banded oriented links in S^3.

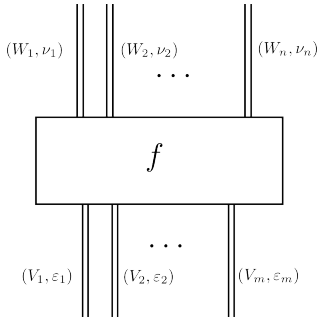


Figure 13: The colouring of a coupon.

Assume that $\mathcal{V}$ is a monoidal category with duality. A colouring[2] of a ribbon graph is an association of an object of $\mathcal{V}$ to every (non-coupon) band and every annulus, and association of a morphism $f : V_1^{\varepsilon_1} \otimes \cdots \otimes V_m^{\varepsilon_m} \to W_1^{\nu_1} \otimes \cdots \otimes W_n^{\nu_n}$ to every coupon meeting bands with colours V_i, W_j as in Fig. 13. Here, the $\varepsilon_i, \nu_j \in \{-1, 1\}$, and $V^1 = V$, $V^{-1} = V^*$ for objects V. For every V_i, W_j, the appropriate sign is determined by the orientation of the bands in question, the sign being positive, if the band is directed downwards, and negative otherwise. This turns the set of coloured ribbon graphs into a monoidal category $\mathrm{Rib}_{\mathcal{V}}$. Objects are sequences $((V_1, \varepsilon_1), \ldots, (V_m, \varepsilon_m))$, $\varepsilon_i \in \{-1, 1\}$, and morphisms $\eta \to \eta'$ are *isotopy types* of coloured ribbon graphs meeting in $\mathbb{R}^2 \times \{0\}$ the sequence η, and similarly in $\mathbb{R}^2 \times \{1\}$ the sequence η', where the numbers ε_i determine the orientations of the bands. Here with composition being given by stacking and tensor product given by placing graphs next to each other.

Theorem 60. *Let $\mathcal{V}$ be a ribbon category. There is a covariant functor $F = F_{\mathcal{V}} : \mathrm{Rib}_{\mathcal{V}} \to \mathcal{V}$ preserving tensor product transforming (V, ε) to V^{ε}, and transforming morphisms to* what it should*; in other words, the diagrams of graphical calculus are sensible and can be treated as if they were ribbon graphs. Furthermore, imposing a few conditions, it is unique.*

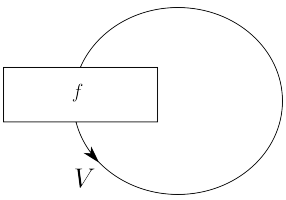


Figure 14: The coloured ribbon graph Ω_f corresponding under F to $\mathrm{tr}(f)$.

Example 61. For a morphism $f : V \to V$ in a ribbon category, define $\mathrm{tr}(f) : \mathbf{1} \to \mathbf{1}$ by $\mathrm{tr}(f) = d_V c_{V,V^*}((\theta_V f) \otimes \mathrm{id}_{V^*})$. Then $F(\Omega_f) = \mathrm{tr}(f)$, where Ω_f is the coloured ribbon graph shown in Fig. 14. It follows immediately that $\mathrm{tr}(fg) = \mathrm{tr}(gf)$ and $\mathrm{tr}(f \otimes g) = \mathrm{tr}(g \otimes f)$. Define $\dim(V) = \mathrm{tr}(\mathrm{id}_V)$. Then $\dim(V \otimes W) = \dim(V) \dim(W)$.

[2] In the language of [Tur94], this is a v-colouring.

5.3.3 Modular categories

As mentioned above, our quest is to find invariants of 3-manifolds. The invariants of coloured ribbon graphs constructed above are too general to be useful for this purpose; rather, the ribbon categories used in the construction should satisfy several other conditions, summed up in the notion of a modular category.

Definition 62. An *Ab-category* is a monoidal category $\mathcal{V}$ such that the set $\mathrm{Hom}(V, W)$ of morphisms $V \to W$ has the structure of an additive abelian group and such that composition is bilinear. In particular, this holds for $\mathrm{End}(\mathbf{1}, \mathbf{1})$, and this group obtains the structure of a (commutative) ring, with multiplication being given by composition; it is denoted K and called the *ground ring* of $\mathcal{V}$.

Definition 63. An object V of a ribbon Ab-category $\mathcal{V}$ is called *simple*, if for any $k \in K$, the map $k \mapsto k \otimes \mathrm{id}_V$ defines a bijection $K \to \mathrm{Hom}(V, V)$.

Definition 64. Let $\{V_i\}_{i\in I}$ be a family of objects in a ribbon Ab-category $\mathcal{V}$. An object V of $\mathcal{V}$ is *dominated* by $\{V_i\}$, if there exists a finite subfamily $\{V_{i(r)}\}_r$ and morphisms $f_r : V_{i(r)} \to V, g_r : V \to V_{i(r)}$ such that $\mathrm{id}_V = \sum_r f_r g_r$.

Definition 65. A *modular category* is a ribbon Ab-category $\mathcal{V}$ with a finite set of simple objects $\{V_i\}_{i\in I}$ satisfying the following axioms:

1. $\mathbf{1} \in \{V_i\}_{i\in I}$.
2. For any $i \in I$, there exists $i^* \in I$ such that V_{i^*} is isomorphic to $(V_i)^*$.
3. All objects of $\mathcal{V}$ are dominated by $\{V_i\}_{i\in I}$.
4. The matrix S with entries $S_{i,j} = \mathrm{tr}(c_{V_j,V_i} c_{V_i,V_j}) \in K$ is invertible over K.

Let $(\mathcal{V}, \{V_i\}_{i\in I})$ be a modular category. We are now in a position to define the link invariant that will give rise to our 3-manifold invariant. Let L be an oriented[3] banded m-component link in S^3, viewed as a link in $\mathbb{R}^2 \times [0, 1]$. Let $\mathrm{col}(L)$ be the finite set of all possible colourings of L with elements of $\{V_i\}_{i\in I}$. For $\lambda \in \mathrm{col}(L)$, let $\Gamma(L, \lambda)$ be the coloured ribbon $(0, 0)$-graph obtained from L by colouring it according to λ. Now $F(\Gamma(L, \lambda))$ is an element of K, and we define

$$\{L\} = \sum_{\lambda\in \mathrm{col}(L)} \prod_{n=1}^{m} \dim(\lambda(L_n)) F(\Gamma(L, \lambda)) \in K.$$

5.3.4 Surgery and 3-manifold invariants

Let L be an m-component banded link in S^3. Choose a regular closed neighbourhood U of L, consisting of m disjoint solid tori $U_1, \ldots, U_m$. Each of these are homeomorphic to $S^1 \times D^2$ with boundary homeomorphic to $S^1 \times S^1$. Choose homeomorphisms $h_i : S^1 \times S^1 \to S^1 \times S^1$ and form the space $M_L = (S^3 \setminus U) \cup_{h_i} (\sqcup_{i=1}^m D^2 \times S^1)$ which is the disjoint union of $S^3 \setminus U$ and m copies of solid tori $D^2 \times S^1$, these two spaces being identified along their common boundary $\sqcup_{i=1}^m S^1 \times S^1$ using the homeomorphisms h_i. The resulting topological space M_L is a closed orientable manifold, and we say that M_L is obtained by *surgery* on S^3 along L.

Theorem 66 (Lickorish, Wallace)**.** *Any closed connected oriented 3-manifold can be obtained by surgery on S^3 along a banded link.*

Remark 67. The space M_L constructed above depends of course on the homeomorphisms involved in the gluing. Using the banded structure of L, one obtains a special class of surgeries (i.e. special choices of homeomorphisms) called *integral surgeries*, and the above theorem can be rephrased to the stronger statement that integral surgery will suffice. This will not be needed below.

[3]One should note that the resulting invariant does not depend on the orientations of the components.

Let L be an oriented m-component banded link. Let $\sigma(L)$ be the signature of the matrix $A_{ij} = \mathrm{lk}(L_i, L_j)$ consisting of linking numbers of the components. Here, the linking number of a banded knot $L_i = K \times [0,1]$ with itself is the linking number of its boundary knots $K \times \{0\}$ and $K \times \{1\}$. Note that $\sigma(L)$ is independent of the orientations of the components.

Let as before $(\mathcal{V}, \{V_i\}_{i\in I})$ be a modular category. Assume that there is an element $\mathcal{D} \in K$ called a *rank* satisfying $\mathcal{D}^2 = \sum_{i\in I}(\dim(V_i))^2$. Since the V_i are simple, each twist θ_{V_i} can be identified with an element $v_i \in K$, which is invertible, since θ_{V_i} is an isomorphism. Now, set $\Delta = \sum_{i\in I} v_i^{-1}(\dim(V_i))^2 \in K$. The dimensions $\dim(V_i)$ are invertible because of axiom (4) above, and it follows that both $\mathcal{D}$ and Δ are invertible in K.

Now, we can define the invariant of 3-manifolds.

Theorem 68 (Reshetikhin–Turaev)**.** *Let M be a closed connected oriented 3-manifold obtained by surgery on S^3 along a banded link L. Then*

$$\tau(M) = \Delta^{\sigma(L)}\mathcal{D}^{-\sigma(L)-m-1}\{L\} \in K$$

is a topological invariant of M.

Finally, we extend the invariant to an invariant of 3-manifolds containing coloured ribbon $(0,0)$-graphs. Let M be a closed connected oriented 3-manifold containing a coloured ribbon graph Ω. Assume that M is the result of surgery along an m-component oriented link L in S^3, and assume by applying isotopy that Ω does not meet the regular neighbourhood U of L used in the surgery, so we can view Ω as a ribbon graph in $S^3 \setminus U$. Now, let

$$\{L, \Omega\} = \sum_{\lambda\in \mathrm{col}(L)} \prod_{n=1}^{m} \dim(\lambda(L_n))F(\Gamma(L,\lambda) \cup \Omega) \in K.$$

As before, this will not depend on the orientation of L or the numbering of the components.

Theorem 69. *Let (M, Ω) be as above. Then*

$$\tau(M, \Omega) = \Delta^{\sigma(L)}\mathcal{D}^{-\sigma(L)-m-1}\{L, \Omega\} \in K$$

is a topological invariant of the pair (M, Ω).

The proof of this theorem uses the theorem by Kirby stating that closed oriented manifolds obtained by (integral) surgery along banded links L and L' are homeomorphic (by orientation preserving homeomorphism) if and only if L and L' are related by a special sequence of moves on banded links called Kirby moves.

5.4 Jakob

The topic for today will be TQFT. Compact oriented $n+1$-dimensional oriented manifolds will be denoted M, and X will be closed oriented n-dimensional manifolds. Homeomorphisms $f : X \to Y$ between n-manifolds will always be orientation preserving.

We say that (M, X, Y) is a *cobordism*, if $\partial M = X \sqcup -Y$.

Let K be a commutative ring with unit. A *modular functor* T with ground ring K assigns to an n-manifold X a projective k-module $T(X)$, and to a homeomorphism $f : X \to Y$ a k-isomorphism $f_\sharp : T(X) \to T(Y)$ satisfying the following axioms.

1. If $g : X \to Y$, $f : Y \to Z$, then $(fg)_\sharp = f_\sharp g_\sharp$.
2. For disjoint unions, $T(X \sqcup Y) = T(X) \otimes_K T(Y)$ satisfying several properties; the splitting is commutative, associative, and natural; see [Tur94].

A modular functor is called *self-dual* in the following sense: If X is a closed oriented n-manifold, there exists a nondegenerate bilinear pairing $d_X : T(X) \otimes T(-X) \to K$, natural with respect to orientation preserving homeomorphism of X.

Definition 70. A *TQFT based on $n+1$-cobordisms* is a modular functor T with ground ring K and a map τ assigning to a cobordism (M, X, Y) a K-homomorphism $\tau(M, X, Y) : T(X) \to T(Y)$ satisfying the following:

1. If M_1 and M_2 are cobordisms and $f : M_1 \to M_2$ an orientation preserving homeomorphism preserving the bases, then the association τ is natural.
2. If $M = M_1 \sqcup M_2$, then $\tau(M) = \tau(M_1) \otimes \tau(M_2)$.
3. If M is obtained from the disjoint union of M_1 and M_2 by gluing along $g : \partial_+(M_1) \to \partial_-(M_2)$, then for some $k \in K^*$, $\tau(M) = k\tau(M_2) \circ g_\sharp \circ \tau(M_1)$. Here, k is called the anomaly.
4. For each X, $\tau(X \times [0,1], X \times \{0\}, X \times \{1\}) = \mathrm{id}_{T(X)}$.

Theorem 71. *Let (T, τ) be a TQFT with ground ring K. Then T is self-dual. It follows that* $\mathrm{Dim}(T(X)) = \mathrm{Dim}(T(-X))$*, where* Dim *is the dimension from the previous lecture.*

Theorem 72. *If M is obtained from a disjoint union of M_1 and M_2 by gluing along $g : \partial M_1 \to -\partial M_2$. Let $-g : -\partial M_1 \to \partial M_2$. If (T, τ) is anomaly-free (i.e. all anomalies are 1), then*

$$\begin{aligned}\tau(M) &= d_{\partial M_1}(\tau(M_1) \otimes (-g)_\sharp^{-1}(\tau(M_2))) \\ &= d_{\partial M_2}(\tau(M_2) \otimes g_\sharp(\tau(M_1))).\end{aligned}$$

Theorem 73. *If (T, τ) is anomaly-free, then for any closed oriented n-manifold X,*

$$\mathrm{Dim}(T(X)) = \tau(X \times S^1) \in K.$$

An M with *X-parametrized boundary* is a pair $(M, f) = (M, f : X \to \partial M)$, where f is an orientation-preserving homeomorphism. If (m, f) has X-parametrized boundary, and (N, g) has $-X$-parametrized boundary, then $M \cup_X N$ is the closed $n+1$-manifold obtained by gluing along $(-g) \circ f^{-1} : \partial M \to -\partial N$. Let $\tau^X(M, f) = f_\sharp^{-1}(\tau(M)) \in T(X)$. We say that (T, τ) is *non-degenerate*, if for any X, the set $\{\tau^X(M, f)\}_{(M,f)} \subset T(X)$ generates $T(X)$ over K.

A *homomorphism* resp. *isomorphism* between TQFTs (T_1, τ_1), (T_2, τ_2) is something that assigns to every closed oriented n-manifold X a homomorphism resp. isomorphism $T_1(X) \to T_2(X)$, satisfying some natural axioms.

Theorem 74. *Let $(T_1, \tau_1), (T_2, \tau_2)$ be non-degenerate anomaly-free TQFTs. If $\tau_1(M) = \tau_2(M)$ for every cobordism M, then (T_1, τ_1) and (T_2, τ_2) are isomorphic.*

Let τ_0 be a function assigning to every closed $n+1$-dimensional manifold considered up to orientation preserving homeomorphism a value in K. Let X be an n-manifold and assume that we have triples $\{(k_s, M_s, N_s)\}$, $k_s \in K$, M_s a compact oriented $(n+1)$-manifold with X-parametrized boundary, and N_s compact oriented $(n+1)$-manifold with $-X$-parametrized boundary. This is called a *splitting system* for X with respect to τ_0 if for any M, N, we have

$$\tau_0(M \cup_X N) = \sum_s k_s \tau_0(M \cup_X N_s)\tau_0(M_s \cup N).$$

For instance, $(1, \emptyset, \emptyset)$ for $X = \emptyset$, if τ_0 is multiplicative with respect to disjoint union.

A *quantum invariant* is a K-valued function τ_0 defined on closed oriented $n+1$-manifold up to orientation preserving homeomorphism satisfying the following:

1. It is normalized: $\tau_0(\emptyset) = 1$.
2. It is multiplicative: $\tau_0(M \sqcup N) = \tau_0(M)\tau_0(N)$.
3. Any X should have a splitting system with respect to τ_0.

Lemma 75. *If (T, τ) is a non-degenerate anomaly-free TQFT, then $M \mapsto \tau(M)$ is a quantum invariant.*

Lemma 76. *Every quantum invariant τ extends to a non-degenerate anomaly-free TQFT, and we have 1-1-correspondence between quantum invariants and non-degenerate anomaly-free TQFTs.*

From now on, we consider the construction of a 3-dimensional TQFT. Fix a (strict) modular category $(V, \{V_i\}_{i\in I})$ with ground ring K and rank $\mathcal{D} \in K$. A *marked arc* on a surface Σ is a simple arc endowed with an object W in V together with a sign ± 1. A surface Σ is *decorated*, if we have a finite totally ordered set of marked arcs. The surface Σ is called a d-surface. The d-surface $-\Sigma$ is the surface Σ with reversed orientation and the signs of the marked arcs multiplied by -1. The labelling and ordering is fixed. The type of a d-surface Σ is $t(\Sigma) = (g; (W_1, \nu_1), \dots (W_m, \nu_m))$, where $g \in \mathbb{N}$ is the genus of Σ. A standard d-surface Σ_t of type t is given as the boundary of a closed regular neighbourhood U_t of a partly coloured ribbon graph R_t in $\mathbb{R}^2 \times [0, 1]$, I was too lazy to draw but is gives a particular surface of type t – see [Tur94]. A parametrized d-surface is a connected surface Σ of type t with a homeomorphism $f : \Sigma_t \to \Sigma$.

The modular functor is now constructed as follows. It is defined on parametrized d-surfaces. For every type $t = (g; (W_1, \nu_1), \dots (W_m, \nu_m))$, define a projective module ψ_t by for $i = (i_1, \dots, i_g) \in I^g$ letting

$$\Phi(t; i) = W_1^{\nu_1} \otimes \cdots \otimes W_m^{\nu_m} \otimes \bigotimes_{r=1}^{g} (V_{i_r} \otimes V_{i_r}^*),$$

where $W_i^1 = W_i$, and $W_i^{-1} = W_i^*$ and putting

$$\psi_t = \bigoplus_{i \in I^g} \mathrm{Hom}(\mathbf{1}, \Phi(t; i)).$$

For example, if $g = 1$, $m = 0$, then $\psi_t = K^{\mathrm{card}(I)}$. If Σ is a parametrized d-surface, let $T(\Sigma)$ be the unordered tensor product of ψ_t's corresponding to connected components of Σ.

Lemma 77. *This defines is a modular functor.*

A *decorated 3-manifold* is a compact oriented 3-manifold containing a ribbon graph with boundary parametrized d-surfaces and free ends of the ribbon graph being attached to the boundary. Pick $x \in \mathrm{Hom}(\mathbf{1}, \Phi(t; i))$ and colour the coupon of R_t by x. This gives a decorated 3-manifold U_t, denoted $H(U_t, R_t, i, x)$. We could reflect the ribbon graph R_t and obtain a decorated 3-manifold $H(U_t^-, -R_t, u, x)$, and we let $\psi_t^* = (T(\Sigma_t))^*$. Let $(M, \partial_- M, \partial_+ M)$ be a cobordism, where M is decorated, and the boundaries are parametrized d-surfaces, we define $\tau(M) : T(\partial_- M) \to T(\partial_+ M)$ by gluing in U_t for each component of $\partial_- M$, and gluing in U_t^- to components of $\partial_+ M$. This gives a closed 3-manifold $\tilde{M}$ containing a ribbon graph $\tilde{\Omega}$. Using the invariant τ from the previous lecture, we get an element $\tau(\tilde{M}, \tilde{\Omega}) \in K$. This number depends on the colouring y of the ribbon graph chosen, so really obtain a map $T(\partial_- M) \otimes T(\partial_+ M)^* \to K$. This defines a TQFT.

11th lecture, April 6th 2011

6 Conformal field theory

As we have seen previously, the use of geometric quantization to obtain a topological quantum field theory is somewhat incompletely; we can associate vector spaces to boundaries of $n+1$-manifolds, but associating morphisms to cobordisms requires use of the heuristic path integral.

The next few weeks will be a brief introduction to conformal field theory. The reference is [Koh02].

6.1 Loop groups and their representations

A reference for this section is [Pre88]. Throughout this section, G is a connected compact Lie group, e.g. $U(n), SU(n), \ldots$, and $G_{\mathbb{C}}$ denotes its complexification, $U(n)_{\mathbb{C}} = GL(n,\mathbb{C}), SU(n)_{\mathbb{C}} = SL(n,\mathbb{C})$. Here, if G is a compact Lie group with Lie algebra $\mathfrak{g}$ with negative semi-definite Killing form obtained by letting $\mathfrak{g}$ acting on itself by the adjoint representation. That is, we think of any element of $\mathfrak{g}$ as an operator on the finite dimensional vector space $\mathfrak{g}$, and the Killing form is

$$\langle x, y\rangle := \mathrm{Tr}(\mathrm{ad}(x)\cdot \mathrm{ad}(y)).$$

In the case where G is semi-simple, the form is negative definite. The Lie algebra, we can complexify to $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$. This is a complex Lie algebra with bracket

$$[x\otimes\lambda, y\otimes\mu] = [x,y]\otimes\lambda\mu.$$

This Lie algebra comes from a unique connected complex Lie group $G_{\mathbb{C}}$ such that G is a subgroup of $G_{\mathbb{C}}$. If G is compact, then $G_{\mathbb{C}}$ is reductive. Conversely, every complex reductive Lie group $G_{\mathbb{C}}$ is the complexification of a unique compact one G – here, G is a maximal compact subgroup of $G_{\mathbb{C}}$ which is the fixpoint set of some involution on $G_{\mathbb{C}}$.

We could also start with a non-compact Lie group and obtain a reductive one by complexifying; for example, $GL(n,\mathbb{R})_{\mathbb{C}} = GL(n,\mathbb{C})$. In the compact case, the importance of complexification stems from the fact that a compact Lie group and its complexification virtually have the same representation theory. That is, there is a 1-1 relation between complex representations of a compact G and complex representations of $G_{\mathbb{C}}$.

Definition 78. The *loop group* LG of G is the set of (smooth) maps $S^1 \to G$.

This is an infinite-dimensional Lie group itself. Let $L\mathfrak{g} = \mathrm{Lie}(LG)$.

Proposition 79. *We have* $(L\mathfrak{g})_{\mathbb{C}} = L\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}\otimes\mathbb{C}((t))$, *where* $\mathbb{C}((t))$ *is the set of Laurent series in* t. *In particular,*

$$[x\otimes f, y\otimes g] = [x,y]\otimes fg.$$

For various reasons, we really need a slight variation on this theme; namely, we need to consider central extensions of the loop groups. That is, we want to consider a sequence of Lie algebras

$$0\to\mathbb{C}\to\widetilde{L\mathfrak{g}_{\mathbb{C}}}\to L\mathfrak{g}_{\mathbb{C}}\to 0,$$

where $\mathbb{C}$ is a Lie algebra with the trivial bracket, and we want all maps to be morphisms. Equivalently, we consider

$$1\to U(1)\to\widetilde{LG}\to LG\to 1.$$

For the Lie algebra, the simplest way to do this would be to simply let $\widetilde{L\mathfrak{g}_{\mathbb{C}}} = L\mathfrak{g}_{\mathbb{C}}\oplus\mathbb{C}c$ and simply extend the bracket. Here the c denotes a generator of the kernel in the exact sequence, commuting with everything. As vector spaces, this is exactly what we want, but we will define another bracket on the direct sum by

$$[\alpha+\mu c, \beta+\mu c] = [\alpha,\beta] + \omega(\alpha,\beta)c.$$

For this to be a Lie bracket, we must have ω is bilinear, and satisfies $\omega(\alpha,\beta) = -\omega(\beta,\alpha)$ and $\omega([\alpha,\beta],\gamma) + (\text{cyclic}) = 0$. Every ω satisfying these conditions defines a bracket on $L\mathfrak{g}_{\mathbb{C}}\otimes\mathbb{C}c$.

Proposition 80. *There is a correspondence between bilinear forms (up to isomorphism) as above, called cocycles, and elements of* $H^2(L\mathfrak{g}_{\mathbb{C}},\mathbb{C})$, *the second Lie algebra cohomology group.*

Proposition 81. *There is a 1-1 correspondence between central extensions of* $L\mathfrak{g}_{\mathbb{C}}$ *up to isomorphism and* $H^2(L\mathfrak{g}_{\mathbb{C}},\mathbb{C})$. *The isomorphism of central extensions is given by identifying cocycles* $\omega(x,y)\equiv\omega'(x,y)$, *if* $\omega(x,y) = \omega'(x,y) + \mu([x,y])$ *for some function* $\mu : L\mathfrak{g}_{\mathbb{C}}\to\mathbb{C}$.

Thus, we have classified all central extensions of the loop algebra. There is another classification which is more relevant for us.

Proposition 82. *If G (or $\mathfrak{g}_{\mathbb{C}}$) is semi-simple with Killing form $\langle\cdot,\cdot\rangle$, then $H^2(L\mathfrak{g}_{\mathbb{C}},\mathbb{C}) \cong \mathbb{C}$ generated by*

$$\omega(X\otimes f, Y\otimes g) = \langle X,Y\rangle \cdot \operatorname{Res}_{t=0}(df\, g).$$

In particular, if we use this ω, we have

$$[X\otimes t^m, Y\otimes t^n] = [X,Y]t^{mn} + \langle X,Y\rangle m\delta_{m+n,0}c$$

Definition 83. The corresponding central extension $\hat{\mathfrak{g}}_{\mathbb{C}}$ of the loop algebra is called the *affine Lie algebra*.

Consider the group $\mathrm{Diff}^{(+)}(S^1)$ of (orientation preserving) diffeomorphisms of S^1. Let $A = \mathrm{Lie}(\mathrm{Diff}(S^1))_{\mathbb{C}}$ be its complexified Lie algebra. Then $A = \{f(z)\frac{d}{dz} \mid f(z) \text{ Laurent series}\}$. The bracket is defined as

$$[f(z)\frac{d}{dz}, g(z)\frac{d}{dz}] = (f(z)g'(z) - g(z)f'(z))\frac{d}{dz}.$$

Typically, the basis chosen for this Lie algebra is $L_n = -z^{n+1}\frac{d}{dz}$. On S^1, we think of L_n as $L_n = ie^{in\theta}\frac{\partial}{\partial z}$, $z = e^{i\theta}$, and $[L_m, L_n] = (m-n)L_{m+n}$.

We consider as before central extensions of A.

Proposition 84. *As before, $H^2(A,\mathbb{C}) = \langle\alpha\rangle$, where α is given by the cocycle*

$$\omega(f\frac{d}{dz}, g\frac{d}{dz}) = \frac{1}{12}\operatorname{Res}_{t=0}(f'''g dz).$$

The reason for choosing this particular scaling is that we want it to lift to a $U(1)$-extension of the Lie group.

Definition 85. Let $\mathcal{V}$ be the central extension of A corresponding to α. This is referred to as the *Virasoro algebra*.

So, we have obtained extensions of the Lie algebra and want to consider now extensions of the groups. We consider the case of loop groups of simply-connected semi-simple Lie groups. Recall that if $\mathfrak{g}$ is a semi-simple complex (thus reductive) Lie algebra, then there is a unique simply-connected complex semi-simple Lie group G such that $\mathrm{Lie}(G) = \mathfrak{g}$. As before, there is a 1-1 correspondence between finite-dimensional complex representations of $\mathfrak{g}$ and G. The centre ZG of G is a finite abelian group, and if $H \subseteq ZG$ is a subgroup, then G/H is a semi-simple complex Lie group with the same Lie algebra as G, but it is no longer simply-connected if H is non-trivial. For example, if $SL(n,\mathbb{C})$ is simply connected, and $PGL(n,\mathbb{C}) = SL(n,\mathbb{Z})/\mathbb{Z}_n$ is not. Similarly, $\mathrm{Spin}(n,\mathbb{C})$ is simply-connected, but $SO(n,\mathbb{C}) = \mathrm{Spin}(n,\mathbb{C})/\mathbb{Z}_2$ is not.

Theorem 86. *Now, assume that G is simply-connected. Let*

$$0 \to \mathbb{C} \to \widetilde{L\mathfrak{g}_{\mathbb{C}}} \to L\mathfrak{g}_{\mathbb{C}} \to 0$$

be a central extension of $L\mathfrak{g}_{\mathbb{C}}$. Then there exists a central extension

$$1 \to U(1) \to \widetilde{LG} \to LG \to 1$$

if and only if $\frac{\omega}{2\pi} \in H^2(G,\mathbb{C})$ represents an integral class. If so, $\widetilde{LG}$ is completely determined by the Lie algebra.

Theorem 87. *If $\langle h_\alpha, h_\alpha \rangle$ is even for all co-roots h_α, then the condition of the previous theorem is satisfied. In particular, this holds for*

$$\omega(X \otimes f, Y \otimes g) = \langle X, Y \rangle \operatorname{Res}_{t=0}(df \cdot g).$$

We turn now to the representation theory for compact/reductive Lie algebras. Recall that when discussing the representation theory of a Lie algebra $\mathfrak{g}$, we choose a Cartan algebra c; that is, a maximal abelian subalgebra of $\mathfrak{g}$. For example, for $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, c is the set of diagonal matrices.

Proposition 88. *If $\mathfrak{g}$ is compact (in the Lie group sense; i.e. the Killing form is negative definite) or reductive, then all maximal abelian subalgebras are conjugate. The rank of the Lie algebra is the dimension of the subalgebra.*

The representation theory of c is the representation of an abelian Lie algebra which is in some sense trivial, as such split into a direct sum one-dimensional Lie algebras, and representations of these are determined by weights. Here, the weight is a map from c to $\mathbb{R}$ or $\mathbb{C}$. For a representation of $\mathfrak{g}$, we restrict it to a representation of c and obtain a finite number of weights. By studying these, we are able to say everything about the original representation.

12th lecture, April 11th 2011

We continue our discussion of loop groups.

- Last time we considered the affine Lie algebra, corresponding to any given semi-simple Lie algebra or Lie group. They are (special examples of) Kac–Moody algebras, obtained by relaxing some of the conditions on the Cartan matrix in the classification of semi-simple Lie algebras.
- We discussed the Virasoro algebra V, which is a central extension of the complexified Lie algebra A of $\mathrm{Diff}^+(S^1)$. Recall that there exist real Lie groups that have no complexification; for example a double cover of $SL(n, \mathbb{R})$, but we can always complexify the Lie algebra.
- We considered central extensions of loop groups

  $$1 \to U(1) \to \widetilde{LG} \to LG \to 1,$$

  for G simple and simply-connected. Such extensions are classified by elements of $H^2(\mathfrak{g}, \mathbb{Z}) \cong \mathbb{Z}$. Choosing a generator of $\mathbb{Z}$, one stands out (the minimal non-trivial one). All others are referred to as "central extensions at level l".

6.1.1 Digression on representation theory

We recall the representation theory for compact (or complex reductive) Lie algebras. Let G be a compact connected Lie group. Suppose we choose a maximal torus T_G for this Lie group, i.e. a connected compact abelian subgroup.

Recall that all finite-dimensional complex representations of G are completely reducible. That is, any representation V can be written as a direct sum $V = V_1 \oplus \cdots \oplus V_n$ of reducible ones. The reason is that G is compact; we choose an inner product $\langle \cdot, \cdot \rangle$ on V and average over G to get another inner product $\langle \cdot, \cdot \rangle$ that is invariant under the G-action. Thus, for any subrepresentation of V, its orthogonal complement is again a subrepresentation.

For T_G, all irreducible representations are 1-dimensional, so a representation is given by a morphism $T_G \to U(1)$. Such a morphism is called a *character* or a *weight* (of T_G).

The characters of T_G form a lattice in $(\mathrm{Lie}(T_G))^*$ known as the *character lattice* or the *weight lattice*.

Consider the adjoint representation $\mathrm{Ad} : G \circlearrowright (\operatorname{Lie} G)_{\mathbb{C}}$. Restricting Ad to T_G gives the *Cartan decomposition* of $(\operatorname{Lie} G)_{\mathbb{C}}$ as irreducible representations of the T_G,

$$(\operatorname{Lie} G)_{\mathbb{C}} \cong (\mathfrak{t}_G)_{\mathbb{C}} \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha,$$

where the α denote the weights of the action of T_G through Ad, and $(\mathfrak{t}_G)_{\mathbb{C}}$ is the weight 0 part, $\mathfrak{t}_G = \operatorname{Lie}(T_G)$. The non-zero α occuring is a finite subset of the character lattice called the *roots* of G (or $\mathfrak{g}$). Different groups with the same Lie algebra give rise to the same decomposition. Note that all $\mathfrak{g}_\alpha$ are 1-dimensional.

The set $\{\alpha\} \subseteq (\operatorname{Lie} T_G)^*$ is a *reduced root system.*

If G is semi-simple, then the root system determines the Lie algebra, as the roots span $(\operatorname{Lie} T_G)^*$.

We also have the *Weyl group* $W_G = N(T_G)/T_G$ which is a finite group acting on T_G, thus on $\operatorname{Lie} T_G$ and $(\operatorname{Lie} T_G)^*$, and it preserves the set of roots. If α is a root, so is $-\alpha$. Given an α, pick a non-zero element e_α in $\mathfrak{g}_\alpha$. One checks that $e_{-\alpha} := \overline{e_\alpha} \in \mathfrak{g}_{-\alpha}$, and that $h_\alpha := -i[e_\alpha, e_{-\alpha}] \in \operatorname{Lie}(T_G)$. We can normalize e_α so that $[h_\alpha, e_{-\alpha}] = 2ie\alpha$, $[h_\alpha, e_{-\alpha}] = -2ie_\alpha$, and $[e_\alpha, e_{-\alpha}] = ih_\alpha$. These are the same commutation relations as for the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These generate $(2, \mathbb{C}) \cong \mathfrak{su}(\not{2})_{\mathbb{C}}$. We call h_α the *coroot* (also sometimes written as $\alpha^\vee$). We can think of them as morphisms $U(1) \to T_G$. Moreover, we can cohose them collectively such that

$$[e_\alpha, e_\beta] = \begin{cases} \eta_{\alpha\beta} e_\alpha & \text{if } \alpha + \beta \text{ is also a root} \\ ih_\alpha & \text{if } \alpha + \beta = 0 \\ 0 & \text{if } \alpha + \beta \text{ is not a root} \end{cases}$$

where here $\eta_{\alpha\beta} \in \mathbb{Z}$.

For every root α, we get a transformation $s_\alpha : \operatorname{Lie} T_G \to \operatorname{Lie} T_G$ mapping $\xi \mapsto \xi - \alpha(\xi)h_\alpha$ (one thinks of this as reflecting along the hyperplane in the dual perpendicular to α). The Weyl group W_G is generated by $\langle s_\alpha \rangle$. We have a finite number of hyperplanes going through the origin. A *Weyl chamber* is a connected component of the hyperplanes.

Similarly, we have $s_{\alpha^\vee} : \mathfrak{t}_G^* \to \mathfrak{t}_G^*$, given by $\nu \mapsto \nu - h_\alpha(\alpha)\alpha$. The Weyl group W_G acts transitively on the Weyl chambers. Choose a Weyl chamber called the *positive* one. We get a partial order on the character lattice, letting $\rho \geq 0$, if and only if $\rho(\text{positive chamber}) > 0$. The roots α are divided into positive and negative ones. A positive root is called *simple* if it cannot be written as a sum of two other positive roots. The positive simple roots form a basis $\{\alpha_1, \ldots, \alpha_r\}$ of $\mathfrak{t}_G^*$, where r is the rank of the group, $r = \operatorname{rk} G = \dim T_G$.

Example 89. For $SU(2)$, T_G is one-dimensional, and the root lattice $\langle \alpha \rangle$ is a proper subset of the weight lattice. Here $W_{SU(2)} = \mathbb{Z}/2\mathbb{Z}$ is reflection about 0.

For $SU(3)$, we have two positive simple roots α_1, α_2, and six roots $-\alpha_1, \alpha_2, \alpha_1 + \alpha_2, -\alpha_1 - \alpha_2, -\alpha_2, \alpha_1$. Here $W_G = S_3$.

So far, we have considered the structure theory of the Lie algebras and now consider the general representation theory. Let V be a finite-dimensional irreducible complex representation of G. As before, we can restrict the representation to a maximal torus T_G and find that $V = \oplus_\beta V_\beta$, where T_G acts with weight β, but the dimensions of V_β might be greater than 1. The $\{\beta\}$ are called the *weights* and form a finite subset of the character lattice. The set $\{\beta\}$ is always invariant under W_G, and any difference $\beta_i - \beta_j$ is in the root lattice.

There exists a weight ρ called the *highest weight* such that $\rho + \alpha_i$ is not a weight for any positive simple α_i. The highest weight always occurs with multiplicity 1 and always is in the positive Weyl chamber. Conversely, a character is called *dominant*, if it is in the closure of the positive Weyl chamber, and we have the following.

Theorem 90. *Every dominant character is the highest weight for an irreducible representation, unique up to isomorphism.*

The existence part of the theorem is given by constructing the representations. There are a number of ways of doing this.

- We can classify all simple Lie algebras, getting the families A_r,B_r,C_r,D_r and the exceptional ones E_6, E_7, E_8, G_2, F_4, and we can study the representations in each case.
- There is a purely algebraic construction of the representations as *Verma modules*.
- There is a geometric construction given by the Borel–Weil theorem, stated below.

Theorem 91 (Borel–Weil theorem)**.** *We can construct all highest weight representations of G as vector spaces of holomorphic sections of some holomorphic line bundle over a flag manifold.*

13th lecture, April 13th 2011

We begin with a brief sketch of Borel–Weil theory. Let G be compact. Given λ a dominant weight, there is a unique V_λ irreducible representation of G such that λ is the highest weight, and it is unique up to isomorphism. The Borel–Weil theorem gives a way to construct these in a geometric fashion.

Proposition 92. *There is an isomorphism $G/T \cong G_{\mathbb{C}}/B^+$, where T is a maximal torus of G, and B^+ is the positive Borel subgroup containing $T_{\mathbb{C}}$.*

We refer to G/T as the (full) *flag variety*. One way of classifying B^+ is that $\mathrm{Lie}(B^+) = \langle \mathfrak{h}_{\mathbb{C}}, e_\alpha \mid \alpha \text{ positive} \rangle$ where $h_{\mathbb{C}}$ comes from the Cartan decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \text{ root}} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \langle e_\alpha \rangle$. For example, if $G = U(n)$, then $G_{\mathbb{C}} = GL(n, \mathbb{C})$, then T consists of diagonal unitary matrices, and B^+ consists of upper triangular matrices. Notice that B^+ is not itself reductive and not a complexification.

Corollary 93. *G/T is a compact complex manifold and a projective variety.*

Suppose that $\lambda : T \to U(1)$ is a dominant weight, giving a map $T_{\mathbb{C}} \to \mathbb{C}^*$, uniquely extending to a map $B^+ \to \mathbb{C}^*$ (but not to $G_{\mathbb{C}} \to \mathbb{C}^*$ or $G \to U(1)$). Put $L_\lambda = G_{\mathbb{C}} \times \mathbb{C}/B^+$, where B^+ acts on $G_{\mathbb{C}}$ by multiplication and on $\mathbb{C}$ through λ. Then $L_\lambda \to G_{\mathbb{C}}/B^+$ is a holomorphic line bundle, and we put $V_\lambda = H^0(G_{\mathbb{C}}/B^+, L_\lambda)$. This is an irreducible representation of G of $G_{\mathbb{C}}$, which is irreducible and has highest weight λ. We can think of this space as a quantization of the flag variety or as the quantization of a co-adjoint orbit.

6.1.2 Back to loop groups

we want to study the representation theory of $\widetilde{LG}, \tilde{\mathfrak{g}}$, and so on. We restrict our attention to "positive energy representations". From now on G is simple and simply connected (e.g. $SU(n)$). Consider the group of rotations of $U(1)$, $\mathrm{Rot}(U(1)) \cong U(1)$, which acts on LG through $(R_\theta(\rho))(t) = \rho(t + \theta)$, for $R_\theta \in \mathrm{Rot}(U(1))$. Consider $\mathrm{Rot}(U(1)) \ltimes LG$.

Definition 94. A positive energy representation of LG is a projective unitary representation of $\mathrm{Rot}(U(1)) \ltimes LG$ on a Hilbert space $\mathcal{H}$ such that the projective representation of $\mathrm{Rot}(U(1))$ lifts to a linear representation with positive spectrum (choose 0 as the lowest eigenvalue), and $\mathcal{H} = \oplus \mathcal{H}(n)$, where the $\mathcal{H}(n)$ are finite-dimensional corresponding to the representations of $U(1)$; that is, every integer, corresponding to representations of $U(1)$, has finite multiplicity.

Now, we have

$$1 \to U(1) \to U(\mathcal{H}) \to PU(\mathcal{H}) \to 1,$$

and a map $LG \to PU(\mathcal{H})$. We pull back to get a linear representation of some central extension of LG at level l. We have $G \subseteq LG$ (as the constant loops), and $[\mathrm{Rot}(U(1)), G] = 1$, and since G is simply connected G acts linearly on $\mathcal{H}(0)$ (a priori only some covering of G does this).

Proposition 95. *If a representation of* $\mathrm{Rot}(U(1)) \ltimes LG$ *is irreducible, then the representation of* G *on* $\mathcal{H}(0)$ *is also irreducible given by a highest weight* λ*.*

Proposition 96. *We have* $\langle , \alpha_{max}\rangle \leq l$*, where* α_{max} *is the highest weight for the adjoint representation; that is, it is the unique* α *such that* $\alpha + \alpha_i$ *is not a root for any positive root* α_i*, and* $\langle \cdot, \cdot \rangle$ *is the Killing form, i.e. an invariant pairing normalized such that* $\langle \alpha_{max}, \alpha_{max}\rangle = 2$*.*

We thus think of the level l as "capping off" a part of the positive Weyl chamber. At level $l = 1$, we get *Weyl alcove*.

Proposition 97. *The irreducible positive energy representations are classified (up to isomorphism) by the level* $l \geq 0$ *and a dominant weight* ρ *with* $\langle \rho, \alpha_{max} \leq l$*.*

For G compact, the closure of the positive Weyl chamber is a fundamental domain for the action of the Weyl group. There is a corresponding picture for $\widetilde{LG}$ and $\tilde{\mathfrak{g}}$. We have an affine Weyl group (at level 1) $W_{\mathrm{aff}} = W \ltimes$ weight lattice, and the Weyl alcove is a fundamental domain of the action of the affine Weyl group. Thus we can still think of the representation theory as being classified by elements in the weight lattice up to the action of (the now affine) Weyl group.

6.1.3 Segal–Sugawara construction

Given a positive energy representaiton of $\widetilde{LG}$, we always get a representation of $\mathcal{V}$, the Virasoro algebra. Recall that $\mathcal{V}$ was spanned by $\langle L_n, c\rangle$. The representation of $\mathcal{V}$ on $\mathcal{H}$ is defined as follows: Let X_i be a basis for $\mathfrak{g}_{\mathbb{C}}$, and X^i a "dual" basis with repsect to $\langle \cdot, \cdot \rangle$. Identify elements in $L\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g}_{\mathbb{C}} \otimes \mathbb{C}((t))$ with their operators on $\mathcal{H}$. Then define

$$L_0 = \frac{1}{2(l + \mathrm{cox})} \left(\sum_i (X_i \otimes t^0)(X^i \otimes t^0) + 2 \sum_{n>0} \sum_i (X_i \otimes t^{-n})(X^i \otimes t^n) \right), L_n = \frac{1}{2(l + \mathrm{cox})} \sum_{m \in \mathbb{Z}} \sum_i (X_i \otimes t^{-m})(X^i \otimes t$$

Here, cox is the dual Coxeter number of $\mathfrak{g}$,

$$\mathrm{cox} = 1 + \langle \alpha_{\max}, \tfrac{1}{2} \sum_i \alpha_i \rangle.$$

Exercise 98. We have

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{l \dim \mathfrak{g}}{l + \mathrm{cox}} \frac{1}{12}(m^3 - m)\delta_{m+n,0}\mathrm{id}_{\mathcal{H}},$$

which defines the action of $c \in \mathcal{V}$.

6.1.4 Borel–Weil for loop groups

The rôle of the flag manifold in the affine version is played by

$$LG_{\mathbb{C}}/B^+G_{\mathbb{C}},$$

where $LG_{\mathbb{C}} = G_{\mathbb{C}} \otimes \mathbb{C}((t))$, where

$$B^+G_{\mathbb{C}} = \{\varphi \,\mathrm{Hol}(D, G_{\mathbb{C}}) \mid \varphi(0) \in B^+\} \subseteq LG_{\mathbb{C}}$$

Here D is the disk of radius 1. Recall that the central extensions of $LG_{\mathbb{C}}$ correspond to integral second cohomology classes which themselves corresponded to line bundles on $LG_{\mathbb{C}}$. We think of $\widehat{LG}_{\mathbb{C}}$ at level l as the line bundle without the zero locus. We have maps

$$B^+G_{\mathbb{C}} \to B^+ \xrightarrow{\lambda} \mathbb{C}^*,$$

where the first map is evaluation at 0. We put $L_\lambda^l = \widehat{LG}_{\mathbb{C}} \times \mathbb{C}^*/B^+G_{\mathbb{C}}$, and

$$V_\lambda^l = H^0(LG_{\mathbb{C}}/B^+G_{\mathbb{C}}, L_\lambda^l).$$

14th lecture, April 18th 2011

6.2 Algebraic approach to conformal field theory

We begin our discussion of conformal theory and begin with a formal algebraic approach. The reference for this is Ueno's *Conformal field theory with gauge symmetries*, [Uen]. Later we give an approach that is closer related to physics and gauge theory using the Wess–Zumino–Witten model.

We first consider so-called stable pointed curves with formal neighbourhoods

Definition 99. A *nodal singularity* in a curve is locally analytically of the form $x^2 = y$.

Definition 100. Let X be a stable pointed curve, i.e. a curve with at worst nodal singularities (double points) and a set of N smooth marked points, and every component of genus 0 or 1 has at least 3 or 1 "special points" that are either nodal or marked.

We will actually assume that every component has at least one marked point. A *formal neighbourhood* around a marked point Q_i is an isomorphism $\varprojlim_m \mathcal{O}_{Q_i}/\mathfrak{m}_{Q_i}^{n+1} \cong \mathbb{C}[[\xi_i]]$, where $\mathfrak{m}_{Q_i}$ is the maximal ideal in $\mathcal{O}_{Q_i}$ of functions vanishing at Q_i.

Definition 101. Let

$$\hat{\mathfrak{g}}_N = \bigoplus_{j=1}^{N} (\mathfrak{g} \otimes \mathbb{C}((\xi_j))) \oplus \mathbb{C}_c,$$

where $\mathfrak{g}$ is a complex simple Lie algebra.

This becomes a Lie algebra with brakcet

$$[\sum_i (x_i \otimes f_i), \sum_j (y_j \otimes g_j)] = \sum_i [x_i, y_i] \otimes f_i g_i + \sum_i (x_i, y_i) \operatorname{Res}_{\xi_i = 0}(g_i df_i)c,$$

where $(\cdot, \cdot)$ is a basic inner product on $\mathfrak{g}$, which we have since $\mathfrak{g}$ is simple. Inside of this Lie algebra, we have a certain subalgebra. Put

$$\mathcal{O}_c(* \sum Q_i) = \varinjlim_n \mathcal{O}(n \sum Q_i),$$

where $\mathcal{O}(n \sum Q_i)$ consists of those functions having poles of order at most n only at Q_i. In other words, if D is a divisor determining a line bundle on X, $\mathcal{O}(D)$ consists of holomorphic sections of this line bundle, or equivalently as meromorphic functions on X with poles only along components of D. Put

$$\hat{\mathfrak{g}}(X) = \mathfrak{g} \otimes H^0(\mathcal{O}(* \sum_i Q_i)),$$

and there is an obvious inclusion $\hat{\mathfrak{g}}(X) \hookrightarrow \hat{\mathfrak{g}}_N$.

Choose a level l. For each $1 \leq 1 \leq N$, choose a highest weight λ_i in the Weyl alcove, i.e. $\langle \lambda_i, \alpha_{\max} \rangle \leq l$. Put

$$\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_N},$$

which is a representation of $\hat{\mathfrak{g}}_N$ as we have seen, and we get a dual representation

$$\mathcal{H}^\dagger_{\vec{\lambda}} = \mathcal{H}^\dagger_{\lambda_1} \otimes \cdots \otimes \mathcal{H}^\dagger_{\lambda_N}.$$

Definition 102. Let

$$\mathcal{V}_\lambda(X) = \mathcal{H}_{\vec{\lambda}} / \hat{\mathfrak{g}}(X) \mathcal{H}_{\vec{\lambda}},$$

be the *space of covacua of* X, and let

$$\mathcal{V}^\dagger_\lambda(X) = \mathrm{Hom}_{\mathbb{C}}(\mathcal{V}_{\vec{\lambda}}(X), \mathbb{C})$$

be the *space of vacua* or *conformal block*.

Theorem 103. *These spaces are finite-dimensional.*

If X' and X' are the same stable pointed curve with 2 different systems of formal pointed neighbourhoods whose first order neighbourhoods $\mathcal{O}(Q_i)/\mathfrak{m}^2_{Q_i}$ are isomorphic, then we obtain isomorphisms

$$\begin{array}{ccc} \mathcal{O}(Q_i)/\mathfrak{m}^2_{Q_i} & \xrightarrow{\cong} & \mathbb{C}[[\xi_i]]/(\xi_i^2) \\ & \searrow^{\cong} & \downarrow \cong \\ & & \mathbb{C}[[\xi'_i]]/(\xi_i'^2) \end{array}$$

and a canonical isomorphism $\mathcal{V}^\dagger_{\vec{\lambda}}(X) \cong \mathcal{V}^\dagger_{\vec{\lambda}}(X')$.

We list the basic properties and fusion rules of conformal blocks.

We first talk about desingularization of nodal curves. In $\dim_{\mathbb{C}} X = 1$, the notions of normality and non-singularity are equivalent, and there always exists a canonical procedure of *normalization*. That is, there is always a normal variety X^{norm} and a birational morphism $X^{\mathrm{norm}} \to X$. Recall that X is called normal if all local rings are integrally closed. For nodal curves, the normalization looks something like Figure **??**. In particular, the map is $2 - 1$ on the singularity.

Theorem 104. *Suppose* $X = (X, Q_1, \ldots, Q_n, \xi_1, \ldots, \xi_N)$ *and* $\tilde{X} = (X, Q_1, \ldots, Q_{N+1}, \xi_1, \ldots, \xi_{N+1})$. *Then canonically*

$$\mathcal{V}^\dagger_{\vec{\lambda},0}(\tilde{X}) \cong \mathcal{V}^\dagger_{\vec{\lambda}}(X).$$

In the picture of moduli spaces of curves with parabolic structure, this corresponds to the fact that a trivial parabolic weight at a points gives no extra parabolic structure.

Now assume that we have a nodal singularity P with pre-image P', P'' in the normalization $\tilde{X}$ as in Figure **??**. Recall that the Weyl group W was generated by reflections $W = \langle s_\alpha \rangle$, and W acts transitively on the Weyl chambers. For a choice of positive Weyl chamber, taking the negative of it will again be a Weyl chamber. In particular there is a $w \in W$, referred to as the *longest element of the Weyl group*, such that $w(\text{pos. Weyl chamber}) = -\text{pos. Weyl chamber}$, but, in general, w does not act by multiplication by -1 (even if this is the case for $(2, \mathbb{C})$). Consider the map $(\cdot)^\dagger$ on the positive Weyl chamber mapping $\lambda \mapsto \lambda^\dagger = -w(\lambda)$. Note that this preserves the Weyl alcoves. Let P_l be the Weyl alcove for the level l. For the double point P, we have a canonical isomorphism

$$\mathcal{V}^\dagger_{\vec{\lambda}}(X) \to \bigoplus_{\mu \in P_l} \mathcal{V}^\dagger_{\mu, \mu^\dagger, \vec{\lambda}}(\tilde{X}).$$

Here P' and P'' are marked points in $\tilde{X}$ with new formal neighbourhoods and weights $\mu, \mu^\dagger$.

Proposition 105. *The dimensions* $\dim \mathcal{V}^\dagger_{\vec{\lambda}}(X)$ *is invariant under deformations.*

Here, a deformation corresponds to deforming a closed curve in X to a double point. For a curve X, using the fusion rules, by repeatedly deforming and normalizing, we end up with a copy of $\mathbb{P}^1$ with a number of marked points.

15th lecture, May 2nd 2011

6.3 Space of conformal blocks vs geometric quantization

Today, we will sketch why, given a Riemann surface, there is a canonical isomorphism between the space of conformal block and the geometric quantization of the moduli space.

Recall that the space of conformal blocks is constructed as follows: Let C be a Riemann surface with marked points $Q_1, \dots, Q_n$ and associated formal coordinates ζ_i at each marked point. Let X be the curve together with the marked points. Let

$$\hat{\mathfrak{g}}_N = \bigoplus_{i=1}^{N} (\mathfrak{g} \otimes \mathbb{C}((\zeta_i))) \oplus \mathbb{C}_c$$

be the vector space with the Lie bracket defined previously. This itself does not depend on the Riemann surface, but it has a subalgebra that does; let

$$\hat{\mathfrak{g}}(X) = \mathfrak{g} \otimes H^0(C, \mathcal{O}_C(* \sum Q_i)).$$

The space of conformal blocks is defined as follows. Given $\vec{\lambda} = (\lambda_1, \dots, \lambda_N)$, a sequence of lattice points in the Weyl alcove at level l, we get a representation $\mathcal{H}_{\vec{\lambda}}$ of $\hat{\mathfrak{g}}_N$.

Definition 106. The space of covacua is

$$\mathcal{V}_{\vec{\lambda}}(X) = \mathcal{H}_{\vec{\lambda}} / \hat{\mathfrak{g}}(X) \mathcal{H}_{\vec{\lambda}},$$

and the space of vacua is the dual space $\mathcal{V}_{\vec{\lambda}}^{\dagger}$.

The space $\mathcal{V}_{\vec{\lambda}}^{\dagger}$ corresponds to geometric quantization of the moduli space of parabolic bundles with parabolic weights λ_i at Q_i. If there are no marked points, we pick an arbitrary point and use $\lambda = 0$.

Remark 107. We obtain from $\mathcal{V}_{\vec{\lambda}}^{\dagger}(X)$ a bundle of conformal blocks over Teichmüller space or the moduli space of (pointed) Riemann surfaces. This bundle comes with a natural projectively flat connection. This is referred to as the "KZ connection" or the "KZ equation", after Knizhnik and Zamolodchikov. It's also called the KZB connection or the WZW connection. The B is for Bertrand.

The moduli space can be thought of in terms of conformal blocks, as we will see in the following.

Proposition 108. *A holomorphic bundle on a Riemann surface can be trivialized over the complement of any point in the Riemann surface.*

The references here are [BL94] and [Fal94].

The first step is an alternative description of the moduli stack of bundles. Using the Proposition above, we can describe a given bundle using a single transition function, as the surface can be covered with two charts over which the bundle trivializes. This transition function of course depends on the choice of charts. We fix the determinant of all bundles. Let A_p be the meromorphic functions on C with a pole of arbitrary order at P. Formally, given an "open disk", the transition functions are elements of $SL_r(\mathbb{C}((z)))$, where z is a formal parameter around p. Here, $\mathbb{C}((z))$ are formal Laurent series in z, and r is the rank of the bundle. Morally, $SL_r(\mathbb{C}((z)))$ determines the bundle, but not uniquely: It depends on the choice of trivialization. Instead, we consider the double coset

$$SL_r(A_p) \backslash SL_r(\mathbb{C}((z))) / SL_r(\mathbb{C}[[z]]).$$

This idea goes back to Weil.

Theorem 109. *This double coset is canonically isomorphic to the moduli stack of fixed determinant rank r bundles.*

Note that the quotient $Q := SL_r(\mathbb{C}((z)))/SL_r(\mathbb{C}[[z]])$ is independent of the Riemann surface; this space is (essentially) referred to as the "affine Grassmannian".

The left quotient defines a bundle $\pi : Q \to \mathcal{M}^0(C)$ over the moduli stack. The moduli stack has a determinant line bundle $\mathcal{L} \to \mathcal{M}^0(C)$. We can pull back this line bundle to obtain $\pi^*\mathcal{L} \to Q$.

Now, $SL_r(\mathbb{C}((z)))$ (which is the loop group of SL_r) acts on Q. The action of $SL_r(\mathbb{C}((z)))$ does not lift to $\pi^*\mathcal{L}$, the action of the central extension $\widehat{SL_r(\mathbb{C}((z)))}$ does act on $\pi^*\mathcal{L}$.

Theorem 110 (Kumar,Mathieu)**.** *The space of global sections $H^0(Q, \pi^*\mathcal{L})$ of $\pi^*\mathcal{L} \to Q$ though of as a representation of $\widehat{SL_r(\mathbb{C}((z)))}$ corresponds exactly to the positive energy representation (called the basic representation) of highest weight $\lambda = 0$.*

Now, $SL_r(A_p)$-invariant sections of $\pi^*\mathcal{L} \to Q$ correspond to sections of $\mathcal{L} \to \mathcal{M}^0(C) = SL_r(A_p)\backslash Q$ through pull-back. Remark that if G is a Lie group, $\mathrm{Lie}(G) = \mathfrak{g}$, and V is a G-representation, $\vec{v} \in V$, then $G \cdot \vec{v} = \vec{v}$, if and only if $\mathfrak{g}\vec{v} = 0$. Recall that

$$\begin{aligned}\mathcal{V}_\lambda^\dagger(X) &= (\mathcal{H}_\lambda/\hat{\mathfrak{g}}(X)\mathcal{H}_\lambda)^\dagger \\ &= \ker(\hat{\mathfrak{g}}(X) \curvearrowright \mathcal{H}_\lambda^*).\end{aligned}$$

Theorem 111.

$$\mathcal{V}_{\lambda=0}^\dagger(X) \cong H^0(Q, \pi^*\mathcal{L})^{SL_r(A_p)} \cong H^0(\mathcal{M}^0(C), \mathcal{L}).$$

Theorem 112.

$$H^0(\mathcal{M}^0(C), \mathcal{L}) \cong H^0(\mathcal{M}^{0,stable}(C), \mathcal{L}).$$

15th lecture, May 4th 2011

6.4 The Wess–Zumino–Witten model

The reference for this is [Oht02].

The WZW model is a particular example of a non-linear Σ-model, where the source is a Riemann surface, and the target is a Lie group.

Assume that X be a closed Riemann surface, and let G be a connected simply-connected complex (or occasionally compact) Lie group. A *field* is a smooth map $f : X \to G$. The action of the theory consists of two terms; an energy term and the WZW term. The *energy term* is

$$E_X(f) = \frac{i}{4\pi}\int_X \mathrm{tr}(f^{-1}\,\partial f \wedge f^{-1}\,\overline{\partial} f).$$

The WZW term is

$$\frac{i}{12\pi}\int_M \mathrm{tr}(\tilde{f}^{-1}d\tilde{f} \wedge \tilde{f}^{-1}d\tilde{f} \wedge \tilde{f}^{-1}d\tilde{f}),$$

where M is a 3-manifold bounded by X, and $\tilde{f} : M \to G$ is an extension of f. The last term is independent of M and $\tilde{f}$ up to $2\pi i\mathbb{Z}$. Let $S_X(f)$ denote the sum of the two terms. The partition function for the WZW model is given by the path integral

$$Z_k^{\mathrm{WZW}}(X) = \fint_{f\in \mathrm{Map}(X,G)} \exp(kS_X(f))\mathcal{D}f$$

where k is some integer. Though ill-defined, this is an invariant of the Riemann surface but not a topological one.

Note that $\mathrm{Map}(X, G)$ is a group under pointwise multiplication. The *Polyakov–Wiegmann formula* says that

$$S_X(fg) = S_X(f) + S_X(g) + \frac{i}{2\pi}\int_X \mathrm{tr}(f^{-1}\,\overline{\partial} f \wedge \partial g\, g^{-1})$$

In the case where X has boundary, we want to associate a vector space to the boundary, as we did in Chern–Simons theory. The vector spaces will arise as spaces of sections of a line bundle over the loop group.

For simplicity assume that $\partial X \cong S^1$. We will think of LG as $\mathrm{Map}(S^1, G)$. We want to define a line bundle $\mathcal{L} \to LG$ explicitly as follows. Look at

$$\left(\mathbb{C} \times \bigcup_{X, \partial X \cong S^1} \mathrm{Map}(X, G)\right) / \sim,$$

where $(\alpha, f) \in \mathbb{C} \sim (\alpha', f')$, if $f|_{\partial X} = f'|_{\partial X'}$, and

$$\alpha = \alpha' \exp(\int_M g^{-1}\, dg \wedge g^{-1}\, dg \wedge g^{-1}\, dg)$$

where M is a 3-manifold such that $\partial M = X \cup_{S^1} -X'$, the gluing being with respect to f and reversing the orientation of X', and g is an extension of $f \cup f'$ to M.

Claim 113. *This defines a complex line bundle $\pi : \mathcal{L} \to LG$ with $\pi(\alpha, f) = f|_{\partial X}$.*

Now, if $\partial X \cong S^1$, define

$$Z_k^{\mathrm{WZW}}(X)(\gamma) := \fint_{f \in \mathrm{Map}(X,G)_\gamma} \exp(kS_X(f))\mathcal{D}f$$

where $\gamma \in LG$, and

$$\mathrm{Map}(X, G)_\gamma = \{f : X \to G, f|_{\partial X} = \gamma\}.$$

Claim 114. *This defines a section $Z_k^{\mathrm{WZW}}(X) : LG \to \mathcal{L}^k$.*

We now want to see that there is a product structure on the total space of $\mathcal{L}$ (and all $\mathcal{L}^k$), such that throwing away the zero section, we get a new group $\pi : \mathcal{L} \setminus (LG \times 0) \to LG$, which is nothing but the central extension of LG.

Let $(\alpha_i, f_i) \in \mathcal{L}_{\gamma_i}$, $i = 1, 2$ be two elements of $\mathcal{L}$, and assume without loss of generality that they are represented by maps f_i of the *same* Riemann surface X. Define

$$(\alpha_1, f_1) \cdot (\alpha_2, f_2) = (\alpha_1\alpha_2 \exp(\delta(f_1 f_2)), f_1 f_2) \in \mathcal{L}_{\gamma_1\gamma_2},$$

where

$$\delta(f_1 f_2) = -E_X(f_1) - E_X(f_2) + E_x(f_1 f_2) - \frac{i}{2\pi}\int_X \mathrm{tr}(f_1^{-1}\,\overline{\partial} f_1 \wedge \partial f_2\, f_2^{-1}).$$

Claim 115. *Using the Polyakov–Wiegmann formula, one sees that this is well-defined. In particular, it is well-behaved with respect to $\sim$.*

Claim 116. *The space $\widehat{LG} := \mathcal{L} \setminus (LG \times 0)$ is a group for $\cdot$. The identity is given by $(1, X \to G : x \mapsto 1_G)$, and it is not hard to cook up inverses. We have a central extension*

$$1 \to \mathbb{C}^* \to \widehat{LG} \to LG \to 1.$$

Claim 117. *$\widehat{LG}$ acts on $\Gamma(\mathcal{L} \to LG)$. This is the basic representation at level 1 of $\widehat{LG}$.*

16th lecture, May 9th 2011

7 Donaldson theory

In the last two lectures, we will see a brief introduction to Donaldson–Witten theory and its relation to topological quantum field theory.

We begin by giving some general remarks on 4-manifolds. Let M be a simply connected 4-manifold. Then $H^i(M,\mathbb{Z})$ is $\mathbb{Z}$, 0, $\mathbb{Z}^{b_2}$, 0, $\mathbb{Z}$ for $i = 0,1,2,3,4$ respectively. On $H^2(M)$, we have the intersection pairing

$$H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \to \mathbb{Z}$$

given by $(\alpha,\beta) \mapsto (\alpha \cup \beta)[M]$. This is a symmetric bilinear form with signature $\sigma = b_2^+ - b_2^-$, and $b_2 = b_2^+ + b_2^-$.

Theorem 118 (Freedman, '82)**.** *Any signature can occur as the intersection pairing of a closed, simply connected topological 4-manifold.*

Theorem 119 (Donaldson, '83)**.** *Restrictions apply in the smooth world. In particular, not all topological 4-manifolds are smoothable.*

The proof uses instantons which we will now review. Suppose we have a smooth principal G-bundle $P \to M$. For the Donaldson invariants, it suffices to look at $G = SU(2)$ or $G = SO(3)$. Let $\mathcal{A}$ be the affine space of connections on P, and let $A \in \mathcal{A}$ be generic. Then

$$\int_M c_s(P) = -\frac{1}{8\pi^2}\operatorname{Tr}(F_A \wedge F_A) \in \mathbb{Z}.$$

Now, assume we have a Riemannian metric g on M. We then have a Hodge star operator $* : H^2(M) \to H^2(M)$. We have the Yang–Mills functional

$$S_{\mathrm{YM}}(A) = \int_M (F_A \wedge *F_A) = \|F_A\|^2.$$

The Hodge star $* : H^2(M) \to H^2(M)$ or $\Omega^2(M) \to \Omega^2(M)$ satisfies $*^2 = 1$, and we can decompose it into its $+1$ and -1 eigenspaces, $\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M)$, and we for the curvature $F = F_A$ we write $F = F_+ + F_-$. Taking this into account,

$$\begin{aligned} S_{\mathrm{YM}}(A) &= \int_M \operatorname{Tr}(F_+ \wedge *F_+ + F_- \wedge *F_-), \\ \int_M c_2(P) &= \frac{1}{8\pi^2}\int_M \operatorname{Tr}(F_+ \wedge *F_+ - F_- \wedge *F_-). \end{aligned}$$

Therefore, $S_{\mathrm{YM}}(A) \geq 8\pi^2 |\int_M c_2(P)|$. The absolute minimum of $S_{\mathrm{YM}}(A)$ is reached when $F = *F$, i.e. $F = F_+$, if $\int_M c_2(P) > 0$ or when $F = -*F$, i.e. $\overline{F} = F_-$ if $\int_M c_2(P) < 0$. The equation $F = *F$ is called the self-duality equation, and $F = -*F$ is the anti-self duality (ASD) equation.

From now on, assume $\int_M c_2(P) < 0$ so that we are interested in the ASD case.

Definition 120. A *Yang–Mills instanton* is a connection satisfying the anti-self duality equation.

We are interested in the moduli space of instantons. Let $\mathcal{A}^*$ be the irreducible connections on P. The moduli space of instantons is

$$\mathcal{M}_P = \{A \in \mathcal{A} \mid *F = -F\}/\mathcal{G}_P,$$

where $\mathcal{G}_P$ is the group of (unitary) gauge transformations. If we restrict to $A^* \in \mathcal{A}$, we get an open subspace of the moduli space of instantons which is a smooth manifold. In general, the moduli space of instantons is neither smooth nor compact, in contrast with the moduli space of

flat connections on a Riemann surface. However, there exists a canonical compactification called the *Uhlenbeck compactification*, given by adding so-called "ideal instantons" that have curvature at isolated points (referred to as "bubbling").

By using the index formula, one finds that

$$\dim(\mathcal{M}_P) = 8|\int_M c_2(P)| - \frac{3}{2}(\chi_M + \sigma_M)$$

A priori, $\mathcal{M}_P$ depends on the choice of the metric g. But Donaldson showed that if $b_2^+ > 1$, for a generic 1-parameter family of metrics, none have reducible instantons. That is, if g is generic, and $b_2^+ > 1$, then $\mathcal{M}_P$ is smooth and independent of g.

7.1 Donaldson invariants

Let $\mathcal{B} = \mathcal{A}/\mathcal{G}$. This space is infinite dimensional and depends only on the topology of M. Then $\mathcal{M}_P \subseteq \mathcal{B}$. We consider Donaldson's μ: On $\mathcal{A} \times M$, we have a "canonical" G-principal bundle obtained by pulling back the bundle over M. We take the adjoint bundle and quotient out by $\mathcal{G}$ to obtain a bundle $\tilde{P} \to \mathcal{B} \times M$. Then $\mu : H_i(M) \to H^{4-i}(\mathcal{B})$ is given by

$$\mu : \alpha \mapsto p_1(\tilde{P})/\alpha,$$

where / denotes the slant product defined for $\alpha \in H_i(N)$, $\beta \in H^j(M \times N)$ gives an element $\beta/\alpha \in H^{j-i}(M)$ using the Künneth formula to write $\beta \in H^*(M) \otimes H^*(N)$ and capping with α.

Theorem 121. *We have $\mu(z) = 1$ if $z \in H_4(M)$. Elements of the form $\mu(x)$ for $x \in H_0(M), H_2(M)$ generate $H^*(\mathcal{B})$ freely.*

Choose generators $H(M, \mathbb{Q}) = \langle x \rangle$, $H_2(M, \mathbb{Q}) = \langle \gamma_1, \dots, \gamma_{b_2} \rangle$. Then the Donaldson invariants

$$\mathbb{Q}[x, \gamma_1, \dots, \gamma_{b_2}] \to \mathbb{Q}$$

are formal polynomials

$$p(x_1, \gamma_1, \dots, \gamma_{b_2}) \mapsto \int_{\mathcal{M}_P} p(\mu(x), \mu(\gamma_1), \dots, \mu(\gamma_{b_2})) \in H^*(B),$$

where the integral is defined using the compactification mentioned earlier. If $b_2^+ > 1$, this depends only on the smooth structure. If $b_2^+ = 1$ it depends on the metric.

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We end the main part of the course, giving a very sketchy overview of Donaldson–Witten theory.

Suppose we have a (4-dimensional) field theory on a Riemannian (4-)manifold, determined by an action $S(\varphi) = \int_M \mathcal{L}(\varphi(x))dx$. If we have observables $\mathcal{O}_i(\varphi)$, then the "vacuum expectation value" is determined, formally, by the path integral

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_r \rangle = \int \mathcal{D}\varphi \mathcal{O}_1(\varphi) \cdots \mathcal{O}_r(\varphi) e^{iS(\varphi)},$$

with partition function $Z = \langle 1 \rangle$. Witten now argues that if we have a symmetry generated by Q, with $Q^2 = 0$, $\{Q, S\} = 0$, such that the energy-momentum tensor satisfies

$$T_{\mu\nu} = \{Q, \Lambda_{\mu\nu}\}$$

for some symmetric tensor $\Lambda_{\mu\nu}$, then the partition function is topological, and moreover, if $\{Q, \mathcal{O}_i\}$, the vacuum expectation value $\langle \mathcal{O}_i \rangle$ is topological.

Witten furthermore argues, that if we take $N = 2$ supersymmetric Yang–Mills theory in 4 dimensions and apply what is known as "topological twist", we still have a field theory in four dimensions with fields having a specific nature: 1 gauge field (i.e. a connection), 1 one-form, 1 self-dual two-form, 1 "fermionic" zero-form, and 2 "bosonic" zero forms. There is a specific action for these behaving in such a way, that the fields don't satisfy the spin-statistics theorem. This topologically twisted theorem then satisfies the previous condition, so Z is topological, and in particular it is independent of the value of the coupling constant. We can take the limit where the coupling constant becomes very small, and in this limit, the path integral "localizes" onto the anti-self dual connections, and there exist observables $\mathcal{O}_i$ such that $\{Q, \mathcal{O}_i\} = 0$, whose vacuum expectation values $\langle \mathcal{O}_1 \dots \mathcal{O}_r \rangle$ are exactly the Donaldson invariants.

8 More student seminars

8.1 Shehryar

We begin discussing some pre-requisites of complex tori. Let V be a $\mathbb{C}$-vector space, $\dim_{\mathbb{C}} V = g$, let $\Lambda \subseteq V$ be a (maximal) lattice:

Definition 122. $\Lambda \subseteq V$ is a lattice if Λ is a discrete subgroup and co-compact.

Associated to a lattice Λ, we have a complex torus $X = V/\Lambda$ of dimension g. If $(e_1, \dots, e_g)$ span V, and $\lambda_1, \dots, \lambda_{2g}$ span Λ, then $\lambda_j = \sum_{i,j} \lambda_{ij} e_i$. Let $\pi \in \mathrm{Mat}_{g \times 2g}(\mathbb{C})$ be the matrix $(\pi)_{ij} = \lambda_{ij}$. If $h : X \to X'$ is a holomorphic map between complex tori, then there exists a unique group homomorphism $f : X \to X'$ such that $h(x) = f(x) + h(0)$, and there exists a unique $\mathbb{Z}$-linear map $F : V \to V'$ such that $F|_\Lambda \subseteq \Lambda'$. This means that we have injective maps

$$\begin{array}{ccc} \mathrm{Hom}(X, X') & \longrightarrow & \mathrm{Hom}_{\mathbb{C}}(V, V') \\ & \searrow & \\ & & \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda') \end{array}$$

If the image of $f \in \mathrm{Hom}(X, X')$ under these two mappings are A, B respectively, then $A\pi = \pi' B$.

As a real manifold,

$$X \cong \times_{i=1}^{2g} S^1 \cong \times_{i=1}^{2g} \lambda_i \mathbb{R}/\lambda_i \mathbb{Z}$$

with fundamental group $\pi_1(X) = \Lambda = H_1(X, \mathbb{Z})$. Also, by the universal coefficient theorem, $H^1(X, \mathbb{Z}) \cong \mathrm{Hom}(\Lambda, \mathbb{Z})$. The cup product map defines an isomorphism $H^n(X, \mathbb{Z}) \cong \wedge^n H^1(X, \mathbb{Z})$.

Define $\mathrm{Alt}^n(\Lambda, \mathbb{Z}) = \wedge^n \mathrm{Hom}(\Lambda, \mathbb{Z})$.

We will be interested in line bundles $V \to X$. We note that $H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X) = \{\text{FOAs} = \text{Factors of automorphy}\}$, where $f : \Lambda \times V \to \mathbb{C}^*$ is a FOA, if $f(\lambda + \mu, v) = f(\lambda, \mu + v) f(\mu, v)$, $\mu, \lambda \in \Lambda, v \in V$. Consider also $(\lambda, v) \mapsto h(\lambda + v)(h(v))^{-1}$, where $h \in H^0(\mathcal{O}_X^*)$. We obtain line bundles by $V \times \mathbb{C}/\Lambda \to X$, where Λ acts by $\lambda \circ (v, t) = (\lambda + v, f(\lambda, v)t)$.

We have a short exact sequence

$$1 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1,$$

which induces a long exact sequence

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \cong \mathrm{Alt}^2(\Lambda, \mathbb{Z}),$$

where c_1 is the first Chern class. Let $L \in H^1(X, \mathcal{O}_X^*)$, with $c_1(L) = E$. Extend to $\tilde{E} : V \times V \to \mathbb{Z}$, such that $\mathrm{Im}\, \tilde{E}|_\Lambda \subseteq \mathbb{Z}$. Conversely, if $E : V \times V \to \mathbb{R}$ is an alternating form such that $\mathrm{Im}\, E|_\Lambda \subseteq \mathbb{Z}$, and $E(iv, iw) = E(v, w)$, then there exists $L \in \mathrm{Pic}(X)$ such that $E = \widetilde{c_1(L)}$. Moreover, there is a 1-1 correspondence between E satisfying these two conditions, and Hermitian forms $H : V \times V \to \mathbb{C}$ such that $\mathrm{Im}\, H|_\Lambda \subseteq \mathbb{Z}$ given by taking the imaginary part of the Hermitian form.

Definition 123. Let $\mathrm{NS}(X) = \mathrm{Im}(c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}))$.

To every FOA, we associated a line bundle, but we want to figure out what a FOA is for a given line bundle. Let $H \in \mathrm{NS}(X)$, and define a *semi-character* $\chi : \Lambda \to \mathbb{C}_1 := \{z \in \mathbb{C} \mid |z| = 1\}$ to be something satisfying

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)e^{\pi i \operatorname{Im} H(\lambda,\mu)}.$$

Let $P(\Lambda) = \{(H, X) \mid H \in \mathrm{NS}(X), \chi \text{ semi-character}\}$. We have mappings

$$\begin{array}{ccccc} \mathrm{Hom}(\Lambda, \mathbb{C}_1) & \longrightarrow & P(\Lambda) & \longrightarrow & \mathrm{NS}(X) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{NS}(X) \end{array}$$

Here, we let $a_L(\lambda, v) = \chi(\lambda)e^{\pi H(v,\lambda)+\frac{\pi}{2}H(\lambda,\lambda)}$. We obtain an isomorphism $\mathrm{Hom}(\Lambda, \mathbb{C}) \cong \mathrm{Pic}^0(X)$, where to a line bundle L we associate the FOA $L = L(H, \chi)a_L$. If $x \in \Lambda$, then

$$t_x^* L = L(H, \chi e(2\pi i \operatorname{Im} H(V, \cdot))),$$

where t_x denotes translation by x. The last thing we need is $\varphi_L : X \to \mathrm{Pic}^0(X)$ given by

$$\varphi_L(x) = t_x^* L \otimes L^{-1} = L(0, e^{2\pi i \operatorname{Im} H(v,\cdot)}).$$

Let $\Lambda(L) = \{v \in V \mid \operatorname{Im} H(v, \Lambda) \subseteq \mathbb{Z}\}$ and $k(L) = \Lambda(L)/\Lambda$.

A *character of a line bundle* is defined as follows. Fix (X, H), X a complex torus, H a positive Hermitian form on X (X is called a polarized abelian variety if this H exists, and H a choice of polarization), $\operatorname{Im} H = E$. Then there is a basis $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ of Λ such that

$$E = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where $D = \mathrm{diag}(d_1, \ldots, d_g)$, $d_i > 0$, $d_i \in \mathbb{Z}$. Moreover $\Lambda_1 = \mathrm{span}(\lambda_1, \ldots, \lambda_g)$ and $\Lambda_2 = \mathrm{span}(\mu_1, \ldots, \mu_g)$, are maximal isotropic subgroups, $\Lambda = \Lambda_1 \oplus \Lambda_2$ and extend it to $V = V_1 \oplus V_2$.

Given a polarization H, we want to pick out a special line bundle as follows: Define $X_0 : V \to \mathbb{C}_1$ by

$$X_0(v) = e^{\pi i E(v_1, v_2)}$$

, where $v_1 + v_2 = v$. Then $X_0|_\Lambda$ is a semi-character, $L_0 = L(H, X_0) \in \mathrm{Pic}^H(X)$.

Proposition 124. *We have $X_0|_{\Lambda_{1,2}} = 0$, and for all $L \in \mathrm{Pic}^H(X)$, there is a $c \in \Lambda$ unique up to $\Lambda(L)$, such that*

$$L = t_c^* L_0$$

and

$$L = L(H, X_0 e^{2\pi i E(c,\cdot)}).$$

Definition 125. c is the *character* of $L \in \mathrm{Pic}^H(X)$.

We will define the theta-function for a line bundle using the character. A *theta-function* associated to L (with FOA f) is a map $\theta : V \to \mathbb{C}$ such that

$$\theta(v + \lambda) = f(\lambda, v)\theta(v).$$

When $H > 0$, writing $V = V_2 \oplus iV_2$, then $H|_{V_2}$ is symmetric, since $\operatorname{Im} H|_{V_2} = 0$. Let B be the induced symmetric form. If c is the character of L, put

$$\theta^c(v) = \exp(\pi H(v, c) - \pi/2H(c, c) + \pi/2B((v + c), (v + c))) \cdot \sum_{l \in \mathbb{Z}^g} \exp(\pi(H - B)(v + c, \lambda) - \tfrac{\pi}{2}(H - B)(\lambda, \lambda)).$$

For all $\omega \in k(L)$, put $\theta^c_{\overline{w}} = a_L(\omega, \dot{})^{-1}\theta^c(\cdot + w)$.

Theorem 126. *Suppose L is a positive definite line bundle, with a choice of decomposition $V = V_1 \oplus V_2$ and c a character L. Then the set of theta-functions $\{\theta^c_{\overline{w}} \mid \overline{w} \in k(L)_1\}$ form a basis for $H^0(L)$, and $\#k(L)_1 = \mathrm{PF}(E) = \mathrm{Det}(E)$. Here, $k(L)_1 = \Lambda_1(L)/\Lambda_1$.*

We now turn to moduli of polarized abelian varieties. Fix (X, H), $H > 0$, $V = V_1 \oplus V_2$, a basis $\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g$, let $e_v = \frac{1}{d_v}\mu_v$ be a complex basis for V. Then the period matrix is $\pi = (Z, D)$, where

$$Z \in \mathfrak{h}_g = \{Z \in M_{g\times g}(\mathbb{C}) \mid {}^t Z = Z, \operatorname{Im} Z > 0\}$$

Conversely, for fixed D, and a $Z \in \mathfrak{h}_g$, we define a variety (X_Z, H_Z), where $X_Z = \mathbb{C}^g/\Lambda_Z$, where $\Lambda_Z = Z\mathbb{Z}^g \oplus D\mathbb{Z}^g$, and $H_z = (\operatorname{Im} Z)^{-1}$).

Suppose $Z, Z' \in \mathfrak{h}_g$, and $(X_z, H_z) \cong (X_{Z'}, H_{Z'})$, then there is a matrix $M \in \mathrm{Sp}_{2g}(\mathbb{Q})$ such that $M(Z) = M(Z')$, and ${}^t M\Lambda_D \subseteq \Lambda_D$, where

$$\Lambda_D = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & D \end{pmatrix} \mathbb{Z}^{2g}.$$

Let G_D be the set of matrices $M \in \mathrm{Sp}_{2g}(\mathbb{Q})$ satisfying these two conditions. Then $\mathfrak{h}_g/G_D$ is the moduli space of polarized abelian varieties. Let (X, H) be a principal polarized abelian variety (meaning that $d_i = 1$ for all i), and let $\Lambda_1 = Z\mathbb{Z}^g$, $\Lambda_2 = \mathbb{Z}^g$. Then for $c \in V$, write $c = Zc_1 + c_2$, $c_i \in V_i$. With respect to a character c, the classical theta function is

$$\theta \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} (v, Z) = \sum_{l\in\mathbb{Z}^g} \exp(\pi i^t (l + c_1)Z(l + c_1) + 2\pi i^t (v + c_2)(l + c_1)).$$

The connection between this theta function and the canonical one we defined is

$$\theta_Z^{Zc_1+c_2} = e^{\pi/2B(\cdot,\cdot)-\pi i^t c_1 c_2}\theta \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} (\cdot, Z).$$

Theorem 127. *The classical theta function*

$$\theta \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} (v, Z)$$

is holomorphic on $\mathbb{C}^g \times \mathfrak{h}_g$ and satisfies the second order heat equation

$$\frac{\partial^2\theta \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}{\partial z_i \partial z_j}\frac{1}{4\pi i} = \frac{\partial\theta \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}{\partial z_{ij}}.$$

For $\sigma \in \mathfrak{h}_g$, consider the line bundle $L_\sigma = L(H_\sigma, X_0) \to (X, H_\sigma)$. We obtain then a bundle over $\mathfrak{h}_g$ with fiber $H^0(X, L_\sigma)$ at σ. The theorem determines a connection in this bundle.

For a Riemann surface X, consider the moduli space $\mathcal{M}$ of flat unitary connections in $L \to X$ is equivalent to the moduli space of line bundles on X, which is the Jacobian $J(X)$ of X, which on the other hand is the space of principal polarized abelian varieties. Varying the complex structure on X defines a map $J(\cdot) \hookrightarrow \mathfrak{h}_g$ (called the Torelli map); determining the image of this map is an open problem.

8.2 Amit

Today we will be discussing Chern–Simons theory with finite gauge group Γ. The reference is Freed and Quinn [FQ93].

Fix an integer $d \in \mathbb{Z}$, $d > 0$. As always, spacetimes are $(d+1)$-dimensional, with d the dimension of space. Let X be a compact oriented spacetime. To X we associate a category $\mathcal{C}_X$ called the space of fields. That is, $\mathcal{C}_X$ is the space of principal Γ-bundles. Since Γ is finite, connections in

Γ-bundles are canonical and we will not consider them. Let $B\Gamma = E\Gamma/\Gamma$ be the classifying space of Γ, so $B\Gamma = K(\Gamma, 1)$.

We consider now the *action* of the theory. Let $\alpha \in H^{d+1}(B\Gamma; \mathbb{R}/\mathbb{Z})$, and consider a singular cocycle $\hat{\alpha} \in C^{d+1}(B\Gamma, \mathbb{R}/\mathbb{Z})$ such that $\hat{\alpha}$ and α represent the same coset of $B^{d+1}(B\Gamma, \mathbb{R}/\mathbb{Z})$. The simplest case is when $\hat{\alpha} = \alpha = 0$.

Now let X be a closed oriented $(d+1)$-dimensional manifold, and fix a principal bundle $P \in \mathcal{C}_X$. We have canonical maps $F : P \to E\Gamma$, $\overline{F} : X \to B\Gamma$. We define the action by

$$S_X(P) = \alpha(\overline{F}_\alpha[X]) \in \mathbb{R}/\mathbb{Z},$$

where $[X] \in H_{d+1}(X)$ is the fundamental class.

Proposition/Definition 128 (Metrized "integration line")**.** Let Y be a closed oriented d-manifold. The *metrized integration line* (MIL) is defined, and if X is a compact oriented $(d+1)$-manifold with boundary $i : \partial X \hookrightarrow X$ and $\alpha \in C^{d+1}(X, \mathbb{R}/\mathbb{Z})$, put

$$I = \exp(2\pi i \int_X \alpha) \in I_{\partial X, i^*\alpha}.$$

Then I has unit norm and satisfies a bunch of properties such as functoriality and a gluing property.

Let $\hat{\alpha} \in C^{d+1}(B\Gamma, \mathbb{R}/\mathbb{Z})$. Let $Q \to Y$ be a principal Γ-bundle. Now we defined the *metrized line* L_Q. Define a category $\mathcal{C}_Q$ whose objects are classifying maps $f : Q \to E\Gamma$, and morphisms are $f \overset{h}{\to} f'$ is a homototopy class h relative to the boundary. For an object f in $\mathcal{C}_Q$, we put

$$\begin{aligned} \mathcal{F}_Q(f) &= I_{Y, \overline{f}^*\hat{\alpha}}, \\ \mathcal{F}_Q(f \overset{h}{\to} f') &= \exp(2\pi i \int_{[0,1]\times Y} \overline{h}^*\hat{\alpha}) : I_{Y, \overline{f'}^*\alpha} \to I_{Y, \overline{f}^*\alpha}. \end{aligned}$$

Consider as before a principal Γ-bundle $P \to X$. For all classifying maps $F : P \to E\Gamma$, put

$$e^{2\pi i S_X(P,F)} = \exp(2\pi i \int \overline{F}^*\hat{\alpha}) \in L_{\partial X, \mathcal{F}F^*\hat{\alpha}}.$$

Theorem 129. *Let Y be a closed oriented d-manifold, X a compact oriented $(d+1)$-manifold. Consider the associations*

$$\begin{aligned} &Q \mapsto L_Q, \ Q \in \mathcal{C}_Y \\ &P \mapsto e^{2\pi i S_X(P)}, \ P \in \mathcal{C}_X. \end{aligned}$$

This satisfies the following properties

(a) Functoriality: For a bundle map $\psi : Q' \to Q$ covering an orientation preserving diffeomorphism $\overline{\psi} : Y' \to Y$, then $\psi_* L_{Q'} \to L_Q$ is an isometry. Similarly, if $\varphi : P' \to P$, $\overline{\varphi} : X' \to X$, then

$$(\partial\varphi)_*(e^{2\pi i S_{X'}(P')}) = e^{2\pi i S_X(P)}.$$

(b) Orientation: $L_{Q,-Y} \cong \overline{L_{Q,Y}}$ and

$$e^{2\pi i S_{-X}(P)} = e^{\overline{2\pi i S_X(P)}}.$$

(c) Additivity: If $Y = Y_1 \sqcup Y_2$, and $X = X_1 \sqcup X_2$, with bundles $Q_i \to Y_i$, $P_i \to X_i$, then

$$\begin{aligned} L_{Q_1 \sqcup Q_2} &\cong L_{Q_1} \otimes L_{Q_2}, \\ e^{2\pi i S_X(P_1 \sqcup P_2)} &= e^{2\pi i S_{X_1}(P_1)} \otimes e^{2\pi i S_{X_2}(P_2)} \end{aligned}$$

(d) *Gluing: Assume $Y \hookrightarrow X$ is a submanifold, and let X^{cut} be the manifold obtained by cutting X along Y with a principal bundle P^{cut} over X^{cut}, and let $Q = P|_Y$. Then*

$$e^{2\pi i S_X(P)} = \mathrm{Tr}_Q(e^{2\pi i X^{\mathrm{cut}}(P^{\mathrm{cut}})}),$$

where the trace is given by contraction,

$$\mathrm{Tr}_Q : L_{\partial P^{\mathrm{cut}}} \cong L_{\partial P} \otimes L_Q \otimes \overline{L_Q} \to L_{\partial P}.$$

Given a manifold M, we have the space $\mathcal{C}_M$ of Γ-bundles as before. We define a measure

$$\mu_P = \frac{1}{\#\mathrm{Aut}P},$$

where $\mathrm{Aut}P$ is the group of automorphisms of P. If $P' \cong P$, then $\mu_P = \mu_{P'}$. If M is a manifold with boundary, $Q \in \mathcal{C}_{\partial M}$, we define

$$\mathcal{C}_M(Q) = \{\langle P, \theta\rangle, P \in \mathcal{C}_M, \partial P \to Q \text{ an isomorphism}\}.$$

A morphisms is a $\varphi : \langle P', \theta'\rangle \to \langle P, \theta\rangle$, $\varphi : P' \xrightarrow{\cong} P$, $\theta' = \theta \circ \partial\varphi$. We write $\langle P', \theta'\rangle \cong \langle P, \theta\rangle$ if there exists a morphism. We write $\overline{\mathcal{C}_M}(Q)$ for the space of equivalence classes. If $N \hookrightarrow M$ is a codimension 1 submanifold, and M^{cut} is M cut along N with boundary $\partial M^{\mathrm{cut}} = \partial M \sqcup N \sqcup -N$. Fix a bundle $Q \to N$. Consider the category $\mathcal{C}_{M^{\mathrm{cut}}}(Q \sqcup Q)$ which is nothing but the category of triples $\langle P^{\mathrm{cut}}, \theta_1, \theta_2\rangle$, where $\theta_i : P^{\mathrm{cut}}|_N \to Q$ are isomorphisms. We have the gluing map

$$g_Q : \overline{\mathcal{C}}_M(Q \sqcup Q) \to \overline{\mathcal{C}}_M,$$
$$\langle P^{\mathrm{cut}}, \theta_1, \theta_2\rangle \to P^{\mathrm{cut}}/(\theta_1 = \theta_2).$$

Lemma 130. *The gluing map g_Q satisfies the following.*

(a) *g_Q maps onto the set of bundles of M whose restriction to N is isomorphic to Q.*

(b) *Let $\varphi \in Aut(Q)$ act on $\langle P^{\mathrm{cut}}, \theta_1, \theta_2\rangle$ by*

$$\varphi\langle P^{\mathrm{cut}}, \theta_1, \theta_2\rangle = \langle P^{\mathrm{cut}}, \varphi \circ \theta_1, \varphi \circ \theta_2\rangle.$$

Then $Aut(Q)$ acts transitively on $g_Q^{-1}([P])$ for $[P] \in \overline{\mathcal{C}}_M$

(c) *We have*

$$\mu_{[P]} = \mathrm{vol}(g_Q^{-1}[P])\mu_Q,$$

where

$$\mathrm{vol}(g_Q^{-1}[P]) = \frac{\#g_Q^{-1}([P])}{\#Aut\langle P^{\mathrm{cut}}, \theta_1, \theta_2\rangle}.$$

To a closed d-manifold Y, we associate the inner product space $E(Y) = L^2(\overline{\mathcal{C}}_Y$, and to a compact oriented manifold X an element $X \in E(\partial X)$. If X is closed, then Z_X is a complex number. If $\alpha = \hat{\alpha} = 0$ and X closed, we have

$$Z_X = \int_{\overline{\mathcal{C}}_X} d\mu([P]) = \mathrm{vol}(\overline{\mathcal{C}}_X)$$

Here, the integral is just a finite sum. When X has boundary,

$$Z_X(Q) = \int_{\overline{\mathcal{C}}_X(Q)} d\mu([P]) = \mathrm{vol}(\overline{\mathcal{C}}_X, Q),\ Q \in \mathcal{C}_X.$$

Theorem 131. *Let Γ be a finite group, $\hat{\alpha} \in C^{d+1}(B\Gamma, \mathbb{R}/\mathbb{Z})$, with assocations $Y \mapsto E(Y)$, $X \mapsto Z_X$, then*

(a) Functoriality: If $f : Y' \to Y$ is an orientation preserving diffeomorphism, we have an induced isometry $f_* : E(Y') \to E(Y)$. If $F : X' \to X$, $(\partial F)_*(Z_{X'}) = Z_X$, where $\partial F = F|_{\partial X}$.

(b) Orientation: $E(-Y) \cong \overline{E(Y)}$, $Z_{-X} = \overline{Z_X}$.

(c) If $Y = Y_1 \sqcup Y_2$, $E(Y) = E(Y_1) \otimes E(Y_2)$, and if $X = X_1 \sqcup X_2$, $Z(X) = Z(X_1) \otimes Z_(X_2)$.

(d) Gluing: If $Y \hookrightarrow X$ and X^{cut} as before, then $Z_X = \mathrm{Tr}_Y(Z_{X^{\mathrm{cut}}})$, where Tr_Y is a contraction as before.

We specialize now to the case $d = 2$. Let Y be a compact 2-manifold. Let ∂Y_i be the components of ∂Y. Fix diffeomorphisms $\varphi_i : S^1 \to \partial Y_i.$, called "+" if φ_i preserves orientation and "−", if it reverses it.

Theorem 132. *If Y is a compact oriented 2-manifold with a parametrized boundary, we have a functor*

$$Q \to L_Q,\ Q \in \mathcal{C}'_Y.$$

Here, $\mathcal{C}'_Y$ means bundles with fixed basepoint in ∂Y. This functor satisfies the gluing property: If $S \hookrightarrow Y$ has codimension 1, $\partial Y^{\mathrm{cut}} = \partial Y \sqcup S \sqcup -S$, and if $Q \in \mathcal{C}'_Y$, $Q^{\mathrm{cut}} \in \mathcal{C}'_{Y^{\mathrm{cut}}}$, then $L_Q \cong L_{Q^{\mathrm{cut}}}$.

Theorem 133. *The assignment $Y \to E(Y)$ satisfies*

(a) An orientation preserving diffeomorphism $f : Y' \to Y$ preserving the boundary parametrizations gives an induced isometry $f_* : E(Y') \cong E(Y)$.

(b) $E(-Y) \cong \overline{E(Y)}$.

(c) If $Y = Y_1 \sqcup Y_2$, then $E(Y) = E(Y_1) \otimes E(Y_2)$.

(d') Coalgebra: If S is a parametrized closed 1-manifold, then $A_s = E([0,1] \times S)$ is a unitary coalgebra*, and $A_{S_1 \sqcup S_2} \cong A_{S_1} \times A_{S_2}$.*

(e) Comodule: $E(Y)$ is a unitary $A_{\partial Y}$-comodule.

(d) If $S \hookrightarrow Y$ is a closed embedded circle, $\partial Y^{\mathrm{cut}} = \partial Y \sqcup S \sqcup -S$, then $E(Y^{\mathrm{cut}})$ is an A_S-comodule, $E(Y) \cong \frac{1}{\#\Gamma} \mathrm{Inv}_{A_S} E(Y^{\mathrm{cut}})$, where Inv *is an involution on A_S.*

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